Deformation Theory and DGLAs

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What is deformation theory?

- Deformation theory is the study of infinitesimal deformations of algebraic or geometric objects.
- Equivalently, it's the local study of a family of objects near some given thing we want to deform (infinitesimal geometry of moduli spaces).
- For example, given $X \xrightarrow{f} Y$, and some 'infinitesimal thickenings' $X \to \tilde{X}$ and $Y \to \tilde{Y}$, we would like to know whether f lifts to the thickenings.

Infinitesimals: Artin rings

- Setup: everything will be over an algebraically closed field k of characteristic zero.
- Our infinitesimals will be the objects of the category $\mathbf{Art}_k$ of commutative local Artinian k-algebras with residue field k.
- Geometrically, if $R \in \mathbf{Art}_k$ then $\operatorname{Spec} R$ is a **fat point**: a point together with some information about its formal neighbourhood.
- R is Noetherian, and as a vector space, $R \cong k \oplus \mathfrak{m}_R$. The ideal $\mathfrak{m}_R$ is nilpotent, since $\mathfrak{m}_R = J(R) = \mathrm{nil}(R)$. Each vector space $\mathfrak{m}_R^j / \mathfrak{m}_R^{j+1}$ is finite-dimensional and there are only finitely many of them. So R is finite-dimensional.

The dual numbers

- The ring of **dual numbers** is the ring $k[\epsilon] := k[t]/t^2 \in \mathbf{Art}_k$.
- Deformations over $k[\epsilon]$ are called **first-order deformations**.
- We can think of $\mathrm{Spec}(k[\epsilon])$ as a point with an infinitesimal tangent vector attached:
  $\mathrm{Hom}_{k-\mathbf{Sch}}(\mathrm{Spec}(k[\epsilon]), X) \cong \{(p, v) : p \in X, v \in T_pX\}$
- We'll now see some examples of first-order deformation problems.

Associative algebras

- Let A be an associative noncommutative k-algebra. We wish to deform the multiplication on A; set $x \odot y = xy + \epsilon f(x, y)$ where $f \in \mathrm{Hom}(A \otimes A, A)$.
- Associativity of $\odot$ forces the equation

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z = 0$$

- We should regard deformations as equivalent if they differ by an automorphism $\phi(x) = x + \epsilon g(x)$ of A, with $g \in \mathrm{Hom}(A, A)$.
- One can check that setting $x \boxdot y := \phi(\phi^{-1}x \odot \phi^{-1}y)$ sends $f(x, y)$ to $f(x, y) - g(x)y + g(xy) - xg(y)$.

Hochschild cohomology

- A cohomology theory for A-bimodules. $HH^n(M, N) := \mathrm{Ext}^{n+1}_{A-\mathbf{bimod}}(M, N)$. Write $HH^n(A)$ for $HH^n(A, A)$.
- $HH^n(A)$ can be computed with the **bar complex** $\mathrm{Bar}(A)$, a free resolution of A of the form

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A$$

with boundary maps

$$d(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

- The complex $\mathrm{Hom}(\mathrm{Bar}(A), A)[1]$ whose cohomology is $HH^n(A, A)$ is the **Hochschild complex** $\mathrm{Hoch}(A)$.

Hochschild cohomology

- One can check that in $\mathrm{Hoch}(A)$, the 1-cocycles are exactly the maps f satisfying

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y)z = 0$$

  and the 1-coboundaries are the maps of the form

$$xg(y) - g(xy) + g(x)y$$

- So we obtain an isomorphism

$$\frac{\text{(first order deformations)}}{\text{(isomorphism)}} \cong HH^1(A)$$

Commutative algebras

- The setup is very similar, but now we ask for everything to be commutative.
- We obtain a subcomplex of the Hochschild complex, the **Harrison complex** $\mathrm{Harr}(A)$. It can be characterised as those cochains vanishing on the sums

$$\sum_{\sigma} \mathrm{sgn}(\sigma) a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)}$$

  where σ is a (p, q)-**shuffle**: a permutation $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ of $(1, \ldots, p+q)$ with $x_i < \cdots < x_p$ and $y_1 < \cdots < y_q$.
- The first order commutative deformations are $\mathrm{HHar}^1(A)$.

