

Singularity categories

Matt Booth

October 23, 2025

These are the notes for the TCC course I am running on singularity categories (<http://mattbooth.info/tcc.html>). They are an evolving document and are based on an earlier set of notes which can be found at <http://mattbooth.info/notes/sgnotes.pdf>.

There are several scattered exercises which I encourage doing. A star in the exercises indicates a hard part.

Contents

1	Introduction	3
2	Derived categories	6
2.1	Chain complexes and quasi-isomorphisms	6
2.2	Projective resolutions and chain homotopies	9
2.3	Homotopy categories and Ext	11
3	Triangulated categories	13
3.1	Axioms	13
3.2	Sums and subcategories	15
3.3	Verdier quotients and D_{sg}	17
4	Notions of regularity	19
4.1	Global dimension	19
4.2	Commutative rings I: the local setting	20
4.3	Commutative rings II	21
4.4	Depth	22
5	Buchweitz's stable category	24
5.1	Gorenstein rings	24
5.2	The stable category	26
5.3	Stable Ext	28

Chapter 1

Introduction

Let A be a ring (not necessarily commutative, but noetherian¹). Associated to A are many interesting invariants, which tell us about the geometry, homological algebra, representation theory etc. of A . Often one thinks of an invariant as being a number, or maybe a vector space, or at least something reasonably concrete. But one can attach categories (with various kinds of structure on top of them) to A , and consider these as invariants. For example:

- The (abelian) categories $\mathbf{mod}\text{-}A$ and $\mathbf{Mod}\text{-}A$. These are rather strong invariants: Morita theory tells us that when A and A' have equivalent module categories, then $Z(A) \cong Z(A')$. In the non-affine setting, one has the Gabriel-Rosenberg theorem, which says that if X is a quasi-separated (e.g. noetherian) scheme then X can be recovered from $\mathbf{Coh}(X)$.
- The triangulated category $D^b(\mathbf{mod}\text{-}A)$ and variants; in particular the big unbounded derived category $D(\mathbf{Mod}\text{-}A)$. These are looser invariants that still know about the homological properties of A . Recovery theorems are fewer and far between here. One famous example in the geometric setting is the Bondal-Orlov reconstruction theorem, which says that if X is a smooth projective variety with (anti)ample canonical bundle then $D^b(X)$ recovers X . For a nice exposition of this see [Că105].
- The triangulated category $\mathbf{per}(A)$ of perfect A -modules, i.e. those complexes of A -modules which are quasi-isomorphic to bounded complexes of finitely generated projectives.
- DG enhancements of the above triangulated categories. Triangulated categories have bad formal properties: for example, the category of triangulated categories doesn't have internal homs. Mapping cones are not functorial. Triangulated categories don't satisfy any reasonable form of geometric descent. Nobody knows how to recover invariants like the Hochschild

¹The point of this assumption is to make $\mathbf{mod}\text{-}A$ an abelian category. It is enough to assume that A is coherent and work with $\mathbf{coh}\text{-}A$ instead.

cohomology $HH^*(A)$ from just the triangulated structure on $D^b(A)$. All of these problems are solved when passing to pretriangulated dg categories. For more on why you should like dg categories, see [Toë11].

- If A is reasonably commutative², one can equip $D(A)$ or $\mathbf{per}(A)$ with the standard monoidal structure given by the (derived) tensor product. This is the starting point for the subject of tensor triangular geometry.

Here is one natural question to ask. As we will see, it leads to a rich theory. It is clear that $\mathbf{per}(A)$ is a subcategory of $D^b(\mathbf{mod}\text{-}A)$. What sort of difference is there between these two things?

Example 1.1. Let k be a field and consider the ring $A := k[x]/x^2$. The module k has a projective resolution given by

$$\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A$$

from which it is clear that $\mathrm{Ext}^i(k, k) \cong k$ for all $i \geq 0$. Since perfect complexes must have bounded self-Exts, k cannot be perfect.

So we want to consider the difference between the triangulated category $D^b(\mathbf{mod}\text{-}A)$ and its triangulated subcategory $\mathbf{per}(A)$. In other words, we are interested in the (Verdier) quotient $D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$. We will soon see the following result, whose main ingredient is the Auslander–Buchsbaum–Serre theorem:

Theorem 1.2. *Let A be a commutative noetherian ring of finite Krull dimension. Then A is regular if and only if $D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$ vanishes.*

Recall that every smooth commutative ring is regular³. So for example, the ring $\mathbb{C}[x_1, \dots, x_n]$ is regular, since it is the coordinate ring of complex n -space, which is smooth. The rings $\mathbb{C}[x, y]/xy$, $\mathbb{C}[x, y]/(x^2 - y^3)$, and $\mathbb{C}[x, y]/(x^3 + x^2 - y^2)$ are not smooth, since they are the coordinate rings of the coordinate axes $xy = 0$, the cuspidal cubic $x^2 = y^3$, and the nodal cubic $y^2 = x^2 + x^3$ respectively, all of which have singular points (to see this, either draw a picture or use calculus).

With this in mind we will call $D_{\mathrm{sg}}(A) := D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$ the singularity category of A , and regard it as a homological invariant that detects the singularities of A (even when A is noncommutative!). Along the way we will see a purely homological characterisation of smoothness in terms of global dimension⁴ - it should already be clear that the existence of finitely generated modules without a bounded projective resolution is an obstruction to the vanishing of $D_{\mathrm{sg}}(A)$.

² E_2 is enough. A more down to earth example is when $A = B \otimes B^{\mathrm{op}}$ is the enveloping algebra of an algebra B , so that $D(B)$ is the derived category of B -bimodules.

³The converse is true if one works over a perfect field, and in particular a field of characteristic zero.

⁴Be warned that there is a terminology clash here. This homological characterisation of smoothness is not equivalent to ‘homological smoothness’; i.e. asking that A be perfect as an A -bimodule.

Remark 1.3. The above motivates our choice of $D^b(\mathbf{mod}\text{-}A)$ as opposed to the sometimes more natural choice of $D(\mathbf{Mod}\text{-}A)$: the quotient $D(\mathbf{Mod}\text{-}A)/\mathbf{per}(A)$ fails to tell us much about the singularities of A , since $D(\mathbf{Mod}\text{-}A)$ is far too big of an object. From the perspective of homotopy theory, we may want A to be a differential graded algebra, in which case $D^b(A)$ is not necessarily well behaved (e.g. A need not be an object of $D^b(A)$). There are some fixes one can make here which we may discuss later.

In the next part of the seminar we will see two important alternate constructions of the singularity category.

The first description has a representation-theoretic flavour. Suppose that A is (Iwanaga)–Gorenstein; i.e. A has finite injective dimension over itself. Buchweitz [Buc86] noticed that the singularity category of A has a description as the stable category of maximal Cohen–Macaulay modules over A . Recall that a finitely generated A -module X is MCM if $\mathrm{Ext}_A^i(X, A)$ vanishes for $i > 0$ (there is a more general characterisation in terms of depth). Loosely, the stable category of MCM modules is what one gets by taking the category of MCM modules and quotienting out by projective modules. The shift of an MCM module X is its (inverse) syzygy.

The second description has a more geometric flavour. Suppose that $R = k[[x_1, \dots, x_n]]/f$ is a complete local hypersurface singularity. A matrix factorisation of f is a pair of free finite rank $k[[x_1, \dots, x_n]]$ -modules M and N together with ‘differentials’ $d : M \rightarrow N$ and $d : N \rightarrow M$ such that $d^2 = f$. One can organise the collection of matrix factorisations into a category, and after modding out by a suitable notion of homotopy the category of matrix factorisations of f becomes equivalent to the singularity category of R . There is much literature in this direction, which we will mention later.

As a general reference for this part, see [Boo19, Chapter 6] or [Boo21, Sections 4 and 5] and the references contained therein. A good general reference is [Sym22].

Chapter 2

Derived categories

We begin with our basic object of study, the derived category of a ring. For far more comprehensive treatments than this section provides, see [Wei94] or [Yek20]. If you are geometrically inclined, you will enjoy [Tho01].

2.1 Chain complexes and quasi-isomorphisms

Let A be a ring. All modules are right modules unless otherwise specified. A **cochain complex** is a $\mathbb{Z}$ -indexed sequence of A -modules $\{M^n\}_{n \in \mathbb{Z}}$ together with differentials $d : M^n \rightarrow M^{n+1}$ such that $d^2 = 0$. A **chain complex** is defined similarly, except that the differentials lower the degree; in this case we typically write the indices as subscripts. One can convert between homological and cohomological notation by setting $M^n = M_{-n}$. For us, the term **complex** will always mean a cochain complex. Observe that the complexes concentrated in degree zero are precisely the A -modules.

Remark 2.1. The **Koszul sign rule** says that when an object of degree p moves past an object of degree q , then a sign change of $(-1)^{pq}$ is required. All of the sign conventions in these notes can be worked out with a careful (if, sometimes, non-obvious!) use of the Koszul sign rule.

