A_∞-algebras

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1 Definitions

Work over a field k of characteristic zero. All complexes are cochain complexes; i.e. the differential has degree 1. A **differential graded algebra** or **dga** for short is a complex A with a chain map $\mu : A \otimes A \to A$ that's associative and unital. Equivalently, a dga is a graded ring together with a differential d satisfying the (graded) Leibniz rule $d(ab) = d(a)b+(-1)^{|a|}d(b)$. A **morphism** of dgas is a morphism of complexes respecting the unit and multiplication; equivalently it's a morphism of graded rings respecting the differential. Say a dga is **commutative** or a **cdga** if all graded commutators $[a, b] = ab - (-1)^{|b||a|}ba$ vanish. Note that if at least one of a, b have even degree, then $ab = ba$, whereas if both are of odd degree then $ab = -ba$. In particular, if a is of odd degree then $a^2 = 0$ – so the even degree part of a cdga behaves like a symmetric algebra, whereas the odd degree part behaves like an exterior algebra. Given a dga A, we may form its cohomology HA, which is also a dga under the multiplication induced from A. We have an obvious k-linear quasi-isomorphism $A \to HA$; is it an algebra map?

Example 1.1 ([Hes07]). Let A be the cdga $k[u, v, w]$ where u, v have degree 3 and w has degree 5, and $d(w) = uv$. So A is finite-dimensional, with basis $\{1, u, v, w, uv, uw, vw, uvw\}$. One can check that $H^0(A) = H^{11}(A) = k$, $H^3(A) = H^8(A) = k \oplus k$, and all other cohomology groups are zero. If $\phi : A \to HA$ is an algebra map, then $\phi(w)$ must be zero. Hence, $\phi(uw)$ must be zero, but uw is one of the generators of $H^*(A)$, so that ϕ cannot be a quasi-isomorphism.

In fact, the above example shows something stronger: A admits no dga quasi-isomorphism to HA. In general, a dga A is called **formal** if it's quasi-isomorphic (possibly via a zig-zag of quasi-isomorphisms) to HA. A dga is called **minimal** if $d = 0$; clearly a minimal dga is formal. What algebra information about A can we transfer along quasi-isomorphisms? The answer is that HA admits the structure of an A_∞-algebra. We'll follow the treatment of Keller in [Kel01].

Definition 1.2. An A_∞-algebra over l is a complex A together with, for each $n \geq 1$, a k-bilinear map $m_n : A^{\otimes n} \to A$ of degree $2 - n$ satisfying for all n the coherence equations (or the **Stasheff identities**)

$$\mathrm{St}_n : \quad \sum (-1)^{r+st} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

where 1 indicates the identity map, the sum runs over decompositions $n = r + s + t$, and all tensor products are over k. We're following the sign conventions of [GJ90]; note that other sign conventions exist in the literature (e.g. in [LH03]).

Remark 1.3. The original motivation for the definition came from Stasheff's work on A_∞-spaces in [Sta63]. If X is a pointed topological space and ΩX its loop space, then we have a 'composition of loops' map $\Omega X \times \Omega X \to \Omega X$. It's not associative, but it is associative up to homotopy. Similarly, one can bracket the product of four loops $a.b.c.d$ in five different ways, and one obtains five homotopies fitting into the Mac Lane pentagon. These homotopies are further linked via higher homotopies; we get an infinite-dimensional polytope K the **associahedron** with $(n-2)$-dimensional faces K_n corresponding to the homotopies between compositions of n loops. An **A_∞-space** is a topological space Y together with maps $f_n : K_n \to Y^n$ satisfying the appropriate coherence conditions. For example a loop space is an A_∞-space. Then, if Y is an A_∞-space, then the singular chain complex of Y is an A_∞-algebra.

For readability, I'll often write $a_1 \cdot a_2$ to mean $a_1 \otimes a_2$ (multiplication in the tensor algebra). Suppose that A is an A_∞-algebra. Then St_1 simply reads as $m_1^2 = 0$, in other words that m_1 is a differential on A. Hence we may define the cohomology HA. The next identity St_2 tells us that $m_1 m_2 = m_2(m_1 \cdot 1 - 1 \cdot m_1)$, or in other words that m_2 is a derivation on (A, m_1). The third identity St_3 yields

$$m_2(1 \cdot m_2 - m_2 \cdot 1) = m_1 m_3 + m_3(\sum_{i+j=2} 1^{\cdot i} \cdot m_1 \cdot 1^{\cdot j})$$

