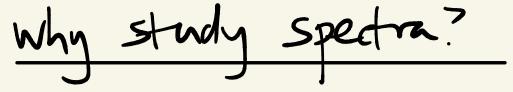
An introduction to spectra Matt Booth Antwerp Algebra Colloguium 31/5/21



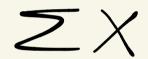
 Spectra are like
topological spaces 6-t simpler and more algebraic

Spectra control
generalised cohomology
theories

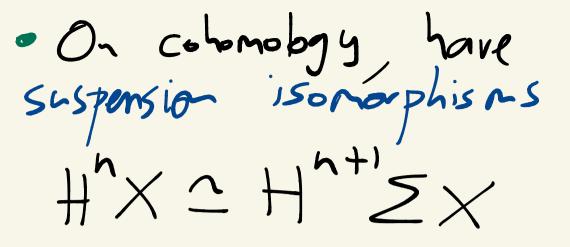
 Spectra are
interesting objects
their own right j∧

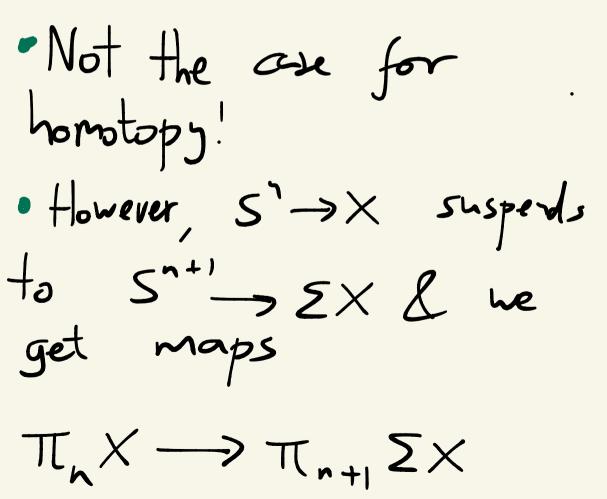
· Spectra are the base objects for spectral alge6ra ~> spectral geometry

Homotopy Vs. homobyy • If X is a topological space, have its suspension



All spaces are pointed! Example ≥5° 2 5"+1





· Frendenthal Suspension theorem : if X is a finite CW complex then  $\pi_{A} \times \neg = \pi_{A+1} \times \neg = \pi_{A+2} \times \neg = \pi_{A+2} \times \neg = \dots$ eventually stabilises thope These stable homotopy groups are less complicated than the Lustable ones.

More generally, if X is a space put  $\pi^{S}_{n} \times := \lim_{n \to \infty} \pi_{n+m} \times \mathbb{Z}^{m}_{n+m} \times$ then the stable honotopy group of X Note that it's an abelian group.

• If  $\Sigma' \times \simeq \Sigma' Y$ then  $\pi_{*}^{s} \times \hat{-} \pi_{*}^{s} \vee$ · ldea: there should be some category of 'space-like objects' where one can invert E, & some notion of wear equivalences' Jich are détected by stable homotopy groups.

Constructing spectra A spectrum 'is an N-graded seguence of spaces X: together with structure maps  $ZX_i \rightarrow X_{i+1}$ • A morphism of spectra is a sequence of maps Xi -> Yi making the obvious diagrams compute.

· Spectra have homotopy groups:  $\pi_{n} X_{i} \longrightarrow \pi_{n+1} \Sigma X_{i} \longrightarrow \pi_{n+1} X_{i+1}$ 



 $\pi_n X := \lim_{M} \pi_{n+M} X_{M}$ 

Example X a space. Have a suspension spectrum EX with  $(\Sigma^{P}X) = \Sigma^{V}X$ Then  $\pi_n \mathcal{E}^{\mathcal{E}} \times \hat{\pi}_n^{\mathcal{E}} \times \mathcal{E}_{\mathcal{A}}^{\mathcal{E}}$ . Sub-example sphere speatin \$=5° so  $S_i \circ S^i$ Sisosi

Example A an abelian group. Have an Eilenberg-Mar Lane spectrum HA with  $(HA) \simeq K(A,i)$ & TXA ~ SA \*=0 Zo else More generally if A is a chain cplx of alelian groups have a spectron HA with  $\pi_{\star}HA \simeq H_{\star}A$ 

The stable homotopy category · Say X-> Y is a Stable équivalence i+  $\pi_* \times \rightarrow \pi_* \times is an$ isonorphism. • The stable homotopy category SHC is the localisation Spectra [stable equivs.]

• The functor  $A \rightarrow HA$ gives an embedding  $D(z) \longrightarrow Stic$ 

 SHC is a triangulated
category & the above enbedding is a triangle functor.

