Nonsmooth CY structures for algebras & coalgebras

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§1. Motivation

Def. A smooth proj. variety X is **Calabi–Yau** if $\omega_X \simeq \mathcal{O}_X$.

Examples smooth degree $n+1$ hypersurface in $\mathbb{P}^n$

<u>Property</u> if X is CY then

$[n]$ is a Serre functor

on $D^b(\mathrm{Coh}\, X)$

[Grothendieck duality]

<u>Def.</u> (Ginzburg '06 ish)

A dg algebra A is <u>Calabi-Yau</u> if

i) A is <u>homologically smooth</u>

($A \in \operatorname{per} A^e$)

ii) There is an A-bilinear quasi-isomorphism $A \simeq A^\vee[n]$

where $A^\vee$ is the inverse dualising complex

is $\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)$.

Prop. (Ginzburg). If A is CY

then $D_{fd}(A) := \{X \in D(A) : \dim H^* X < \infty\}$

has $[n]$ as a Serre functor.

Prop: Let X be a smooth projective variety. Then $\exists$ a dg algebra A & a triangle equivalence $D^b(\mathrm{Coh}\,X) \simeq \mathrm{per}\,A$

(idea: A is $\mathbb{R}\mathrm{End}$ of a compact generator)

& X is CY $\iff$ A is CY.

Def. A dg algebra A is nonsmooth Calabi-Yau if there is a bimodule quasi-isomorphism $A \simeq A^\vee[n]$

Loose idea behind Koszul duality:

There's an equivalence of ∞-categories

$$\left\{ \begin{matrix} \text{augmented} \\ \text{dg algebras} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{conilpotent dg} \\ \text{coalgebras} \end{matrix} \right\}$$

(this underlies derived noncommutative deformation theory)

Def. 1) A dg algebra A is Frobenius
if there's a left A-linear quasi-isomorphism

$$A \simeq A^*[n]$$

2) A dg algebra A is symmetric if
there's an A-bilinear q.iso

$$A \simeq A^*[n]$$

3) An augmented dg algebra $A \to k$ is **Gorenstein** if there's an A-bilinear quasi-iso $\mathbb{R}\mathrm{Hom}_A(k, A)[n] \simeq k$

rmks
- This is due to Avramov–Foxby
- This is a 'one-sided' version of us CY
- Equivalent to ask for a left A-linear qiso.

Main thm (BCL)

The following properties of dg (co)algebras are Koszul dual:

1) nonsmooth CY $\longleftrightarrow$ symmetric

2) Gorenstein $\longleftrightarrow$ Frobenius .

Examples

1) If X is a irreducible projective variety with Cohen-Macaulay singularities, take a dg algebra A st $\mathrm{per}\, X \simeq \mathrm{per}\, A$

X has trivial ω $\Longleftrightarrow$ A is nonsmooth CY.

Proof: Grothendieck duality!

2) Take R a commutative noetherian local ring.

Then R is nonsmooth CY

$\Longleftrightarrow$ R is Gorenstein.

§2 Koszul duality.

<u>Def</u>. A dg coalgebra is a comonoid in $(\text{dgVect}, \otimes)$

<u>ie</u>. a dg vector space C

with a comultiplication $\Delta : C \longrightarrow C \otimes C$

a counit $\eta : C \longrightarrow k$

satisfying coassociativity & counitality.

<u>Observation</u> if C is a coalgebra then its linear dual C^* is an algebra.

(Algebras don't necessarily dualize to coalgebras!)

↳ This allows one to view a dg coalgebra as a certain kind of topological dg algebra!

If V is a vector space,

$$V = \varinjlim_{\substack{U \subseteq V \\ U \text{ fin dim}}} U \quad . \quad \text{So} \quad V^* = \varprojlim_{\substack{U \subseteq V \\ U \text{ fin dim.}}} U^*$$

<u>Def.</u> A (dg) vector space is **pseudocompact** if it's an inverse limit of fin. dim. (dg) vector spaces.

Get a category $\mathrm{pcVect} \subseteq \mathrm{TopVect}$

<u>Def.</u> A pseudocompact dg algebra is a (formal) inverse limit of fin. dim. dg algebras.

<u>Prop.</u> The linear dual gives an equivalence

$$\mathrm{dgCog} \simeq \mathrm{pcdgAlg}^{op}$$

Ex. The dg algebra $K[[x]]$ $\quad |x| = n$, $dx = 0$

is pseudocompact!

$$K[[x]] = \lim_{\overleftarrow{i}} K[x]/x^i$$

Given an augmented dg algebra A, cook up a dg coalgebra BA the bar construction on A as follows.

