Glasgow AG sem talk. Sep. 2022

§1 DEFORMATION THEORY

Deligne: 'dglas control deformation functors'

differential graded Lie algebras

More precisely: have

$L \mapsto \left[\Gamma \mapsto \dfrac{MC(L \otimes m_\Gamma)}{G_g} \right]$ ← Maurer-Cartan elt's ← Gauges

↑ certain functors $cArt \longrightarrow Set$

Lurie-Pridham ~2010: $\exists$ in char 0 an equivalence (of ∞cats)

$$dgLie \xleftrightarrow{\sim} \text{formal moduli problems}$$

↑ derived def^n functors: certain fctrs $cdgArt^{\leq 0} \to Set$

Not true in char p.

Loosely: this is the Koszul duality between Com & Lie operads

Lurie: in all chars $\exists$ an equivalence (of ∞cats)

$$aug\cdot Alg \xleftrightarrow{\sim} nc \text{ formal moduli problems}$$

$$(\subseteq Fun(dgArt, sSet))$$

Loosely: this is Koszul duality btwn Ass & itself.

[rmk: $E_n^!$-algs control E_n-FMPs; for $n = 1, \infty$ specialises to above]

More concretely: the map $\psi : aug\cdot Alg \xrightarrow{\sim} ncFMPs$ sends

$$\psi(A)(\Gamma) = Map_{aug\,Alg}(\Gamma^!, A)$$

cf. Brantner's partition Lie algs $= E_\infty^!$-algs.

To define $\Gamma^!$ we need to take a detour through coalgebras. Similar formulae/descriptions in Com/Lie setting.

Moral: deformation theory is coalgebraic in nature

[rmk: can also be encoded operadically]

cf. Hinich: a FMP is a formal stack, which is determined by its coalg of distributions via calculus.

§2 COALGEBRAS

if $\mathcal{C}$ is a monoidal category, an <u>algebra</u> is an object A w/ maps $A \otimes A \to A$, $\mathbb{1} \to A$ satisfying unitality & associativity diagrams

Similarly a <u>coalgebra</u> has maps $C \to C \otimes C$, $C \to \mathbb{1}$ satisfying coUn CoAss diagrams.

From now on $\mathcal{C} = \mathrm{dgVect}_k$; get <u>dgas</u> & <u>dgCs</u>

A coalg is <u>coaugmented</u> if $C \to k$ has a section ~~a coaug~~ Get a <u>coaug. coideal</u> $\overline{C}$ w/ reduced comultiplication $\overline{\Delta}$ making $\overline{C}$ into a noncounital coalgebra

A coaug. coalg. is <u>conilpotent</u> if $\forall c\ \exists p \in \mathbb{N}$ st $\overline{\Delta}^p(c) = 0$

e.g. the <u>tensor coalg</u> $T^c(V) = k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$ with deconcatenation coproduct $\Delta(v_1 \cdots v_n) = \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$

It's conilpotent since $\overline{\Delta}^{n+1}(v^{\otimes n}) = 0$.

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if A is an aug. alg, have the <u>bar construction</u>

$$BA := (T^c(\Sigma \overline{A}),\ d_1 + d_2)$$

d_1 = usual differential on $T^c \Sigma \overline{A}$

$$d_2(a_1 \cdots a_n) = \sum_{i=0}^{n} a_1 \cdots a_i a_{i+1} \cdots a_n$$

<u>rmk</u> $T^c V$ is the cofree <u>conil.</u> coalg on V

Similarly there's a <u>cobar construction</u>

$$\Omega : \text{conil. Cog} \longrightarrow \text{aug. Alg}$$

$$\Omega C := (T(\Sigma^{-1} \overline{C}),\ d_1 + d_2)$$

$d_2(c_1 \cdots c_n) = \sum_i c_1 \cdots \overline{\Delta} c_i \cdots c_n$

If C is a coalgebra then $C^\vee := \mathrm{Hom}(C,k)$ is an algebra.

If A is an algebra then $A^\vee$ only need be a coalgebra if A is finite-dimensional!!

If A is an algebra then the <u>Koszul dual</u> of A is $A^! := B^\vee A$.

Bar & cobar are adjoints: $\Omega \dashv B$.