If M is a complex then its **shift** $M[1]$ is the complex with $M[1]^n = M^{1+n}$ and differential $d_{M[1]} = -d_M$. Graphically, this corresponds to shifting the complex *left*. There are analogous shifts $M[n]$ for all integers n , and we have $M[i][j] \cong M[i+j]$.

A **morphism** of complexes $M \rightarrow N$ is simply a collection of maps $M_n \rightarrow N_n$. A **morphism of degree n** is a morphism $M[n] \rightarrow N$. A morphism $f : M \rightarrow N$ is a **chain map** if it is compatible with the differentials in the sense that $d_N f = (-1)^{\deg f} f d_M$.

An **n -cocycle** in a complex M is an element $m \in M^n$ such that $dm = 0$. We denote the A -module of n -cycles by $Z^n(M)$. An **n -coboundary** is an element $m \in M^n$ of the form $m = dn$. We denote the module of n -coboundaries by $B^n(M)$. Clearly we have $B^n M \subseteq Z^n M$, and the quotient $Z^n(M)/B^n(M)$ is

the n^{th} **cohomology** module $H^n(M)$. We often assemble the cohomology of M into a complex $H(M)$ with zero differential.

Exercise 2.2 (Mapping complexes).

1. If M, N are complexes, show that there is a natural complex of $\mathbb{Z}$ -modules $\text{Hom}_A(M, N)$ which in degree n consists of the morphisms of degree n , and whose differential is defined by $f \mapsto d_N f - (-1)^{\deg f} f d_M$. Show that the n -cocycles are precisely the degree n chain maps.
2. If X is a complex and f, g are two endomorphisms of X , their **graded commutator** is $[f, g] = fg - (-1)^{\deg(f) \cdot \deg(g)} gf$. Show that the graded commutator $[d_X, -]$ makes the graded abelian group $\text{End}_A(X)$ into a complex.
3. Show that the obvious inclusion $\text{Hom}_A(M, N) \rightarrow \text{End}_A(M \oplus N)$ is a chain map. This explains the sign in the differential of the hom-complex.
4. *If B is another ring, Y a complex of right B -modules and Z a complex of left B -modules, show that the graded abelian group $Y \otimes_B Z$ which in degree i is given by $\bigoplus_{p+q=i} Y^p \otimes_B Z^q$ is a complex under the differential $d(y \otimes z) = dy \otimes z + (-1)^{\deg(y)} y \otimes dz$. If M is a B - A -bimodule and N an A -module, show that the evaluation pairing $\text{Hom}(M, N) \otimes_B M \rightarrow N$ is a chain map.
5. *Formulate and prove a hom-tensor adjunction for complexes.

Exercise 2.3. Compute the cohomology of the two-term complex

$$0 \rightarrow A \xrightarrow{a} A \rightarrow 0.$$

If $f : M \rightarrow N$ is a chain map, it induces maps $H^i f : H^i M \rightarrow H^i N$ on cohomology groups. A chain map f is a **quasi-isomorphism** if each $H^i f$ is an isomorphism. Two chain complexes are **quasi-isomorphic** if there is a (finite length) zig-zag of quasi-isomorphisms between them.

Exercise 2.4. Let M be the complex of abelian groups $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$, with the rightmost $\mathbb{Z}$ placed in degree zero. Show that the projection $M \rightarrow \mathbb{Z}/n$ is a quasi-isomorphism.

Exercise 2.5. If k is a field, and M is a complex of k -vector spaces, show that M is quasi-isomorphic to $H(M)$. (Hint: if $U \hookrightarrow V$ is a subspace, then U admits a complement $U^\perp \hookrightarrow V$ such that $U \oplus U^\perp = V$.)

Exercise 2.6. Let A be the ring $\mathbb{C}[x, y]$, and consider the two complexes

$$M = A \oplus A \xrightarrow{(x, y)} A$$

and

$$M' = A \xrightarrow{x \mapsto 0, y \mapsto 0} \mathbb{C}$$

Show that $H(M) \cong H(M')$. Show that there is no (A -linear!) quasi-isomorphism $M \rightarrow M'$. *Show that M and M' are not quasi-isomorphic.

A complex is **acyclic** if it is quasi-isomorphic to the zero complex. A complex M is **exact at n** if $H^n(M) \cong 0$.

Exercise 2.7. If M is a complex, show that M is acyclic if and only if M is exact at all $i \in \mathbb{Z}$.

Exercise 2.8.

- When is a two-term complex $0 \rightarrow M_0 \xrightarrow{f} M_1 \rightarrow 0$ acyclic?
- When is a three-term complex $0 \rightarrow M_0 \xrightarrow{f} M_1 \xrightarrow{g} M_2 \rightarrow 0$ acyclic?

If $f : M \rightarrow N$ is a chain map, the **mapping cone** $\text{cone}(f)$ is the complex with $\text{cone}(f)_i = M_{i+1} \oplus N_i$ and differential given by the upper-triangular matrix $\begin{pmatrix} d_M & f \\ 0 & -d_N \end{pmatrix}$.

Exercise 2.9. If $f : M \rightarrow N$ is a chain map, show that there are natural chain maps $M \rightarrow N \rightarrow \text{cone}(f) \rightarrow M[1]$. Show that they induce a long exact sequence $\cdots \rightarrow H^i(M) \xrightarrow{H^i(f)} H^i(N) \rightarrow H^i(\text{cone}(f)) \rightarrow H^{i+1}(M) \rightarrow \cdots$. Deduce that f is a quasi-isomorphism if and only if $\text{cone}(f)$ is acyclic.

{tStructureExer1}

Exercise 2.10. Let $M = 0 \rightarrow M_n \rightarrow \cdots \rightarrow M_m \rightarrow 0$ be a complex with finitely many nonzero terms. Show that M can be obtained from the finite set $\{M^i\}_i$ via a finite sequence of shifts and mapping cones.

A complex M is **strictly bounded above** if $M^i \cong 0$ for all $i \gg 0$ and **strictly bounded below** if $M^i \cong 0$ for all $i \ll 0$. A complex M is **strictly bounded** if $M^i \cong 0$ for all but finitely many i ; this is equivalent to being strictly bounded above and strictly bounded below. We say that M is **bounded** if the complex $H(M)$ is bounded, and similarly for above/below.

Synonyms for bounded above in the literature include **right bounded** and **eventually connective**; bounded below is also referred to as **left bounded** or **eventually connective**. Sometimes we will use **cohomologically bounded** as a synonym for bounded, for emphasis.

If A is a ring then we let $\mathbf{Ch}(A)$ denote the category of chain complexes of A -modules, with morphisms given by the chain maps.

Definition 2.11. Let A be a ring. The **derived category** of A is the localisation $D(A) := \mathbf{Ch}(A)[\text{quasi-iso}^{-1}]$ given by formally inverting $\mathbf{Ch}(A)$ at the quasi-isomorphisms. This has subcategories $D^+(A)$, $D^-(A)$, $D^b(A)$ on those complexes which are bounded below, bounded above, and bounded, respectively. If A is noetherian, we write $D(\mathbf{mod}\text{-}A)$ for the full subcategory of $D(A)$ on the bounded complexes of finitely generated A -modules.

Exercise 2.12. Show that $D(A)$ is also the quotient $\mathbf{Ch}(A)/(\text{acyclic modules})$ where we identify all acyclic modules with zero. (Hint: mapping cones.)

For the rest of this section, we will try to understand $D(A)$. Sometimes - often! - this will be too hard and we restrict ourselves to $D^b(A)$ instead.

2.2 Projective resolutions and chain homotopies

Recall that an A -module is **projective** if it is a summand of a free module.

Exercise 2.13. Let P be a module. Show that P is projective if and only if for all maps $f : P \rightarrow N$ and surjections $\pi : M \twoheadrightarrow N$, there exists a lift of f through π , i.e. a map $\tilde{f} : P \rightarrow M$ such that $\pi\tilde{f} = f$.

Exercise 2.14. If p, q are distinct primes, show that $\mathbb{Z}/p$ is a projective $\mathbb{Z}/(pq)$ -module.

Exercise 2.15. A **short exact sequence** is an exact complex of the form $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$. Let P be a projective module. Show that the functor $\text{Hom}_A(P, -) : \mathbf{mod}\text{-}A \rightarrow \mathbf{Ab}$ is **exact**; i.e. preserves short exact sequences.

If M is a module, a **projective resolution** of M is a complex

$$P = \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$$

of projectives together with a quasi-isomorphism $P \xrightarrow{\sim} M$.

Exercise 2.16. If $P = \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$ is a complex of projectives, show that P is a projective resolution of M if and only if $H^i(P) \cong 0$ for $i \neq 0$, and $H^0(P) \cong M$.

Proposition 2.17. *Projective resolutions exist.*

Proof. Given an arbitrary module M , we can find a free module P_0 and a surjection $P_0 \twoheadrightarrow M$; for example, take P_0 to be the free module on the set M . This yields a short exact sequence

$$0 \rightarrow K_0 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where K_0 denotes the kernel. Find a free module P_1 and a surjection $P_1 \twoheadrightarrow K_0$; by composition this yields an exact sequence

$$0 \rightarrow K_1 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

where K_1 denotes the kernel of $P_1 \rightarrow K_0$. Continuing inductively produces the desired resolution. $\square$

Exercise 2.18. If A is noetherian and M is a finitely generated module, show that M has a resolution by finitely generated projectives.

