An introduction to spectra

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Why study spectra?

- Spectra are like topological spaces but simpler and more algebraic

- Spectra control generalised cohomology theories

- Spectra are interesting objects in their own right

- Spectra are the base objects for spectral algebra
  $\rightsquigarrow$ spectral geometry

Homotopy vs. homology

- If X is a topological space, have its suspension

$$\Sigma X$$

All spaces are pointed!

Example

$$\Sigma S^n \simeq S^{n+1}$$

- On cohomology, have suspension isomorphisms

$$H^n X \cong H^{n+1} \Sigma X$$

- Not the case for homotopy!
- However, $S^n \to X$ suspends to $S^{n+1} \longrightarrow \Sigma X$ & we get maps

$$\pi_n X \longrightarrow \pi_{n+1} \Sigma X$$

• Freudenthal suspension theorem:
if X is a finite CW complex then

$$\pi_n X \to \pi_{n+1} \Sigma X \to \pi_{n+2} \Sigma^2 X \to \cdots$$

eventually stabilises

Hope These stable homotopy groups are less complicated than the unstable ones.

- More generally, if X is a space put

$$\pi_n^S X := \varinjlim_m \pi_{n+m} \Sigma^m X$$

the n^{th} stable homotopy group of X

- Note that it's an abelian group.

- If $\Sigma^r X \simeq \Sigma^r Y$ then $\pi_*^s X \simeq \pi_*^s Y$
- Idea: there should be some category of 'space-like objects' where one can invert Σ, & some notion of 'weak equivalences' which are detected by stable homotopy groups.

Constructing spectra

- A **spectrum** is an $\mathbb{N}$-graded sequence of spaces X_i together with structure maps

$$\Sigma X_i \to X_{i+1}$$

- A **morphism** of spectra is a sequence of maps $X_i \to Y_i$ making the obvious diagrams commute.

- Spectra have homotopy groups:

$$\pi_n X_i \longrightarrow \pi_{n+1} \Sigma X_i \longrightarrow \pi_{n+1} X_{i+1}$$

define

$$\pi_n X := \varinjlim_{m} \pi_{n+m} X_m$$

Example X a space.

Have a suspension spectrum $\Sigma^\infty X$ with

$$(\Sigma^\infty X)_i = \Sigma^i X$$

Then $\pi_n \Sigma^\infty X \simeq \pi_n^S X$.

Sub-example

$\mathbb{S} = \Sigma^\infty S^0$ sphere spectrum

$\Sigma^i S^0 \simeq S^i$ so $\mathbb{S}_i \simeq S^i$

Example A an abelian group. Have an Eilenberg-Mac Lane spectrum HA with $(HA)_i \simeq K(A,i)$ & $\pi_* A \simeq \begin{cases} A & * = 0 \\ 0 & \text{else} \end{cases}$

More generally if A is a chain cplx of abelian groups have a spectrum HA with

$$\pi_* HA \simeq H_* A$$

The stable homotopy category

- Say $X \to Y$ is a **stable equivalence** if $\pi_* X \to \pi_* Y$ is an isomorphism.
- The **stable homotopy category** SHC is the localisation

$$\text{Spectra}[\text{stable equivs.}^{-1}]$$

- The functor $A \longrightarrow HA$ gives an embedding

$$D(\mathbb{Z}) \hookrightarrow SHC$$

- SHC is a triangulated category & the above embedding is a triangle functor.

- On SHC, Σ is invertible

- In particular in SHC we have 'spheres'

$\Sigma^i \mathbb{S}$ for all $i \in \mathbb{Z}$

- SHC is enriched in abelian groups & one has

$[\Sigma^i \mathbb{S}, X] \simeq \pi_i X$

Remark Stable equivalences are the weak equivs. of a model structure on Spectra

Fibrant objects are the Ω-spectra

CW spectra are cofibrant.

Cohomology theories

- A cohomology theory is a functor

$$F : \left\{\begin{array}{c}\text{connected} \\ \text{CW} \\ \text{complexes}\end{array}\right\} \longrightarrow \text{gr. Ab}$$

satisfying:

- homotopy invariance
- F sends wedges to products
- LESs for CW pairs

Examples

- Singular cohomology $H^*(X, R)$
- K-theory (complex & real)
- Cobordism

Brown Representability Th^m

All cohomology theories are representable by spectra:

$$F^*(X) \simeq [\Sigma^\infty X, E]_*$$

$$= [\Sigma^\infty X, \Sigma^{-*} E]$$

- Moreover maps between cohomology theories lift to maps between spectra

Examples

$H^*(-,R) \rightsquigarrow HR$

K-theory $\rightsquigarrow KU, KO$

Cobordism $\rightsquigarrow MO$

In particular

$$\pi_n MO \cong \left\{ \begin{array}{l} \text{cobordism classes} \\ \text{of compact smooth} \\ n\text{-manifolds} \end{array} \right\}$$