The left hand side is the associator of m_2, and the right hand side is the boundary of the map m_3 in the complex $\mathrm{Hom}(A^{\otimes 3}, A)$. Hence, m_2 is a homotopy associative 'multiplication' on A. In particular, we obtain:

Lemma 1.4. *Suppose that A is an A_∞-algebra with $m_3 = 0$ or $m_1 = 0$. Then (A, m_1, m_2) is a dga. Similarly, if A is any A_∞-algebra, then $(HA, [m_2])$ is a graded algebra. Conversely, if (A, d, μ) is a dga, then $(A, d, \mu, 0, 0, 0, \cdots)$ is an A_∞-algebra.*

Additional signs arise in the above formulas via the Koszul sign rule when one wants to put elements into them. The following Lemma is extremely useful:

Lemma 1.5. *Fix positive integers $n = r + s + t$ and n homogeneous elements $a_1, \ldots, a_n$ in A. Then*

$$(1^{\cdot r} \cdot m_s \cdot 1^{\cdot t})(a_1 \cdots a_n) = (-1)^{\epsilon} a_1 \cdots a_r \cdot m_s(a_{r+1} \cdots a_{r+s}) \cdot a_{r+s+1} \cdots a_n$$

where $\epsilon = s \sum_{j=1}^{r} |a_j|$. In particular, if s is even then the naïve choice of sign is the correct one.

Proof. Using the Koszul sign rule gives a power of $|m_s| \sum_{j=1}^{r} |a_j|$, which has the same parity as ϵ. ☐

An A_∞-algebra A is **strictly unital** if there exists an element $\eta \in A^0$ such that $m_1(\eta) = 0$, $m_1(\eta, a) = m_2(a, \eta) = a$, and if $n > 2$ then m_n vanishes whenever one of its arguments is η.

Definition 1.6. Let A and B be A_∞-algebras. A **morphism** is a family of degree $1-n$ linear maps $f_n : A^{\otimes n} \to B$ satisfying the identities

$$\sum_{n=r+s+t} (-1)^{r+st} f_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = \sum_{i_1+\ldots+i_r=n} (-1)^{\sigma(i_1,\ldots,i_n)} m_r(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

where $\sigma(i_1, \ldots, i_n)$ is the sum $\sum_j (r-j)(i_j - 1)$ (note that only terms with $r-j$ odd and i_j even will contribute to the sign).

In particular, f_1 is a chain map. A morphism f is **strict** if it's a chain map; i.e. $f_n = 0$ for $n > 1$. A morphism f is a **quasi-isomorphism** if f_1 is. One can compose morphisms by setting $(f \circ g)_n = \sum_{i_1+\ldots+i_r=n} (-1)^{\sigma(i_1,\ldots,i_n)} f_r \circ (g_{i_1} \otimes \cdots \otimes g_{i_r})$.

2 Coalgebras and homotopy theory

We give an alternate quick definition of an A_∞-algebra. If V is a dg-vector space, then the reduced tensor coalgebra $\bar{T}^c(V)$ is a dg-coalgebra: the coproduct is the **deconcatenation coproduct** $\bar{T}^c(V) \to \bar{T}^c(V) \otimes \bar{T}^c(V)$ that sends $v_1 \cdots v_n$ to $\sum_i v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$. The differential is built out of the bar differential and the differential on V. It's easy to see that $\bar{T}^c(V)$ is conilpotent: $\Delta^{n+1}(v_1 \cdots v_n) = 0$. In fact, $\bar{T}^c$ is the cofree conilpotent noncounital coalgebra functor: if C is conilpotent then $C \to \bar{T}^c(V)$ is determined completely by the composition $l : C \to \bar{T}^c(V) \to V$. For example, any map $f : \bar{T}^c(W) \to \bar{T}^c(V)$ is determined completely by its **Taylor coefficients** $f_n : W^{\otimes n} \to V$.

Lemma 2.1. *Let f, g be composable coalgebra maps between three reduced tensor coalgebras. Then the Taylor coefficients of the composition $f \circ g$ are given by*

$$(g \circ f)_n = \sum_{i_1 + \ldots + i_r = n} g_r(f_{i_1} \otimes \cdots \otimes f_{i_r})$$

Note the similarity with composition of A_∞-algebra maps.

Definition 2.2. Let C be a dg-coalgebra. A **coderivation of degree p on C** is a linear degree p endomorphism δ of C satisfying $(\delta \otimes 1 + 1 \otimes \delta) \circ \Delta = \Delta \circ \delta$.