As a graded coalgebra, $BA = T^c(\bar{A}[1])$

$\bar{A} = \ker(A \to k)$

$T^c(V) = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$

$$\Delta(v_1 - v_r) = \sum_{i=1}^{r} (v_1 - v_i) \otimes (v_{i+1} - v_r)$$

Differential on BA has two components:

d_1 usual differential ('internal')

d_2 induced from multiplication on $\bar{A}$

Gives a functor $B: \text{aug.Alg} \to \text{con.Cog}$

Has a left adjoint $\Omega: \text{con.Cog} \to \text{aug.Alg}$

the cobar construction

constructed as $\left(T(\bar{C}[-1]), d_1 + d_2\right)$

coaugmented & Δ is suitably nilpotent.

Prop. A an augmented algebra.

There is a quasi-isomorphism

$$(BA)^* \simeq \mathbb{R}\mathrm{End}_A(k, k)$$

$A^!$ 'Koszul dual of A'.

$\underline{\text{Thm}}$ (Quillen, Hinich, Lefèvre-Hasegawa, Positselski)

("Koszul duality.")

There are model structures on aug. dgAlg

con. dgCog

Making $\Omega \dashv B$ into a Quillen equivalence.

<u>Example</u> i) $A = \dfrac{K[\varepsilon]}{\varepsilon^2}$, $|\varepsilon| = 1-n$, $d\varepsilon = 0$

BA is $K[\![x]\!]$ with $|x| = n$, $dx = 0$

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2) $A = K[x]$, $|x| = n$, $dx = 0$, BA is $\dfrac{K[\varepsilon]}{\varepsilon^2}$, $|\varepsilon| = 1-n$, $d\varepsilon = 0$

or $K[\![x]\!]$

Just like an algebra has a derived category, a coalgebra C has a coderived category constructed via the following:

1) A C-comodule is a vector space V & a coassociative 'coaction' map $V \longrightarrow V \otimes C$

2) $D^{co}(C)$ is a localisation of C-comod

It's a triangulated category!

Thm (Module - comodule Koszul duality)

If A is a dg algebra, C the corresponding Koszul dual coalgebra, then $\exists$ a triangle equiv.

$$D(A) \simeq D^{co}(C).$$

$$A \longmapsto K$$

$$K \longmapsto C$$

This allows us to transport things like RHom, linear duals, etc. to $D^\infty(C)$.

This allows us to define what it means for a dg coalgebra to be

nsCY,

Gorenstein,

Frobenius,

symmetric.

Rmk: we need to use <u>bimodule</u> Koszul duality, developed by Guan-Holstein-Lazarev

<u>Main thm</u> The following properties are Koszul dual

C sym $\Longleftrightarrow$ $\exists$ a weak equivalence $C \xrightarrow{\sim} \underbrace{C^{*}[n]}_{\text{Serre object}}$

1) nonsmooth CY $\longleftrightarrow$ symmetric

2) Gorenstein $\longleftrightarrow$ Frobenius

<u>Idea</u> 'linear dual' $\xleftrightarrow{KD}$ '(co)module/bimodule dual'.

$* \longleftrightarrow \mathbb{R}\mathrm{Hom}_C(-, C)$

Rmk Main application of this in our paper is a new characterisation of Poincaré duality spaces:

Given X, $\rightsquigarrow C_\bullet \Omega X$ dg algebra

$\rightsquigarrow C_\bullet X$ dg coalgebra

$C^\bullet X \simeq (C_\bullet X)^*$

$C_\bullet \Omega X$ knows lots of topological information about X !

Moreover, $C_\bullet \Omega X \simeq \Omega C_\bullet X$

(Adams, Rivera-Zeinalian)

Thm (BCL) X a suitably nice topological space

TFAE

easy! 1) X is PD

2) $C_*\Omega X$ is Gorenstein

3) $C_*\Omega X$ is CY

main thm. 4) $C_* X$ is Frobenius

5) $C_* X$ is symmetric.

uses that $C_*\Omega X$ is a Hopf algebra.

standard bar complex.

$$HH^\bullet(A) = R\mathrm{Hom}_{A^e}(A,A) \simeq \underbrace{BA \otimes^\tau A}$$

$BA \otimes A$ with a twisted differential.