Proposition 2.19. *Let $f : M \rightarrow M'$ be a map of modules. If $P \rightarrow M$ and $P' \rightarrow M'$ are projective resolutions, then f lifts to a map $P \rightarrow P'$.*

Proof. Iteratively use the lifting property of projective modules. $\square$

{truncExer}

Exercise 2.20 (Truncations). If M is a right bounded complex, show that there is a strictly right bounded complex M' with a quasi-isomorphism $M \simeq M'$. Show that if M is a complex of projectives, then one can take M' to be a complex of projectives. (Hint: if n is the largest integer for which $H^n(M)$ is nonzero, then one can arrange for M' to agree with M in degrees $< n$ and be zero in degrees $> n$.)

If M is a right bounded complex, a **projective resolution** of M is a right bounded complex of projectives P with a quasi-isomorphism $P \rightarrow M$.

Proposition 2.21. *Projective resolutions of right bounded complexes exist.*

Proof. The construction we give is known as the **Cartan–Eilenberg resolution**. By 2.20 we may assume that $M_i \cong 0$ for $i \gg 0$. First resolve each M_n individually to obtain a projective resolution P_n . The differentials in M lift to maps $P_n \rightarrow P_{n+1}$ and this yields a double complex P . Totalising P gives the desired resolution. $\square$

Remark 2.22. One can also use this construction to give, for every complex M , a complex of projectives P with a quasi-isomorphism $P \rightarrow M$. In general, such a complex P is not ‘nice enough’ from a homotopical viewpoint - for evidence towards this see 2.27 below. The correct generalisation of ‘right bounded complex of projectives’ to the unbounded setting is given by the concept of **h-projective** complexes, which we will not discuss further.

If $f, g : M \rightarrow N$ are two maps of degree n , a **chain homotopy** from f to g is a degree $n-1$ map $h : M \rightarrow N$ such that $\partial h = f - g$, where ∂h denotes the differential of h in the mapping complex $\text{Hom}_A(M, N)$. We write $f \simeq g$.

Say that two complexes M, N are **chain homotopy equivalent** if there exists a pair of maps $f : M \rightarrow N$ and $g : N \rightarrow M$ such that $fg \simeq \text{id}_N$ and $gf \simeq \text{id}_M$. A complex is **nullhomotopic** if it is chain homotopy equivalent to 0.

Exercise 2.23. Show that a complex M is nullhomotopic if and only if there exists a degree -1 map $h : M \rightarrow M$ such that $\partial h = \text{id}_M$.

Exercise 2.24. Show that if M, N are chain homotopy equivalent complexes then they are quasi-isomorphic.

Proposition 2.25. *Let $f : M \rightarrow N$ be a map of modules. If $g, g' : P \rightarrow Q$ are two lifts of f to projective resolutions, then g and g' are chain homotopic.*

Proof. The homotopy is constructed iteratively using the lifting property. $\square$

Corollary 2.26. *If P is a right bounded acyclic complex of projectives, then P is nullhomotopic.*

{unbExer}

Exercise 2.27. Let A be the ring $k[\varepsilon]/\varepsilon^2$ of dual numbers. Let M be the unbounded complex of projective modules

$$\cdots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} \cdots$$

Show that M is acyclic. Show that M is not nullhomotopic. In particular, the previous Corollary is false if one drops the right bounded hypothesis, essentially because one needs a rightmost module at which to begin the induction.

Proposition 2.28. *Let P be two right bounded complexes of projectives. Then $f : P \rightarrow Q$ is a quasi-isomorphism if and only if it is a chain homotopy equivalence.*

Proof. We only need to prove the forwards implication, so assume that f is a quasi-isomorphism. Let C be the cone of f , so that C is acyclic. Hence the previous Corollary yields a nullhomotopy of C . But a chain homotopy equivalence $C \simeq 0$ is the same thing as a chain homotopy equivalence $P \simeq Q$. $\square$

2.3 Homotopy categories and Ext

Definition 2.29. Let A be a ring. The **homotopy category** of A is the category $K^-(\mathbf{Proj}\text{--}A)$ whose objects are the strictly right bounded complexes of projectives. The morphisms are given by homotopy classes of chain maps. We may also replace K^- by K^b to consider only strictly bounded complexes, or **Proj** by **proj** to consider only those complexes of finitely generated projectives (which is only well-behaved when A is noetherian).

Theorem 2.30. *There is an equivalence $K^-(\mathbf{Proj}\text{--}A) \rightarrow D^-(A)$ which is the identity on objects.*

Proof. If $\mathbf{Ch}^-(\mathbf{Proj}\text{--}A)$ denotes the category of strictly right bounded complexes of projectives (without taking homotopy classes), then there is an obvious functor $\mathbf{Ch}^-(\mathbf{Proj}\text{--}A) \rightarrow D(A)$ which sends an object to its equivalence class in the quotient. Since quasi-isomorphic objects in the source are chain homotopy equivalent, it descends to a functor $K^-(\mathbf{Proj}\text{--}A) \rightarrow D(A)$. Since every right bounded complex admits a projective resolution, and these can be taken to be strictly bounded by 2.20, this gives an essentially surjective functor $K^-(\mathbf{Proj}\text{--}A) \rightarrow D^-(A)$. Fully faithfulness of this functor follows from the above Proposition. $\square$

Similarly, we obtain an equivalence $K^-(\mathbf{proj}\text{--}A) \simeq D^-(\mathbf{mod}\text{--}A)$.

Definition 2.31. If X, Y are two objects of $D^-(A)$, then we write

$$\mathrm{Ext}_A^i(X, Y) := \mathrm{Hom}_{D(A)}(X, Y[i]).$$

By the above Theorem, we may compute Ext as follows. First find projective resolutions P, Q of X, Y respectively. We put $\mathbb{R}\mathrm{Hom}_A(X, Y) := \mathrm{Hom}_A(P, Q)$, and it follows that $\mathrm{Ext}_A^i(X, Y) \cong H^i \mathbb{R}\mathrm{Hom}_A(X, Y)$. The complex $\mathbb{R}\mathrm{Hom}_A(X, Y)$ is called the (total) **derived hom-complex** from X to Y .

Exercise 2.32. Show that $\mathbb{R}\mathrm{Hom}_A(X, Y)$ is well-defined as an object of $D(\mathbb{Z})$, i.e. different choices of resolution lead to quasi-isomorphic derived hom complexes. *If R is a commutative ring and A is an R -algebra, show that $\mathbb{R}\mathrm{Hom}_A(X, Y)$ is well-defined as an object of $D(R)$.

In fact, if one runs our above proofs a little more carefully, they show that there is a natural quasi-isomorphism $\mathbb{R}\mathrm{Hom}_A(X, Y) \simeq \mathrm{Hom}_A(P, Y)$, i.e. one only need resolve in the first variable. One can also resolve only in the second variable by taking *injective resolutions*, which we will not treat here.

Definition 2.33. A complex is **perfect** if it is quasi-isomorphic to a strictly bounded complex of finitely generated projectives. We denote the category of perfect complexes by $\mathbf{per}(A) \hookrightarrow D(A)$.

By the previous Theorem, there is an equivalence $K^b(\mathbf{proj}\text{-}A) \simeq \mathbf{per}(A)$.

Exercise 2.34. If P is perfect and X is bounded, show that $\mathbb{R}\mathrm{Hom}_A(P, X)$ is bounded. Let A be the dual numbers and let M be the complex

$$\cdots \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A \xrightarrow{\varepsilon} A$$

with the rightmost A in degree zero. Show that M is a projective resolution of k and use this to compute $\mathrm{Ext}_A^*(k, k)$. Deduce that k is not a perfect A -module.

Chapter 3

Triangulated categories

The notion of triangulated category is an axiomatisation of some of the properties that derived categories. As intimated above, triangulated categories will not be completely sufficient for our uses, so we only give a sketch of the ideas. See [Nee01], [Wei94] or [Yek20] for comprehensive discussions.

3.1 Axioms

Let k be a commutative ring. A k -linear **triangulated category** is a k -linear category $\mathcal{C}$ together with two extra pieces of data. The first piece of data is a linear autoequivalence Σ of $\mathcal{C}$, which we call the suspension or the shift functor. A **triangle** in $\mathcal{C}$ is a sequence of three morphisms

$$X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$$

which we will frequently abbreviate by dropping the ΣX term and letting the rightmost arrow point to nowhere. A **morphism** of triangles is a triple of morphisms which fits into the obvious commutative diagram. The second piece of data is a class of **exact** (or **distinguished**) triangles. The suspension and the shift should satisfy the following axioms:

- TR0: Exact triangles are closed under isomorphisms and under Σ .
- TR1: The triangle $X \xrightarrow{\text{id}} X \rightarrow 0 \rightarrow$ is exact. Every morphism $f : X \rightarrow Y$ has a **cone** Z , which fits into an exact triangle of the form $X \rightarrow Y \rightarrow Z \rightarrow$. We caution that the cone *need not be functorial*.
- TR2: One can rotate triangles: the triangle $X \rightarrow Y \rightarrow Z \rightarrow$ is exact if and only if $Y \rightarrow Z \rightarrow \Sigma X \xrightarrow{-}$ is, where one has to flip the sign on the indicated map.
- TR3: Given a morphism $f \rightarrow g$ in the arrow category (i.e. a commutative square from f to g !) then there is an induced morphism $\text{cone}(f) \rightarrow \text{cone}(g)$

fitting into a morphism of exact triangles. This morphism *need not be unique*.