The graded space $\mathrm{Coder}(C)$ of all coderivations of C is not closed under composition, but is closed under the commutator bracket. Say that $\delta \in \mathrm{Coder}^1(C)$ is a **differential** if $\delta^2 = 0$; in this case $\mathrm{ad}(\delta)$ is a differential on $\mathrm{Coder}(C)$, making $\mathrm{Coder}(C)$ into a dgla. In the special case that $C = \bar{T}^c(V)$, a coderivation is determined by its Taylor coefficients. Coderivations compose similarly to coalgebra morphisms:

Lemma 2.3. *Let δ, δ' be coderivations on $\bar{T}^c(V)$. Then the Taylor coefficients of the composition $\delta \circ \delta'$ are given by*

$$(\delta \circ \delta')_n = \sum_{r+s+t=n} \delta_{r+1+t}(1^{\otimes r} \otimes \delta'_s \otimes 1^{\otimes t})$$

Theorem 2.4. *An A_∞-algebra structure on a graded vector space A is the same thing as a differential δ on $\bar{T}^c(A[1])$.*

Proof. We provide a sketch. Given a coderivation δ we obtain Taylor coefficients $\delta_n : A[1]^{\otimes n} \to A$ of degree 1; in other words, these are maps $m_n : A^{\otimes n} \to A$ of degree $2 - n$. The Stasheff identities are equivalent to δ being a differential. The sign changes occur in the Stasheff identities because of the need to move elements past the formal suspension symbol [1]. □

The following proposition can be checked in a similar manner:

Proposition 2.5. *Let A, A' be two A_∞-algebras with associated differentials δ, δ'. Then an A_∞-morphism $f : A \to A'$ is the same thing as a coalgebra morphism $\bar{T}^c(A[1]) \to \bar{T}^c(A'[1])$ commuting with the coderivations.*

Definition 2.6. Let A, A' be A_∞-algebras and f, g a pair of maps $A \to A'$. Let F, G be the associated maps $\bar{T}^c(A[1]) \to \bar{T}^c(A'[1])$. Say that f and g are **homotopic** if there's a map $H : \bar{T}^c(A[1]) \to \bar{T}^c(A'[1])$ of degree -1 with $\Delta H = F \otimes H + H \otimes G$ and $F - G = \partial H$, where ∂ is the differential in the Hom-complex.

One can unwind this definition into a set of identities on the Taylor coefficients of H; this is done in [LH03], 1.2. Say that A, A' are **homotopy equivalent** if there are maps $f : a \to A'$ and $f' : A' \to A$ satisfying $f'f \simeq \mathrm{id}_A$ and $ff' \simeq \mathrm{id}_{A'}$.

Theorem 2.7 ([Pro11])**.** *Homotopy equivalence is an equivalence relation on the category* $\mathbf{Alg}_\infty$ *of A_∞-algebras. Moreover, two A_∞-algebras are homotopy equivalent if and only if they're quasi-isomorphic.*

The category **dga** of differential graded algebras sits inside the category $\mathbf{Alg}_\infty$. It's not a full subcategory: there may be more A_∞-algebra maps than dga maps between two dgas. However, two dgas are dga quasi-isomorphic if and only if they're A_∞-quasi-isomorphic: this is shown in, for example, [LH03], 1.3.1.3. Abstractly, this follows from the existence of model structures on both **dga** and **cndgc**, the category of conilpotent dg-coalgebras, for which the bar and cobar constructions are Quillen equivalences.

Including $\mathbf{dga} \hookrightarrow \mathbf{Alg}_\infty$ does not create more quasi-isomorphism classes. Indeed every A_∞-algebra is quasi-isomorphic to a dga: one can take the adjunction quasi-isomorphism $\Omega BA \to A$ induced by the bar and cobar constructions. However, we do get new descriptions of quasi-isomorphism class representatives. One nice such representative is the **minimal model** of an A_∞-algebra.

3 Minimal models

An A_∞-algebra is **minimal** if $m_1 = 0$. Clearly a dga is minimal (in the earlier sense) if and only if it's a minimal A_∞-algebra. Every A_∞-algebra admits a minimal model. More precisely:

Theorem 3.1 (Kadeishvili [Kad80])**.** *Let $(A, m_1, m_2, \dots)$ be an A_∞-algebra, and let HA be its cohomology ring. Then there exists the structure of an A_∞-algebra $\mathscr{H}A = (HA, 0, [m_2], p_3, p_4, \dots)$ on HA, and an A_∞-algebra morphism $\mathscr{H}A \to A$ lifting the identity of A. Moreover, the A_∞-algebra structure on HA is unique up to quasi-isomorphism.*