Not too hard to check that if Γ is ~~finite~~ Artinian then $\Gamma^! \simeq \Omega(\Gamma^\vee)$, so that

$$\psi(A)(\Gamma) := \mathrm{Map}_{Alg}(\Gamma^!, A) \simeq \mathrm{Map}_{Alg}(\Omega(\Gamma^\vee), A)$$
$$\simeq \mathrm{Map}_{Cog}(\Gamma^\vee, BA)$$

This is a <u>coalgebraic</u> object!

rmk $Alg \xrightarrow{\sim} FMP$ factors through equivs

$Alg \xrightarrow{\sim} Cog_{conil}$ (Bar)

$Cog_{conil} \xrightarrow{\sim} pcAlg^{op}_{nil}$ (Dualize)

$pcAlg^{op}_{nil} \xrightarrow{\sim} FMP$ $(x \mapsto \mathrm{Map}(x,-))$

Moreover an algebra is <u>pseudocompact</u> if it's the dual of a coalgebra. There's a forgetful functor $pcAlg \to Alg$

Above formalism gives $\psi(A)(\Gamma) \simeq \mathrm{Map}_{pcAlg}(A^!, \Gamma)$

this is a prorepresentability statement.

<u>Key ingredient</u> conil. Cog & aug. Alg are model categories & $\Omega \dashv B$ is a Quillen equivalence.

aug. Alg: WE = quasi-isos & Fib = surjections. (usual Hinich model structure)

conil. Cog: WEs are created by Ω & Cof = injections

weak equivs of coalgebras are q-isos but converse is not true.

§3 GLOBAL KOSZUL DUALITY

There's an adjoint pair

$$\Omega : \mathrm{Cog}_{\mathrm{coaug}} \rightleftarrows \mathrm{Alg}_{\mathrm{aug}} : \check{B}$$

'extended Bar constrⁿ'; uses cofree coalg. functor (nasty!)

Q. Can we put model strs on each side making this a Quillen equivalence?

Such a result would give prorepresentability results in global deformation theory, where the small objects are all fin dim algs (not just Artinian ones)

Neither $\check{B}$ nor Ω preserve q.isos so need a stronger notion of equivalence: MC-equivalence

if E is a dga have a dg category $MC_{dg}(E)$:

- objects are MC elts of E
- $\mathrm{Hom}(x,y) := \mathrm{Hom}_{E\text{-Mod}}(E^x, E^y)$

so this is a subcat of $Tw(E)$

if C a coalg & E an alg then $\mathrm{Hom}(C,A)$ is a dga under

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[rrr, bend right, "f\cdot g"'] & C\otimes C \arrow[r, "f\otimes g"] & A\otimes A \arrow[r, "\mu"] & A \end{tikzcd}$$

& $MC\,\mathrm{Hom}(C,A) \simeq \mathrm{Hom}(\Omega C, A) \simeq \mathrm{Hom}(C, \check{B}A)$

A map $A \to A'$ of algs is an MC equivalence if $\forall C$, $MC_{dg}\mathrm{Hom}(C,A) \to MC_{dg}\mathrm{Hom}(C,A')$ is a quasi-equivalence.

Similar defⁿ for coalgs.

Cofibrations of coalgs are maps that $\mathcal{U}\mathcal{C}_{dg}\mathrm{Hom}(-,A)$ sends to fibrations of dg categories.
Similarly for fibrations of algebras.

<u>Rmk</u> There's a notion of <u>strong homotopy</u> between (co)alg maps.

Let $D^n \in sSet$ be standard simplicial model of n-disc with 2 vertices, 2 edges, ... one n-cell & $I_n := Ch(D^n) \in Cog$.
Then I_n is an interval object for a notion of n-homotopy.

$n=1$: usual chain htpy $\bullet \longrightarrow \bullet$

$n=2$: something weird 

$n \geq 3$: notions are all equivalent. 

Similar notion of n-homotopy for coalgebras using $I_n^\vee \simeq$ cochains on D^n

<u>Fact</u> $\mathrm{Ob}\, H^0 \mathcal{U}\mathcal{C}_{dg}\mathrm{Hom}(C,A) \simeq \dfrac{\mathrm{Hom}(\Omega C, A)}{(3\text{-htpy})} \simeq \dfrac{\mathrm{Hom}(C, \check{B}A)}{(3\text{-htpy})}$

Also a notion of ∞-htpy defined using standard cellular model for S^∞. It's equivalent to 3-htpy.
Technically a bit trickier since one has to think of $I_\infty^\vee$ as a pseudocompact algebra, use $\hat{\otimes}$, etc.