• On SHC, Z is invertible

 In perficular in SHC
we have "spheres" Z'S for all iEZ

 SHC is enriched in abelian groups & one has  $[Z' S, X] \simeq \pi_i X$ 

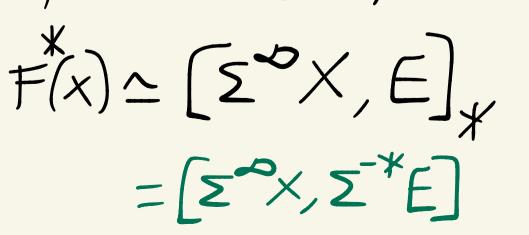
Remark Stable equivalences are the weak equivs. of a model structure on Spectra

Fibrant sojects are the *N*-spectra

CW spectra are cofibrant.

Cohomology theories · A cohomology theory 15 a functor ⇒gr.Ab satisfying: · homotopy invariance · F sends wedges to products · LESs for CW pairs

Examples - Singukr cohomology  $H^*(x, R)$ · K-theory (complex & real) Cobordism Bron Representability Thm All cohomology theories are representable by Specta:



Moreover maps tetueen cohomology theories lift to maps between spectra Examples  $H^{*}(-,R) \longrightarrow HR$ K-theory ~> K4, K0 (shordism ~7 MO

In particular TLNOS Cobordism classes TLNOS for compart smoth n-manifolds

Structured spectra • Spaces have a smash product  $X_{\Lambda Y} = \frac{X_{X}Y}{X_{V}Y}$ 

- 5 ~ 5 ~ 5 m+r More generally, Sh=Z

Unfortunately, A does not
lift to spectra in a
homotopically sensible way
(Lewis)

• The problem : there are nontrivial braiding isomorphisms  $S'_{\Lambda}S' \xrightarrow{\sim} S'_{\Lambda}S'$ that are nontrivial on homotopy (the above is  $-1 \in \pi_2 S^2$ )

· S^ S'A ··· AS' gets a Zn - action via permiting the factors.

• One solution: cang these actions around as part of the data Hovey-Shipley-Smith: A symmetric spectrum is a spectrum  $\{X_i\}$ with actions  $\Sigma_i \cap X_i$ st. the composite maps  $S^{K} \wedge X_{i} \longrightarrow X_{i+K}$ Z<sub>x</sub> x Z<sub>i</sub> - equivariant

• Then the sategory of symmetric spectra admits a symmetric monoridal smash product chose chit is \$ • a ring spectrum a monorid for  $\Lambda$ i S ie. a spectron A with maps  $A \land A \rightarrow A$  $S \longrightarrow A$ satisfying associativity & wit arisms

Example> • S is the initial ring spectrum (cf. Z) · If A is a ring then HA is a ring spectrum

•  $H^* \times \Lambda [\Sigma^{\infty} \times, HA]_{X}$   $Cup \qquad milliplication products <math>\Lambda$  induced HA

· A module over a is a spectrum M with action map R~M->M satisfying some identifies Examples • Every spectrum X is on S-module Ma Lagre Lay.

· Up to homotopy HR-modules are some thing as +he dy R-modules. D(R)~Ho(HR-mod) SHC

 Technical consideration :
Symmetric spectra are the Simplest model of highly strictured spectra 6Lt their honotopy theory 13 more complicated: correct notion of Leck equivalence is not detected on underlying spectra.

Other models: • Orthogonal spectra S-modules · coordinate - free specta (induced by fd. real Innur product spaces) · Excisive functors

Applications of ring spectra Applications of rings:

1) Algebraic geornetry

2) Homological algebra

commutative ring spectra

~ Spectral AG

commutative dgas or simplicial c.mgs

~ derived AG

commutative ~ usual rings AG

7 Spectral geometry

Can derelop many of the concepts of classical AC in the spectral World.

Applications so far are more topological:

e.g. tmf

2) Spectral algebra Fix a ring spectrum A If N,M are A-modules, have  $E_{X+}^{*}(N, M)$ 

 $\xi T_{or_{A}}^{*}(N,M)$ 

via the usual sort

of Construction.

In particular can do Hochschild theory If R is a commutative ring spectrum, A an R-module, put  $A_{R}^{e} := A \wedge_{R} A^{\circ \Gamma}$ derived enveloping algebra

Examples If R, A are discrete rings then  $A_R^e \wedge A_R^o \mathbb{L}_A^{op}$ · R<sub>R</sub> · R • Q <sup>e</sup> · Q · (Z/p) has homotopy the dual Steenrod algebra

The topological Hochschild homology of A relative to R is  $THH^{R}_{*}(A) := Tor^{AR}_{*}(A, A)$ Similarly, have topological Hochschild cohomology  $THH_{R}^{*}(A) := E_{X} + \frac{*}{A_{R}^{e}}(A, A)$ 

• THH<sup>\$</sup>(A) : K-theory K(A)

•  $THH_{q}(A)$ 'non-additive' deformation theory of A

hanks

for 1 listening.