- TR4: the famous octahedral axiom. Loosely this encodes a version of the third isomorphism theorem, if one thinks of cones as homotopy cokernels.

Proposition 3.1. $D(A)$ and $D^b(A)$ are triangulated categories.

Proof. The suspension is given by the shift $[1]$. The exact triangles are precisely those triangles isomorphic to triangles of the form $X \rightarrow Y \rightarrow \text{cone}(f) \rightarrow$. $\square$

The intuition is that an exact triangle behaves like a rolled-up long exact sequence. Indeed, in our main example $D(A)$, given a morphism $f : X \rightarrow Y$ the induced morphism $\text{cone}(f) \rightarrow X[1]$ corresponds precisely to the connecting morphisms in the associated long exact sequence.

Exercise 3.2. Show that cones are unique up to isomorphism. *Show that this isomorphism need not be unique.

If $f : X \rightarrow Y$ is a morphism, with cone C , we will often refer to $\Sigma^{-1}C$ as the **cocone** of f . Observe that this fits into the exact triangle $\Sigma^{-1}C \rightarrow X \xrightarrow{f} Y \rightarrow$.

Exercise 3.3. Show that $X \rightarrow Y \rightarrow Z \rightarrow$ is exact if and only if the rotated triangle $\Sigma^{-1}Z \rightarrow X \rightarrow Y \rightarrow$ is. (Hint: what happens if you rotate a triangle three times?)

{zeroExer}

Exercise 3.4. Show that any two consecutive compositions in an exact triangle are zero.

A **triangle functor** between triangulated categories is a functor that commutes with Σ and sends exact triangles to exact triangles.

Exercise 3.5. If X is a fixed object of $D^b(A)$, show that the functor

$$\mathbb{R}\text{Hom}_A(X, -) : D(A) \rightarrow D(\mathbb{Z})$$

is a triangle functor. Deduce that if $Y \rightarrow Z \rightarrow W \rightarrow$ is an exact triangle in $D(A)$ then there is a long exact sequence

$$\cdots \rightarrow \text{Ext}_A^i(X, Y) \rightarrow \text{Ext}_A^i(X, Z) \rightarrow \text{Ext}_A^i(X, W) \rightarrow \text{Ext}_A^{i+1}(X, Y) \rightarrow \cdots$$

of abelian groups.

We generalise the previous exercise. If $\mathcal{T}$ is a triangulated category, we write $\text{Ext}_{\mathcal{T}}^i(X, Y) := \mathcal{T}(X, \Sigma^i Y)$.

{TriCatLES}

Proposition 3.6. Let $\mathcal{T}$ be a triangulated category. If X is any object of $\mathcal{T}$ and $Y \rightarrow Z \rightarrow W \rightarrow$ is an exact triangle, then there is a long exact sequence

$$\cdots \rightarrow \text{Ext}_{\mathcal{T}}^i(X, Y) \rightarrow \text{Ext}_{\mathcal{T}}^i(X, Z) \rightarrow \text{Ext}_{\mathcal{T}}^i(X, W) \rightarrow \text{Ext}_{\mathcal{T}}^{i+1}(X, Y) \rightarrow \cdots$$

of abelian groups.

Proof. By rotating the triangle and applying shifts it suffices to check exactness at $\text{Ext}_{\mathcal{T}}^0(X, Z) \cong \mathcal{T}(X, Z)$. The composition is zero by 3.4, and exactness follows from completing a commutative square

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

to a morphism of exact triangles. $\square$

Remark 3.7. There is a similar long exact sequence involving the $\text{Ext}_{\mathcal{T}}^i(-, X)$ functors. It can be derived from the previous proposition using the fact that the opposite of a triangulated category is itself triangulated.

Remark 3.8. Say that a triangulated category $\mathcal{T}$ **has functorial cones** if there is a functor $C : \text{Ar}(\mathcal{T}) \rightarrow \mathcal{T}$ such that for each f , the object $C(f)$ is a cone of f . Then a triangulated category $\mathcal{T}$ has functorial cones if and only if $\mathcal{T}$ is semisimple abelian. This fact goes back to Verdier’s thesis [Ver96, 1.2.13], but Greg Stevenson has given a modern proof [Ste]. The loose idea is that having functorial cones actually forces $\mathcal{T}$ to have kernels and cokernels. Then the claim follows because monos and epis split.

Remark 3.9. For homotopy theorists: we will later say that a triangulated category $\mathcal{T}$ has an **enhancement** if it is the homotopy category of a pretriangulated dg category (so, roughly, if one can coherently assign derived hom-complexes $\mathbb{R}\text{Hom}_{\mathcal{T}}(X, Y)$ to every $X, Y \in \mathcal{T}$). A more general notion than this is a **topological enhancement**, namely that $\mathcal{T}$ is the homotopy category of a stable ∞ -category (note that pretriangulated dg categories are precisely the k -linear stable ∞ -categories). The classical stable homotopy category admits a topological enhancement, but not an enhancement. Suppose that $\mathcal{C}$ is a stable ∞ -category, so that the homotopy category $h_0\mathcal{C}$ is canonically triangulated. In $\mathcal{C}$, one can make a functorial choice of cone, giving a morphism $\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \mathcal{C}$. One then obtains a functor $h_0\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow h_0\mathcal{C}$. There is a comparison map $h_0\text{Fun}(\Delta^1, \mathcal{C}) \rightarrow \text{Ar}(h_0\mathcal{C})$, but it fails to be an equivalence, as spelled out in [Lur]. In particular, the ‘functorial cone’ does not factor through a morphism $\text{Ar}(h_0\mathcal{C}) \rightarrow \mathcal{C}$, and so this does not prove that every enhanceable triangulated category is actually abelian.

3.2 Sums and subcategories

Exercise 3.10. An **extension** of A by B is an object X fitting into an exact triangle $B \rightarrow X \rightarrow A \rightarrow$. Show that extensions are classified by $\text{Ext}^1(A, B)$.

Proposition 3.11. *Triangulated categories have finite biproducts.*

Proof. Let $\mathcal{T}$ be a k -linear triangulated category. It has a zero object, so we show that $\mathcal{T}$ has binary biproducts. We define $X \oplus Y$ to be the unique object

fitting into the exact triangle $X \rightarrow X \oplus Y \rightarrow Y \xrightarrow{0}$. Equivalently, $X \oplus Y$ is the unique extension corresponding to $0 \in \text{Ext}_{\mathcal{T}}^1(Y, X)$. By completing the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & X \oplus Y & \longrightarrow & Y & \xrightarrow{0} & \longrightarrow \\ \downarrow \text{id} & & & & \downarrow & & \\ X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & \end{array}$$

to a morphism of exact triangles, we see that X is a retract of $X \oplus Y$. Similarly we see that Y is also a retract. One can further extend this diagram to a diagram with exact rows of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \xrightarrow{\text{id}} & Y & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow \text{id} & & \\ X & \longrightarrow & X \oplus Y & \longrightarrow & Y & \xrightarrow{0} & \longrightarrow \\ \downarrow \text{id} & & \downarrow & & \downarrow & & \\ X & \xrightarrow{\text{id}} & X & \longrightarrow & 0 & \longrightarrow & \\ \downarrow & & \downarrow 0 & & \downarrow & & \end{array}$$

and the Nine Lemma now shows that the middle vertical triangle is exact; in other words we have $X \oplus Y \cong Y \oplus X$.

If Z is a third object, apply the long exact sequence of 3.6 together with the above splittings to deduce that $\mathcal{T}(X \oplus Y, Z) \cong \mathcal{T}(X, Z) \oplus \mathcal{T}(Y, Z)$. The fact that $\oplus$ is a coproduct now follows from the Yoneda lemma. The argument for products is similar. $\square$

Definition 3.12. If $\mathcal{D}$ is a triangulated category, then a **triangulated subcategory** $\mathcal{C}$ is a full subcategory which contains 0 and is closed under shifts and cones. The restriction of the triangulated structure to $\mathcal{C}$ makes $\mathcal{C}$ into a triangulated category and the inclusion into a triangle functor.

Exercise 3.13. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C} \ni 0$ a full subcategory.

1. Show that if $\mathcal{C}$ is closed under cones, then it is a triangulated subcategory that is moreover closed under finite direct sums.
2. Show that if $\mathcal{C}$ is closed under shifts, then it is a triangulated subcategory if and only if it is closed under extensions.