Remark 3.2. While the multiplication on HA is induced by m_2, we need not have $p_n = [m_n]$ for $n > 2$. Indeed, if A is a non-formal dga, then $\mathscr{H}A$ must have nontrivial higher multiplications. We also note that $\mathscr{H}A \to A$ is clearly an A_∞-quasi-isomorphism, since it lifts the identity on A. We also remark that the theorem follows from the essentially equivalent **homotopy transfer theorem**: if A is an A_∞-algebra, and V a homotopy retract of A, then V admits the structure of an A_∞-algebra making the retract into an A_∞-quasi-isomorphism (see [LV12] 9.4 for details). The result follows since, over a field, the cohomology of any chain complex is always a homotopy retract as one can choose splittings.

It's possible to give a constructive proof of Kadeishvili's theorem: Merkulov did this in [Mer99]. One can define the p_n recursively: suppose for convenience that A is a dga. Choose any section $\sigma : HA \to A$ and let $\pi : A \to HA$ be the projection to HA. We'll identify HA with its image under σ. Choose a homotopy $h : \mathrm{id}_A \to \sigma\pi$. Define recursively maps $\lambda_n : (HA)^{\otimes n} \to A$ by $\lambda_2 = m_2$, and

$$\lambda_n := \sum_{s+t=n} (-1)^{s+1} \lambda_2(h\lambda_s \otimes h\lambda_t)$$

where we formally interpret $h\lambda_1 := -\mathrm{id}_A$. Then, $p_n = \pi \circ \lambda_n$. See [Mar06] for some very explicit formulas (whose sign conventions differ).

Definition 3.3. Let G be an abelian group. An A_∞-algebra A is **Adams $G-$graded** or just **Adams graded** if it admits a secondary grading by G such that each higher multiplication map m_n is of degree $(2-n, 0)$.

If an A_∞-algebra is Adams graded, then by making appropriate choices one can upgrade Merkulov's construction to give an A_∞-quasi-isomorphism of Adams graded algebras $A \to \mathscr{H}A$. Moreover, if A is strictly unital, one can choose the morphism to be strictly unital. See Section 2 of [LPWZ09] for more details.

One can sometimes compute A_∞-operations on a dga by means of Massey products. In what follows, $\tilde{a}$ means $(-1)^{1+|a|}a$. We're using the same sign conventions as [Kra66].

Definition 3.4. Let $a_{1\,1}, \ldots, a_{r\,r}$ be any r elements of a dga A. The **r-fold Massey product** $\langle [a_{1\,1}], \ldots, [a_{r\,r}] \rangle$ of the cohomology classes $[a_{1\,1}], \ldots, [a_{r\,r}]$ is defined to be the set of cohomology classes of sums $\tilde{a}_{1\,1}a_{2\,r} + \cdots + \tilde{a}_{1\,r-1}a_{r\,r}$ such that $da_{i\,j} = \tilde{a}_{i\,i}a_{i+1\,j} + \cdots + \tilde{a}_{i\,j-1}a_{j\,j}$ for all $1 \leq i \leq j \leq n$ with $(i, j) \neq (1, n)$. This operation is well-defined, in the sense that it depends only on the cohomology classes $[a_{1\,1}], \ldots, [a_{r\,r}]$.

We'll abuse terminology by referring to elements of $\langle x_1, \ldots, x_r \rangle$ as Massey products.

Remark 3.5. We remark that $\langle x_1, \ldots, x_r \rangle$ may be empty: for example, in order for $\langle x, y, z \rangle$ to be nonempty, we must have $xy = yz = 0$. More generally, for $\langle x_1, \ldots, x_r \rangle$ to be nonempty, we require that each $\langle x_p, \ldots, x_q \rangle$ is nonempty for $0 < q - p < n - 1$. Most sources define $\langle x, y, z \rangle$ only when it's nonempty, and leave it undefined otherwise.

The point is that, when Massey products exist, Merkulov's higher multiplications p_n are all Massey products, up to sign.

Theorem 3.6 ([LPWZ09])**.** *Let A be a dga and let $x_1, \ldots, x_r$ ($r > 2$) be cohomology classes in HA, and suppose that $\langle x_1, \ldots, x_r \rangle$ is nonempty. Give HA an A_∞-algebra structure via Merkulov's construction. Then, up to sign, the higher multiplication $p_r(x_1, \ldots, x_r)$ is a Massey product.*

So, if A is a formal dga, then all Massey products (that exist) will vanish. The converse is not true: formality of a dga cannot be checked simply by looking at its Massey products.

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