Structured spectra

- Spaces have a smash product $X \wedge Y = \dfrac{X \times Y}{X \vee Y}$
- $S^m \wedge S^n \simeq S^{m+n}$

More generally, $S^1 \wedge \simeq \Sigma$

- Unfortunately, $\wedge$ does not lift to spectra in a homotopically sensible way (Lewis)

- The problem: there are nontrivial braiding isomorphisms

$$S^1 \wedge S^1 \xrightarrow{\sim} S^1 \wedge S^1$$

that are nontrivial on homotopy (the above is $-1 \in \pi_2 S^2$)

- $S^n \simeq S^1 \wedge \cdots \wedge S^1$ gets a Σ_n-action via permuting the factors.

- One solution: carry these actions around as part of the data

<u>Hovey–Shipley–Smith:</u>

A **symmetric spectrum** is a spectrum $\{X_i\}$ with actions $\Sigma_i \curvearrowright X_i$ s.t. the composite maps

$$S^k \wedge X_i \longrightarrow X_{i+k}$$

are

$\Sigma_k \times \Sigma_i$-equivariant

- Then the category of symmetric spectra admits a symmetric monoidal smash product whose unit is $\mathbb{S}$

- a ring spectrum is a monoid for $\wedge$

ie. a spectrum A with maps

$$A \wedge A \longrightarrow A$$
$$\mathbb{S} \longrightarrow A$$

satisfying associativity & unit axioms

Examples

- $\mathbb{S}$ is the initial ring spectrum (cf. $\mathbb{Z}$)
- If A is a ring then HA is a ring spectrum
- $H^* X \cong [\Sigma^\infty X, HA]_*$

cup products $\cong$ multiplication induced from HA

- A **module** over a ring spectrum R is a spectrum M with action map

$$R \wedge M \longrightarrow M$$

satisfying some identities

Examples

- Every spectrum X is an $\mathbb{S}$-module in a unique way.

- Up to homotopy, HR-modules are the same thing as dg R-modules:

$$D(R) \simeq Ho(HR\text{-mod}) \hookrightarrow Ho(\mathbb{S}\text{-modules}) \simeq SHC$$

- Technical consideration:

Symmetric spectra are the simplest model of highly structured spectra, but their homotopy theory is more complicated:

correct notion of weak equivalence is not detected on underlying spectra.

Other models:

- Orthogonal spectra
- S-modules
- coordinate-free spectra (indexed by f.d. real inner product spaces)
- Excisive functors

Applications of ring spectra

Applications of rings:

1) Algebraic geometry

2) Homological algebra

1) Spectral geometry

commutative rings	$\rightsquigarrow$	usual AG
commutative dgas or simplicial c. rings	$\rightsquigarrow$	derived AG
commutative ring spectra	$\rightsquigarrow$	spectral AG

Can develop many of the concepts of classical AG in the spectral world.

Applications so far are more topological:

e.g. tmf

2) Spectral algebra

Fix a ring spectrum A

If N, M are A-modules, have $\mathrm{Ext}^*_A(N, M)$

& $\mathrm{Tor}^A_*(N, M)$

via the usual sort of construction.

In particular can do Hochschild theory

If R is a commutative ring spectrum, A an R-module, put

$$A^e_R := A \wedge_R A^{op}$$

derived enveloping algebra

Examples

- If R, A are discrete rings then $A^e_R \simeq A \otimes^{\mathbb{L}}_R A^{op}$
- $R^e_R \simeq R$
- $\mathbb{Q}^e_{\mathbb{S}} \simeq \mathbb{Q}$
- $(\mathbb{Z}/p)^e_{\mathbb{S}}$ has homotopy the dual Steenrod algebra

The topological Hochschild homology of A relative to R is

$$THH^R_*(A) := \mathrm{Tor}_*^{A_R^e}(A, A)$$

Similarly, have topological Hochschild cohomology

$$THH_R^*(A) := \mathrm{Ext}^*_{A_R^e}(A, A)$$

- $THH^{\mathbb{S}}(A)$:

K-theory $K(A)$

- $THH_{\mathbb{S}}(A)$:

'non-additive' deformation theory of A

Thanks

for

listening!