Let $\mathcal{D}$ be a triangulated category. A triangulated subcategory $\mathcal{C}$ of $\mathcal{D}$ is called **thick** (or **épaisse**) if it is closed under direct summands. If S is a set of objects of $\mathcal{T}$, the **thick closure** of S is the smallest thick subcategory of $\mathcal{T}$ containing S . We denote it by $\mathbf{thick}_{\mathcal{D}}(S)$ or just $\mathbf{thick}(S)$ when the context is clear. When $S = \{X\}$ then we write $\mathbf{thick}_{\mathcal{D}}(X)$ or just $\mathbf{thick}(X)$.

Proposition 3.14. $\mathbf{thick}(S)$ exists.

Proof. One sets $\mathbf{thick}_1(S)$ to be the full subcategory on those objects which are direct summands of objects of the form $\bigoplus_{i=1}^n \Sigma^{t_i} S_i$ with $S_i \in S$. One then inductively sets $\mathbf{thick}_{r+1}(S)$ to be the full subcategory given by extending objects of $\mathbf{thick}_1(S)$ by objects of $\mathbf{thick}_r(S)$. Then we take $\mathbf{thick}(S)$ to be the union of all the $\mathbf{thick}_r(S)$. $\square$

{tStructureExer2}

Exercise 3.15 (2.10, derived version). Let A be a ring and M a bounded complex. Show that $M \in \mathbf{thick}\{H^i(M) : i \in \mathbb{Z}\}$. (Hint: induct on i .)

Example 3.16. If A is a ring, the subcategory of $\mathbf{per}(A)$ on the strictly bounded complexes of finitely generated free A -modules is a triangulated subcategory which is not in general thick, since it need not be closed under summands.

Proposition 3.17. *Let A be a noetherian ring. Then there are equalities $\mathbf{per}(A) = \mathbf{thick}_{D(A)}(A) = \mathbf{thick}_{D^b(A)}(A)$.*

Proof. The second equality is easy to see so we concentrate on the first. Certainly $\mathbf{per}(A)$ is a triangulated subcategory of $D(A)$, and moreover it is thick since summands of finitely generated projective modules are finitely generated projective. So we need only show that $\mathbf{per}(A) \subseteq \mathbf{thick}_{D(A)}(A)$. Since the latter is closed under sums and summands we have $\mathbf{proj}\text{-}A \subseteq \mathbf{thick}_{D(A)}(A)$. But by 2.10 we are done. $\square$

Remark 3.18. In more general settings one defines $\mathbf{per}(A) := \mathbf{thick}_{D(A)}(A)$.

3.3 Verdier quotients and D_{sg}

Suppose that $F : \mathcal{C} \rightarrow \mathcal{D}$ is a triangle functor. The **kernel** of F is the full subcategory of $\mathcal{C}$ on those objects x such that $F(x) \cong 0$. We denote the kernel by $\ker(F)$.

Exercise 3.19. Show that $\ker(F)$ is a thick triangulated subcategory of $\mathcal{C}$.

Definition 3.20. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C} \hookrightarrow \mathcal{D}$ a triangulated subcategory. The **Verdier quotient** is the universal triangulated category $\mathcal{D}/\mathcal{C}$ equipped with a functor $\pi : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$ such that $\mathcal{C} \subseteq \ker(\pi)$. In other words, if $F : \mathcal{D} \rightarrow \mathcal{D}'$ is a triangle functor which kills $\mathcal{C}$, then F factors through π .

Proposition 3.21. *Verdier localisations exist. The kernel of the natural projection $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$ is precisely $\mathbf{thick}_{\mathcal{D}}(\mathcal{C})$.*

In particular, if $\mathcal{C}$ is thick, then $\mathcal{C}$ is precisely the kernel of the projection to the Verdier quotient.

Proof sketch. The rough idea is to formally adjoin inverses to morphisms whose cone lies in $\mathcal{C}$. We follow the construction given in [Nee01, §2.1]. The objects of $\mathcal{D}/\mathcal{C}$ will simply be the objects of $\mathcal{D}$. For two such objects X, Y , let $\alpha(X, Y)$ denote the set of roofs $X \xleftarrow{f} Z \xrightarrow{g} Y$ where f has cone in $\mathcal{C}$. We think of such

a roof as a ‘fraction’ g/f . A morphism of roofs $R \rightarrow R'$ is simply a morphism $Z \rightarrow Z'$ making the obvious diagram commute. Declare that two such roofs R, R' are equivalent if they are dominated by a common roof $R \leftarrow R'' \rightarrow R'$. Define $(\mathcal{D}/\mathcal{C})(X, Y)$ to be the set of equivalence classes of roofs. Composition is given by homotopy pullback; a concrete model for the homotopy pullback of $Z \rightarrow Y \leftarrow Z'$ is given by the cocone of the induced map $Z \oplus Z' \rightarrow Y$. This makes $\mathcal{D}/\mathcal{C}$ into a category, with an obvious functor $\pi : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$ which sends $f : X \rightarrow Y$ to the roof $X \xleftarrow{\text{id}} X \xrightarrow{f} Y$. Next, one shows that if the cone of f is in $\mathcal{C}$, then this roof is inverse in $\mathcal{C}/\mathcal{D}$ to the roof $Y \xleftarrow{f} X \xrightarrow{\text{id}} X$. In particular, π inverts all morphisms whose cone is in $\mathcal{C}$, and hence kills $\mathcal{C}$. One then shows that $\mathcal{D}/\mathcal{C}$ inherits a triangulated structure from $\mathcal{D}$ making π into a triangle functor. The universal property follows from the construction. To identify the kernel, one direction is clear since kernels are always thick. For the other direction, one shows that $X \rightarrow 0$ becomes an isomorphism in $\mathcal{D}/\mathcal{C}$ if and only if X is a summand of an object of $\mathcal{C}$. $\square$

Example 3.22. Let $K(A)$ denote the chain homotopy category, where the morphisms are chain maps up to chain homotopy equivalence. This is a triangulated category in the usual way. Let $K_{\text{ac}}(A)$ denote the subcategory of acyclic complexes; this is a thick subcategory. The Verdier quotient $K(A)/K_{\text{ac}}(A)$ is precisely the derived category $D(A)$.

We can finally define the singularity category.

Definition 3.23. Let A be a noetherian ring. Then the singularity category is the Verdier quotient

$$D_{\text{sg}}(A) := \frac{D^b(\mathbf{mod}\text{-}A)}{\mathbf{per}(A)}$$

The singularity category comes equipped with a natural projection map $D^b(\mathbf{mod}\text{-}A) \rightarrow D_{\text{sg}}(A)$ whose kernel is precisely $\mathbf{per}(A)$.

Chapter 4

Notions of regularity

Good references for this part are [Lam99, Eis95, Mat86]. We will mainly restrict ourselves here to two-sided noetherian rings, since their dimension theory avoids numerous pathologies present in the general case.

4.1 Global dimension

Let A be a ring. If M is an A -module, the **projective dimension** of M is the minimal length of a projective resolution of M , where the length means the number of nonzero modules. We denote it by $\mathbf{pd}_A(M)$ or just $\mathbf{pd}(M)$. Evidently, the modules of projective dimension 0 are precisely the projective modules.

Exercise 4.1. Let A be noetherian. Show that a finitely generated module M has finite projective dimension if and only if $M \in D(A)$ is a perfect complex.

Exercise 4.2. If $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ is a short exact sequence, show that $\mathbf{pd}(Y) \leq \max(\mathbf{pd}(X), \mathbf{pd}(Z))$.

If A is noetherian, the **global dimension** of A is defined to be the supremum of the projective dimensions of all finitely generated A -modules. We denote the global dimension by $\mathbf{gldim}(A)$.

Exercise 4.3. If A is a PID, show that $\mathbf{gldim}(A) \leq 1$.

Exercise 4.4. *Show that a noetherian ring has infinite global dimension if and only if it has a module of infinite projective dimension.

Remark 4.5. A priori, there is a left and a right notion of global dimension. However, for two-sided noetherian rings the two concepts agree, and we will use the two notions interchangeably.

Example 4.6. A ring A has global dimension zero if and only if every module is projective. These are precisely the semisimple rings. A commutative semisimple ring is a finite direct product of fields.

Example 4.7. If A has global dimension n , then if M, N are two finitely generated A -modules, we must have $\text{Ext}^i(M, N) \cong 0$ for $i > n$. In particular, when k is a field the ring $k[x]/x^2$ must have infinite global dimension.

Lemma 4.8. *If A has finite global dimension then $D_{\text{sg}}(A)$ vanishes.*

Proof. Take a bounded complex $M = M_p \rightarrow \cdots \rightarrow M_q$ of finitely generated modules. By hypothesis, each M_i has a bounded resolution P_i by finitely generated projectives, and moreover each differential $M_i \rightarrow M_{i+1}$ lifts to a morphism $P_i \rightarrow P_{i+1}$ of complexes. It follows that M is quasi-isomorphic to the totalisation of the double complex $P_p \rightarrow \cdots \rightarrow P_q$, which is clearly perfect. Hence M is quasi-isomorphic to a perfect complex. So $\text{per}(A) = D^b(A)$ and hence $D_{\text{sg}}(A)$ vanishes. $\square$

In the rest of this part we will look for a converse to the above lemma in the setting of commutative rings. To begin with we will restrict ourselves to local rings.