More general deformations

- In general, for $R \in \mathbf{Art}_k$ we could have considered flat R-algebras $\tilde{A}$ together with an isomorphism

$$\tilde{A} \otimes_R k \cong A$$

  regarding two as equivalent if there's a R-linear isomorphism between them which specialises to the identity on A.
- If $\mathcal{M}_n^{\mathrm{com}} \hookrightarrow k^{n^3}$ is the moduli space of associative commutative algebras of dimension n, then a deformation of A over R is the same thing as a map $\operatorname{Spec} R \to \mathcal{M}_n$ where the closed point of R gets sent to A.
- In this sense, we've computed $T_A(\mathcal{M}_n^{\mathrm{com}}) \cong \mathrm{HHar}^1(A) \in k - \mathbf{mod}$.

Deformations of a dg-vector space

- A **dg-vector space** is a cochain complex $V = (V^i, d^i)$ of k-vector spaces. Note that if V, W are dg-vector spaces then so is $\mathrm{Hom}(V, W)$.
- Define a deformation of V over R to be a cochain complex $\tilde{V} = (V^i \otimes_k R, d_R^i)$ of R-modules such that $\tilde{V} \otimes_R k \cong V$.
- Say that two deformations are isomorphic if there's a R-linear isomorphism between them that specialises to id_V on V.
- Similarly to before, the first-order deformations of V correspond to elements of $\mathrm{Ext}^1(V, V)$.

Deformations of a scheme

- If X is a scheme, then a **deformation of** X **over** R is a scheme $\tilde{X}$, flat over $\operatorname{Spec} R$, with an isomorphism $\tilde{X} \times_{\operatorname{Spec} R} k \cong X$.
- If X is smooth then first-order deformations of X are parameterised by $H^1(X, \mathcal{T}_X)$.
- Smooth affine schemes have no nontrivial deformations.

Deformation functors

- Suppose we're given some object X we want to deform. We could try to collect all of the deformation data by studying the assignment

$$\mathrm{Def}_X : R \mapsto \frac{\text{(deformations of } X \text{ over } R)}{\text{(isomorphisms)}}$$

- We should obtain a functor $\mathrm{Def}_X : \mathbf{Art}_k \to \mathbf{Set}$, since we should be able to base change deformations; e.g. if $\tilde{A}$ is a deformation of an algebra A over R, and we have a map $R \to R'$, then $\tilde{A} \otimes_R R'$ is a deformation of A over R'.
- Clearly $\mathrm{Def}_X(k)$ should be a one-element set, since the only deformation over k should be the trivial deformation.

Deformation functors

- Suppose we have a functor $F : \mathbf{Art}_k \to \mathbf{Set}$. If we have a diagram

$$R' \to R \leftarrow R'' \qquad (*)$$

in $\mathbf{Art}_k$, then taking the pullback and applying F we obtain a natural map of sets $\eta : F(R' \times_R R'') \to F(R') \times_{F(R)} F(R'')$.

- A **deformation functor** is a functor $F : \mathbf{Art}_k \to \mathbf{Set}$ such that:
  1. $F(k)$ is a one-element set.
  2. In any diagram * as above, whenever $R' \to R$ is a surjection then η is a surjection.
  3. In *, if $R = k$ then η is a bijection.

Examples of deformation functors

- All of the deformation problems we saw earlier.
- Any representable functor.
- More generally, let $\hat{\mathbf{Art}}_k$ be the category of complete local Noetherian k-algebras with residue field k. A functor $F : \mathbf{Art}_k \to \mathbf{Set}$ is **prorepresentable** if it's the restriction of a representable functor on $\hat{\mathbf{Art}}_k$.

The tangent space

- The **tangent space** to a deformation functor is the set $T^1F := F(k[\epsilon])$. It's a vector space. It's not always finite-dimensional, but it is if F is prorepresentable.
- $T^1\,\mathrm{Hom}(R, -) \cong T_{\mathfrak{m}_R}(\mathrm{Spec}\,R)$.
- If X is a k-scheme,

$$X \text{ is proper over } k \implies T^1\mathrm{Def}_X \text{ is finite-dimensional.}$$

- We know that if A is an algebra, then

$$T^1\mathrm{Def}_A \cong HH^1(A)$$

Claim: every deformation functor we've seen is controlled by a suitable naturally occurring DGLA.