4.2 Commutative rings I: the local setting

If R is a commutative ring, recall that the **Krull dimension** of R is the supremum of the lengths of all chains of prime ideals in R . If M is an R -module, we put $\dim(M) := \dim(R/\text{ann}(M))$. In particular, since taking quotients cannot increase the Krull dimension, we have $\dim(M) \leq \dim(R)$.

If $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring with residue field k , then there is an inequality $\dim(R) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, the dimension of the cotangent space of R . This is the same as the minimal number of generators of the ideal $\mathfrak{m}$, by Nakayama's Lemma.

Say that $(R, \mathfrak{m}, k)$ is **regular** if $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$. In other words, this means that the (co)tangent space of R has the correct expected dimension.

Theorem 4.9 (“Auslander–Buchsbaum–Serre”). *Let R be a commutative local noetherian ring. The following are equivalent:*

1. R is regular.
2. $\text{gldim}(R)$ is finite.
3. $D_{\text{sg}}(R)$ vanishes.

Moreover, if any of the above hold, the global dimension of R is equal to its Krull dimension.

Before we begin the proof, we make a historical remark. The ‘original’ ABS theorem is (1) $\iff$ (2), which long predates the invention of singularity categories. The implication (1) $\implies$ (2) was first noticed by Buchsbaum, and the implication (2) $\implies$ (1) was independently proved by both Serre and Auslander–Buchsbaum. The equivalence of both statements with (3) and the statement about the Krull dimension are in fact easy corollaries of the proof.

Lemma 4.10. *Let R be a commutative noetherian ring and $x_1, \dots, x_n$ a regular sequence in R . Then $\mathbf{pd}_R(R/(x_1, \dots, x_n)) = n$.*

Proof. The proof is an induction on n . At the induction step one uses the Koszul resolution for $R/(x_1, \dots, x_n)$ defined by taking exterior powers of R^n . $\square$

Sketch proof of Auslander–Buchsbaum–Serre. Let $\mathfrak{m}$ be the maximal ideal of R and $k = R/\mathfrak{m}$ the residue field. The proof relies on the key equality

$$\mathbf{gldim}(R) = \mathbf{pd}_R(k)$$

which can be proved via an argument with Tor-dimension. We deduce that (2) and (3) are equivalent: we have already observed one direction of the proof, and the other follows since if $D_{\text{sg}}(R)$ vanishes then certainly k has finite projective dimension.

Assume now that (1) holds, i.e. that R is a regular local ring. Take a minimal set of generators $x_1, \dots, x_d$ for the maximal ideal $\mathfrak{m}$. Because R is regular, we have $d = \dim(R)$ by hypothesis. The x_i in fact form a regular sequence on R , so by the lemma on regular sequences we have $d = \mathbf{pd}_R(k)$. By the key equality, we see that both (2) and the statement about Krull dimension hold.

We are left to show that (2) implies (1). This is the hard part of the proof; we omit the argument which, roughly, is an induction on $\mathbf{gldim}(R)$. $\square$

Corollary 4.11. *If R is a commutative noetherian regular local ring and $\mathfrak{p}$ is a prime ideal of R , then the localisation $R_{\mathfrak{p}}$ is also regular local.*

Proof. Resolutions localise so we have $\mathbf{gldim}(R_{\mathfrak{p}}) \leq \mathbf{gldim}(R) < \infty$. $\square$

4.3 Commutative rings II

Now we move to the global setting.

Theorem 4.12 (“Global Auslander–Buchsbaum–Serre”). *Let R be a commutative noetherian ring. The following are equivalent:*

1. *The localisation $R_{\mathfrak{m}}$ is regular for every $\mathfrak{m} \in \text{MaxSpec}(R)$.*
2. *The localisation $R_{\mathfrak{p}}$ is regular for every $\mathfrak{p} \in \text{Spec}(R)$.*
3. *Every finitely generated R -module has finite projective dimension.*
4. *$D_{\text{sg}}(R)$ vanishes.*

Moreover, if any of the above hold, the global dimension of R is equal to its Krull dimension.

If R satisfies any of the above equivalent conditions, we call R **regular**.

Sketch proof of global ABS. The equivalence of (3) and (4) is clear. Since resolutions localise, if (4) holds then every $D_{\text{sg}}(R_{\mathfrak{p}})$ also vanishes, and hence (2) holds by ABS. Clearly (2) implies (1) so we only need to show that (1) implies (3). One first proves, by a compactness argument, that for every finitely generated R -module M there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathbf{pd}_R(M) = \mathbf{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$. Hence if each $R_{\mathfrak{m}}$ is regular, then (3) holds by ABS again. The statement about Krull dimension follows from the equalities

$$\mathbf{gldim}(R) = \sup_{\mathfrak{m}} \mathbf{gldim}(R_{\mathfrak{m}}) = \sup_{\mathfrak{m}} \dim(R_{\mathfrak{m}}) = \dim(R)$$

where in the second equality we are using ABS for one final time. $\square$

Note that we have not proved that a commutative noetherian regular ring must have finite global dimension. In fact this is false! Nagata gave an example of a commutative noetherian regular ring R with infinite Krull dimension (and hence, by global ABS, global dimension). Each localisation of R must have finite - but arbitrarily large - global dimension. Although $D_{\text{sg}}(R)$ vanishes, it does not vanish in a ‘uniform’ way, in the sense that one cannot uniformly bound the projective dimension of all finitely generated modules.

Example 4.13 (Nagata [Nag62]). Let $I_n \subseteq \mathbb{N}$ denote the interval $[2^{n-1}, 2^n - 1]$, which has length 2^{n-1} . Let $A = \mathbb{C}[x_1, x_2, \dots]$ be the infinite-dimensional polynomial ring and for each $n \in \mathbb{N}$ let $\mathfrak{p}_n$ denote the ideal generated by $\{x_i : i \in I_n\}$. Put $S := A/\cup_n \mathfrak{p}_n$ and put $R := A_S$ the localisation. Since $\mathfrak{p}_n$ has height 2^{n-1} in A it follows that R has infinite Krull dimension. To prove that it is regular, first use the Prime Avoidance Lemma to show that every maximal ideal of R is of the form $\mathfrak{p}_n R$, so that we need to check that each $A_{\mathfrak{p}_n}$ is regular; this holds since it is a localisation of a regular ring. To prove that it is noetherian boils down to checking that each $A_{\mathfrak{p}_n}$ is noetherian; again this holds since it can be written as a localisation of a noetherian ring.

Remark 4.14. Suppose that k is a field and R is a (locally of) finite type k -algebra. If k is perfect (for example, characteristic zero, algebraically closed, or finite) then R is regular if and only if R is smooth over k . Without the perfectness assumption this is no longer true; for example, the variety $x^2 - y^2 = t$ defined over the field $\mathbb{Z}/2(t)$ is regular but not smooth.

4.4 Depth

Whilst we are doing some commutative algebra, we collect some facts about depth which will be useful later. Throughout this section, let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. The **depth** of an R -module M is the smallest number i for which $\text{Ext}^i(k, M)$ is nonzero. Note that $\text{depth}(M)$ may be infinite - clearly the zero module has $\text{depth}(0) = \infty$.

Example 4.15. A module M is depth zero precisely when there exists a nonzero map $k \rightarrow M$. This is equivalent to the existence of $x \in M$ with $x\mathfrak{m} = 0$.

Theorem 4.16 (Rees). *If M is finitely generated, then the depth of M is the length of a maximal M -regular sequence $x_1, \dots, x_n$ with all $x_i \in \mathfrak{m}$.*

Proof. This is by induction on the depth of M - one shows that if $x \in \mathfrak{m}$ is a non-zero-divisor then $\text{depth}(M/x) = \text{depth}(M) - 1$. $\square$

Corollary 4.17. *If $M \neq 0$ is finitely generated, there is an inequality*

$$\text{depth}(M) \leq \dim(M).$$

Proof. If x is not a zero-divisor on M , we have $\dim(M/x) = \dim(M) - 1$. An induction now shows that $\dim(M/(x_1, \dots, x_n)) = \dim(M) - n$ for any M -regular sequence. The left hand side is at least zero, so we have $n \leq \dim(M)$. $\square$

Say that a module M is **Cohen–Macaulay** (or just **CM**) if it satisfies $\text{depth}(M) = \dim(M)$. Say that M is **maximal Cohen–Macaulay** (or **MCM**) if $\text{depth}(M) = \dim(R)$. By convention, we also say that 0 is an MCM module. Say that R is **Cohen–Macaulay** if the R -module R is CM (in which case it is necessarily MCM).

Remark 4.18. One can extend the above definitions to non-local rings by saying that M is (M)CM whenever all of its localisations at primes are so.

Example 4.19. A regular local ring is CM, since in this case $\mathfrak{m}$ is generated by a regular sequence of length $\dim(R)$, which implies that $\text{depth}(R) \geq \dim(R)$.

Example 4.20. A commutative Artinian ring is CM, since it has Krull dimension zero. Every module is MCM.

Theorem 4.21 (Auslander–Buchsbaum formula). *If M is a nonzero module of finite projective dimension, then there is an equality*

$$\text{depth}(M) + \text{pd}_R(M) = \text{depth}(R).$$

Proof. The idea is to induct on the projective dimension of M . At the induction step one uses a characterisation of depth in terms of the Koszul complex for $\mathfrak{m}$. $\square$

Corollary 4.22. *Let R be a Cohen–Macaulay ring and M an MCM R -module. Then M is either projective or has infinite projective dimension.*

Example 4.23. Let R be a commutative Artinian ring. Then every R -module is either projective or has infinite projective dimension. For example, one can take $R = kG$ for G a finite group and k a field; this fact is well known when $\text{char}(k)$ does not divide the order of G , in which case kG is semisimple and hence every module is projective.

Chapter 5

Buchweitz's stable category

Recap: give a better proof of the original ABS theorem.

The original (and very good) reference here is [Buc86]; see also the new and updated version [Buc21]. If you are a representation theorist then see also [Bel00] for some significant generalisations.

5.1 Gorenstein rings

Unfortunately we must finally come to injective modules¹.

Definition 5.1. A module I over a ring A is **injective** if, whenever $I \hookrightarrow M$ is an injection, there exists a module $N \hookrightarrow M$ with $I \oplus N \cong M$.

Exercise 5.2. Show that a module is injective if and only if it has the right lifting property with respect to injections: if $N \hookrightarrow M$ is an injection and $N \rightarrow I$ is any map then there exists an extension $M \rightarrow I$.

Exercise 5.3. *Show that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$ -modules.

Exercise 5.4. If A is a semisimple ring, show that every A -module is injective.

Just like every module has a projective resolution, every module M has an **injective resolution**: a complex $I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \cdots$ which resolves M . Maps of modules lift to maps of injective resolutions, uniquely up to chain homotopy. One can construct Chevalley–Eilenberg type injective resolutions (of bounded below complexes). All of our theorems about projective resolutions dualise to analogous theorems about injectives, and in fact one can prove the following theorem:

Theorem 5.5. *Let A be a ring. There is a triangle equivalence*

$$K^+(\mathrm{Inj}(A)) \simeq D^+(A)$$

which is the identity on objects.

¹Note to writer: in final version, move this discussion upwards. Perhaps move the discussion on depth downwards.

Dually to projective dimension, one can define the **injective dimension** of a module.

Remark 5.6. It is a nontrivial fact that $\mathbf{gldim} A = \sup_M \mathbf{id}(M)$, where the supremum is taken over **all** A -modules M .

Definition 5.7. A two-sided noetherian ring A is **Gorenstein** or **Iwanaga–Gorenstein** if the A -module A has finite injective dimension over A , as both a left and a right module.

Remark 5.8. If A is Gorenstein, then a theorem of Zaks states that the right injective dimension of A must agree with the left injective dimension of A . In general, the injective dimension may be infinite on one side and finite on the other. See [Zak69] for further discussion.

Theorem 5.9. *Let R be a commutative noetherian local ring with residue field k , of Krull dimension n . The following are equivalent:*

1. R is Gorenstein.
2. R has injective dimension n .
3. R is CM and $\mathrm{Ext}_R^n(k, R) \simeq k$.

We omit the proof, which uses an induction on the dimension and some facts about dualising modules. As an immediate corollary, we see that a commutative Gorenstein ring is CM.

Example 5.10. A commutative noetherian ring R is **lci** (locally complete intersection) if for every prime p , the completion $\hat{R}_p$ is of the form $A/(a_1, \dots, a_r)$ where A is regular complete local and the a_i are a regular sequence. The obvious class of examples of lci rings is given by the *global* complete intersections, i.e. the rings of the form $R = k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ where the f_i form a regular sequence (which in this setting is equivalent to $\dim R = n - r$). In particular, hypersurfaces (the case $r = 1$) are lci. All lci rings are Gorenstein: to see this, first note that being Gorenstein is a complete local property, so it suffices to check that a complete intersection is Gorenstein. This is a computation with the Koszul complex.

Remark 5.11. In general there exist CM rings which are not Gorenstein, and Gorenstein rings which are not lci; examples can be found even in Artinian rings.

Definition 5.12. A module M over a Gorenstein ring A is **maximal Cohen–Macaulay** (or just MCM for short) if there is a natural quasi-isomorphism $\mathbb{R}\mathrm{Hom}_A(M, A) \simeq \mathrm{Hom}_A(M, A)$. This is equivalent to the condition that $\mathrm{Ext}_A^i(M, A) \cong 0$ for $i > 0$.

Theorem 5.13. *Let R be a commutative Gorenstein local ring. Then the two definitions of MCM agree for finitely generated R -modules.*

Proof idea. For a local CM ring R with finitely generated module M , local duality tells us that M has maximal depth if and only if $\text{Ext}^i(M, \omega)$ vanishes for all $i > 0$. Here ω denotes a dualising module for R , which exists since R is CM. (The **dualising complex** is then $\omega[\dim R]$). A Gorenstein ring is precisely a ring where R is a dualising module for R . $\square$

5.2 The stable category

Suppose from now on that A is a Gorenstein ring. Let M be a finitely generated module. A **syzygy** of M is a module ΩM which fits into a short exact sequence of the form

$$0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$$

where P is a finitely generated projective. Observe that one can stitch together the syzygy exact sequences for all $\Omega^i M$ into a projective resolution of M .

Exercise 5.14. Show that if ΩM is a syzygy of M then so is $Q \oplus \Omega M$ for any finitely generated projective module Q .

Lemma 5.15. *If M is MCM then so is ΩM .*

Proof. The defining short exact sequence yields an exact triangle

$$\mathbb{R}\text{Hom}_R(M, R) \rightarrow \mathbb{R}\text{Hom}_R(P, R) \rightarrow \mathbb{R}\text{Hom}_R(\Omega M, R) \rightarrow$$

which shows that ΩM must be MCM. $\square$

One can define the **stable category of MCM modules** $\underline{\text{MCM}}(A)$ to have objects the MCM A -modules, and morphisms given by

$$\underline{\text{Hom}}(M, N) := \frac{\text{Hom}(M, N)}{\text{morphisms which factor through a projective module}}.$$

In particular, projective modules go to zero in $\underline{\text{MCM}}(A)$. Observe that $\underline{\text{MCM}}(A)$ is a subcategory of the larger stable category $\underline{\text{mod}}(A)$, which is defined analogously.

Lemma 5.16. *There is a well-defined functor Ω on $\underline{\text{MCM}}(A)$ which sends an object to its syzygy.*

Proof. Any two syzygies X, Y of a module M are **stably equivalent** in that $X \oplus Q \cong Y \oplus Q'$ for some projective (or free) modules Q, Q' . Hence they define the same object in the stable category. Functoriality is easy to see. $\square$

Lemma 5.17. *There is a functor $\iota : \underline{\text{MCM}}(A) \rightarrow D_{\text{sg}}(A)$ which is the identity on objects. It sends the Ω functor to the inverse shift functor $[-1]$.*

Proof. The projection $\mathbf{MCM}(A) \rightarrow D_{\text{sg}}(A)$ clearly kills projective modules so factors uniquely through the stable category; ι is the factoring map. By definition of Ω there is an exact triangle

$$\Omega M \rightarrow P \rightarrow M \rightarrow$$

in $D^b(A)$ which induces an exact triangle

$$\Omega M \rightarrow 0 \rightarrow M \rightarrow$$

in $D_{\text{sg}}(A)$. Hence the natural connecting map $M \rightarrow (\Omega M)[1]$ is an isomorphism. $\square$

{buchthm}

Theorem 5.18. *ι is an equivalence. Hence $\mathbf{MCM}(A)$ is a triangulated category, with shift functor the ‘inverse syzygy’ Ω^{-1} .*

The proof passes through an intermediate category $K_{\text{ac}}(\text{proj} A)$, the homotopy category of acyclic complexes of finitely generated projective modules.

{buchprop}

Proposition 5.19.

1. *There are functors $\Omega_i : K_{\text{ac}}(\text{proj} A) \rightarrow \mathbf{MCM}(A)$ which send a complex X to the cokernel of $d^i : X^{-i-1} \rightarrow X^{-i}$. We have $\Omega_i(X[j]) \simeq \Omega_{i-j}X$.*
2. *There are triangle functors $\sigma_i : K_{\text{ac}}(\text{proj} A) \rightarrow D_{\text{sg}}(A)$ which send a complex to its brutal truncation at i . In other words, $(\sigma_i X)^j$ is X^j for $j \leq i$ and 0 otherwise.*
3. *We have $\iota \Omega_i(X) \cong \sigma_{-i}(X)[-i]$.*

Proof. First observe that we obviously have functors $\Omega_i : \text{Ch}_{\text{ac}}(\text{proj} A) \rightarrow \mathbf{mod}(A)$. If $f : P \rightarrow Q$ is nullhomotopic then the map $\Omega_i f$ factors through a projective module, so that they descend to functors $\Omega_i : K_{\text{ac}}(\text{proj} A) \rightarrow \mathbf{mod}(A)$. The compatibility with shifts is clear. To see that the images are MCM, observe that we have isomorphisms

$$\text{Ext}_A^j(\Omega_i X, A) \cong \text{Ext}_A^{i+k}(\Omega_{j-k} X, A)$$

for all $j > 0$ and $k \geq 0$. In particular we see by taking $k \gg 0$ that these Ext groups vanish and hence $\Omega_i X$ is MCM. Claim (2) is easy to see, and claim (3) follows from the fact that for $X \in K_{\text{ac}}(\text{proj} A)$, the complex $\sigma_i(X)[i]$ is a projective resolution of $\Omega_{-i}X$. $\square$

We remark that part (1) of the preceding proposition tells us that ‘high enough syzygies of an arbitrary module are MCM’.