DGLAs

- A **differential graded Lie algebra**, or **DGLA**, is a dg-vector space with a graded Lie bracket. More precisely, a DGLA L is a $\mathbb{Z}$-graded k-vector space $L = \oplus_{i \in \mathbb{Z}} L^i$, a differential $d : L \to L$ and a bilinear bracket $[-,-] : L \times L \to L$ satisfying the following:
  1. The differential d makes L into a dg-module over k, i.e. L is a cochain complex of k-vector spaces.
  2. $[-,-]$ respects the grading: $[L^i, L^j] \subset L^{i+j}$
  3. $[-,-]$ is graded-anticommutative: $[x,y] = -(-1)^{\bar{x}\cdot\bar{y}}[y,x]$
  4. $[-,-]$ satisfies the graded Jacobi identity: $[[x,y],z] = [x,[y,z]] - (-1)^{\bar{x}\cdot\bar{y}}[y,[x,z]]$
  5. the graded Leibniz rule: $d[x,y] = [dx,y] + (-1)^{\bar{x}}[x,dy]$

Examples of DGLAs

- If L is a DGLA then L^0 and $L^e := \oplus_{i \in \mathbb{Z}} L^{2i}$ are Lie algebras. Conversely we can view any Lie algebra L as a DGLA concentrated in degree 0.
- If (V, d) is a dg-vector space then $A = \mathrm{End}_k(V)$ is a dg-algebra. If we equip A with commutator $[f, g] = fg - (-1)^{\bar{f} \cdot \bar{g}} gf$ and differential $\partial(f) = [d, f]$, then A becomes a DGLA. Every such DGLA is **formal**; i.e. quasi-isomorphic to its cohomology DGLA

$$H^*(\mathrm{End}_k(V)) \cong H^*(\mathbb{R}\mathrm{End}_k(V)) \cong \mathrm{Ext}^*_k(V, V)$$

- More generally, any DGA can be turned into a DGLA using commutators. For example, the de Rham complex of a smooth manifold is a DGLA.

Back to dg-vector spaces

- Let (V, d) be a dg-vector space and set $L := \mathrm{End}_k(V)$. A deformation of V over R is the same thing as a deformation of the differential $d_R = d + \xi$ for $\xi \in L^1 \otimes \mathfrak{m}_R$, and the condition that $d_R^2 = 0$ forces $d\xi + \frac{1}{2}[\xi, \xi] = 0$.
- Two deformations are equivalent if there's an isomorphism ϕ between them satisfying $\phi = \mathrm{id}_V + u$ for $u \in L^0 \otimes \mathfrak{m}_R$.
- One can check that this corresponds to the exponential of the adjoint action $\exp(L^0 \otimes \mathfrak{m}_R) \to GL(L^1 \otimes \mathfrak{m}_R)$.

Back to dg-vector spaces

- So we've shown that

$$\mathrm{Def}_V(R) \cong \frac{\{\xi \in L^1 \otimes \mathfrak{m}_R : \quad d\xi + \frac{1}{2}[\xi, \xi] = 0\}}{\exp(L^0 \otimes \mathfrak{m}_R)}$$

- In general, given a DGLA L, its **Maurer-Cartan functor** is $MC_L(R) = \{\xi \in L^1 \otimes \mathfrak{m}_R : \quad d\xi + \frac{1}{2}[\xi, \xi] = 0\}$
- Its **gauge group functor** is $G_L(R) = \exp(L^0 \otimes \mathfrak{m}_R)$
- Set $\mathrm{Def}_L = MC_L/G_L$. Then Def_L is a deformation functor.

Deformation functors and DGLAs

- One can show that $T^1(\mathrm{Def}_L) = H^1(L)$.
- If L is quasi-isomorphic to L', then Def_L is naturally isomorphic to $\mathrm{Def}_{L'}$.
- We've seen that for deformations of a dg-vector space V, the controlling DGLA is $\mathrm{End}_k(V)$.
- For deformations of an associative algebra A, the controlling DGLA is the Hochschild complex $\mathrm{Hoch}(A)$, equipped with the **Gernstenhaber bracket**.
- If A is commutative, then $\mathrm{Harr}(A)$ is a sub-DGLA of $\mathrm{Hoch}(A)$.

Extending deformation functors

- Question (Deligne): Does every deformation functor come from a DGLA?
- In the derived setting, the answer is yes! The higher cohomology groups $H^n(L)$ measure derived deformations, and also provide obstructions to the lifting of deformations.
- One can also talk about noncommutative deformations; the theory is the same but one needs to use DGAs rather than DGLAs.