{sigma0}

Proposition 5.20. *The functor σ_0 is a triangle equivalence.*

Before we prove this, let us show how this yields a proof of 5.18.

Proof of 5.18. By 5.20 and 5.19 it suffices to prove that Ω_0 is an equivalence. It is faithful by 5.20 so it remains to check that Ω_0 is full and essentially surjective. To do this we introduce the notion of the **complete resolution** of an MCM module M . This is glued together out of the data of

1. A projective resolution $P \rightarrow M$
2. A projective resolution $Q \rightarrow M^\vee$

where we denote $M^\vee := \text{Hom}_A(M, A)$. Since M is MCM, its dual $M^\vee$ is MCM, and moreover M is reflexive: we have a natural isomorphism $M \rightarrow M^{\vee\vee}$. Dualising the projective resolution Q hence yields a quasi-isomorphism $M \rightarrow Q^\vee$. Composing $P \rightarrow M \rightarrow Q^\vee$ hence yields a quasi-isomorphism. A **complete resolution** for M is the corresponding complex

$$\mathbf{CR}(M) := \text{cocone}(P \rightarrow Q^\vee) \in K_{\text{ac}}(\text{proj} A)$$

We caution that this is merely notation, although by choosing functorial projective resolutions $\mathbf{CR}$ can be upgraded to a functor. Clearly we have an isomorphism $\Omega_0 \mathbf{CR}(M) \simeq M$, and hence Ω_0 is essentially surjective. To show fullness, let $f : M \rightarrow M'$ be a map between two MCM modules. By lifting both f and $f^\vee$ to maps of projective resolutions, we can lift f to a map of complete resolutions. $\square$

Proof of 5.20. We need to show that σ_0 is fully faithful and essentially surjective. For essential surjectivity, take an object X of the singularity category, which we may assume is a bounded and strictly right bounded complex of finitely generated projectives. Observe that, for every k , the complexes X and $\sigma_k X$ differ by a perfect complex and hence are isomorphic in the singularity category. For $k \ll 0$, we see that $M := H^k(\sigma_k X) \simeq \sigma_k X$ is MCM since high enough syzygies are MCM as in 5.19(1). Completing $\sigma_k X$ to a complete resolution of M and then shifting yields an acyclic complex Y with $\sigma_0 Y \simeq \sigma_k Y \simeq \sigma_k X \simeq X$, as desired.

For fully faithfulness, we apply a theorem of Verdier stating that it is enough to show that if P is a perfect complex and $X \in K_{\text{ac}}(\text{proj} A)$, then there exists a k such that $\text{Hom}_{D^b(A)}(\sigma_k X, P) \simeq 0$. By taking shifts and cones we can reduce to the case that P is a projective module concentrated in degree $-i$. Taking $k > i$, in this case, the claim amounts to checking that the Ext group $\text{Ext}_A^1(\Omega_{-i-1} X, P)$ vanishes. But this is the case since P is finitely generated projective and the module $\Omega_{-i-1} X$ is MCM. $\square$

5.3 Stable Ext

Since it is a triangulated category, the stable category $\underline{\mathbf{MCM}}(A)$ admits a notion of Ext groups, which we denote by $\underline{\text{Ext}}$ and refer to as the **stable Ext groups**. One pleasing fact is the following:

Proposition 5.21. *Let A be a Gorenstein ring and M, N two MCM R -modules.*

1. For $j > 0$, there are natural isomorphisms $\underline{\text{Ext}}_A^j(M, N) \cong \text{Ext}_A^j(M, N)$.
2. For $j < -1$ there are natural isomorphisms $\underline{\text{Ext}}_A^j(M, N) \cong \text{Tor}_{-j-1}^A(N, M^\vee)$.
3. There is a four-term exact sequence

$$0 \rightarrow \underline{\text{Ext}}_A^{-1}(M, N) \rightarrow N \otimes_A M^\vee \rightarrow \text{Hom}_A(M, N) \rightarrow \underline{\text{Ext}}_A^0(M, N) \rightarrow 0.$$

Proof. In what follows we let $\mathbf{CR}(X)$ denote a complete resolution for X . Since $\mathbf{MCM}(A)$ is equivalent to $K_{\text{ac}}(\text{proj}(A))$, we have isomorphisms

$$\underline{\text{Ext}}_A^i(M, N) \cong H^i \text{Hom}_A(\mathbf{CR}(M), \mathbf{CR}(N))$$

Let P be a projective resolution for N and Q a projective resolution of $N^\vee$. The description of $\mathbf{CR}(N)$ as a mapping cocone of the natural morphism $P \rightarrow Q^\vee$ gives us an exact triangle

$$\text{Hom}_A(\mathbf{CR}(M), \mathbf{CR}(N)) \rightarrow \text{Hom}_A(\mathbf{CR}(M), P) \rightarrow \text{Hom}_A(\mathbf{CR}(M), Q^\vee) \rightarrow$$

in $D(\mathbb{Z})$. Since A is Gorenstein, the complex $\text{Hom}_A(\mathbf{CR}(M), Q^\vee)$ is actually acyclic - morally this is since $Q^\vee$ is close to being an injective resolution of N . Hence we obtain a quasi-isomorphism

$$\text{Hom}_A(\mathbf{CR}(M), \mathbf{CR}(N)) \simeq \text{Hom}_A(\mathbf{CR}(M), P)$$

Now let P' be a projective resolution of M and Q' a projective resolution of $M^\vee$. We obtain an exact triangle

$$\text{Hom}_A(Q'^\vee, P) \rightarrow \text{Hom}_A(P', P) \rightarrow \text{Hom}_A(\mathbf{CR}(M), P) \rightarrow$$

There is a natural map

$$P \otimes_A Q' \cong P \otimes_A Q'^{\vee\vee} \rightarrow \text{Hom}_A(Q'^\vee, P)$$

which is a quasi-isomorphism since P and Q' are bounded above complexes of finitely generated projectives. The exact triangle above becomes an exact triangle

$$N \otimes_A^{\mathbb{L}} M^\vee \rightarrow \mathbb{R}\text{Hom}_A(M, N) \rightarrow \text{Hom}_A(\mathbf{CR}(M), N) \rightarrow$$

which finally yields a long exact sequence

$$\cdots \rightarrow \text{Tor}_{-i}^A(N, M^\vee) \rightarrow \text{Ext}_A^i(M, N) \rightarrow \underline{\text{Ext}}_A^i(M, N) \rightarrow \cdots$$

which gives the desired statements. $\square$

Bibliography

- [Bel00] Apostolos Beligiannis. The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization. *Comm. Algebra*, 28(10):4547–4596, 2000.
- [Boo19] Matt Booth. *The derived contraction algebra*. PhD thesis, The University of Edinburgh, 2019. ArXiv:1911.09626.
- [Boo21] Matt Booth. Singularity categories via the derived quotient. *Adv. Math.*, 381:Paper No. 107631, 56, 2021.
- [Buc86] Ragnar-Olaf Buchweitz. Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings, 1986. Available at <https://tspace.library.utoronto.ca/handle/1807/16682>.
- [Buc21] Ragnar-Olaf Buchweitz. *Maximal Cohen-Macaulay modules and Tate cohomology*, volume 262 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2021] ©2021. With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz.
- [Căł05] Andrei Căldăraru. Derived categories of sheaves: a skimming. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 43–75. Amer. Math. Soc., Providence, RI, 2005.
- [Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [Lam99] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.
- [Lur] Jacob Lurie. Functorial kernel in derived category. MathOverflow. URL:<https://mathoverflow.net/q/386369> (version: 2021-03-13).
- [Mat86] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.
- [Nag62] Masayoshi Nagata. *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.
- [Nee01] Amnon Neeman. *Triangulated Categories*. Princeton University Press, 2001.
- [Ste] Greg Stevenson. On the failure of functorial cones in triangulated categories.
- [Sym22] Julie Symons. On singularity categories, 2022.

- [Tho01] R. P. Thomas. Derived categories for the working mathematician. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 349–361. Amer. Math. Soc., Providence, RI, 2001.
- [Toë11] Bertrand Toën. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011.
- [Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
- [Yek20] Amnon Yekutieli. *Derived categories*, volume 183 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020.
- [Zak69] Abraham Zaks. Injective dimension of semi-primary rings. *J. Algebra*, 13:73–86, 1969.