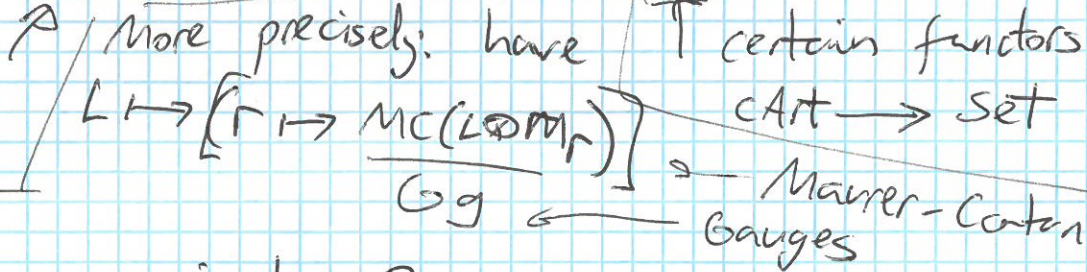


Glasgow AG sem talk. Sep. 2022

§1 DEFORMATION THEORY

Deligne: 'dglas control deformation functors'

differential graded Lie algebras



Lurie-Pridham: \exists in char 0 on equivalence (of ∞ -cats)

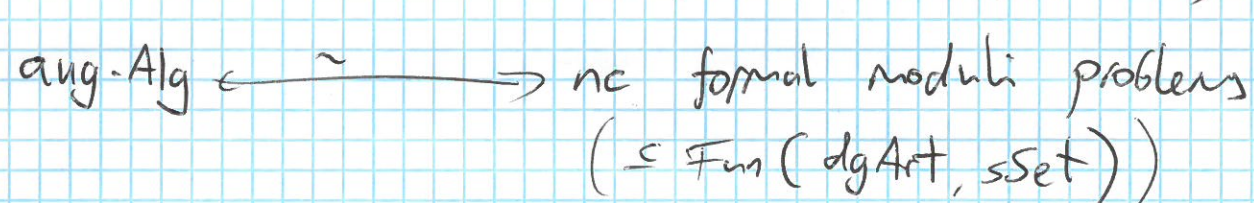
~2010



Not true in char p . \uparrow derived def^m functors: certain fctrs $\text{cdgArt}^{\text{so}} \rightarrow \text{Set}$

Loosely: this is the Koszul duality between Com & Lie operads

Lurie: in all chars \exists on equivalence (of ∞ -cats)



Loosely: this is Koszul duality btwn Ass & itself.

[rmk: $E_n^!$ -algs control E_n -FMPs; for $n=1, \infty$ specialises to above]

More concretely: the map $\Psi: \text{aug-Alg} \xrightarrow{\sim} \text{ncFMPs}$ sends $\Psi(A)(\Gamma) = \text{Map}_{\text{aug-Alg}}(\Gamma^!, A)$

cf. Bourn's partition Lie algs = $E_\infty^!$ -algs.

To define $\Gamma^!$ we need to take a detour through coalgebras. Similar formulae/descriptions in Com/Lie setting.

Moral: deformation theory is coalgebraic in nature

[rmk: can also be encoded operadically]

cf. Hinich: a FMP is a formal stack, which is determined by its coalg of distributions via calculus.

§2 COALGEBRAS

if \mathcal{C} is a monoidal category, an algebra is an object A w/ maps $A \otimes A \rightarrow A$ satisfying unitality & associativity diagrams
 $\mathbb{1} \rightarrow A$

Similarly a coalgebra has maps $C \rightarrow C \otimes C$ satisfying comultiplication diagrams, $C \otimes \mathbb{1}$
 $C \rightarrow \mathbb{1}$

From now on $\mathcal{C} = \text{dVect}_K$; get dgas & dgc's

A coalg. is coaugmented if $C \rightarrow K$ has a section ϵ
~~coaug~~ Get a coaug. coideal \bar{C} w/ reduced comultiplication
 $\bar{\Delta}$ making \bar{C} into a noncounital coalgebra

A coaug. coalg. is conilpotent if $\forall c \exists p \in \mathbb{N}$ st $\sum P(c) = 0$
 e.g. the tensor coalg $T^c(V) = K \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$

with deconcatenation coproduct $\Delta(V_1 \dots V_n) = \sum_{i=0}^{n-1} V_1 \dots V_i \otimes V_{i+1} \dots V_n$
 is conilpotent since $\Delta^{n+1}(V^{\otimes n}) = 0$.

If A is an aug. alg, have the bar construction

~~BA~~ $BA := (T^c(\Sigma \bar{A}), d_1 + d_2)$ | mk $T^c V$ is the cofree conil. coalg on V
 $d_1 =$ usual differential on $T^c \Sigma \bar{A}$
 $d_2(a_1 \dots a_n) = \sum_{i=0}^{n-1} a_1 \dots a_i \otimes a_{i+1} \dots a_n$

Similarly there's a cobar construction

$\Omega: \text{conil. Coalg} \rightarrow \text{aug. Alg}$

$\Omega C := (T(\Sigma^{-1} \bar{C}), d_1 + d_2)$

$d_2(c_1 \dots c_n) = \sum_i c_1 \dots \bar{c}_i \dots c_n$

If C is a coalgebra then $C^\vee := \text{Hom}(C, k)$ is an algebra.

If A is an algebra then A^\vee only need be a coalgebra if A is finite-dimensional!

If A is an algebra then the Koszul dual of A is $A^! := B^\vee A$.

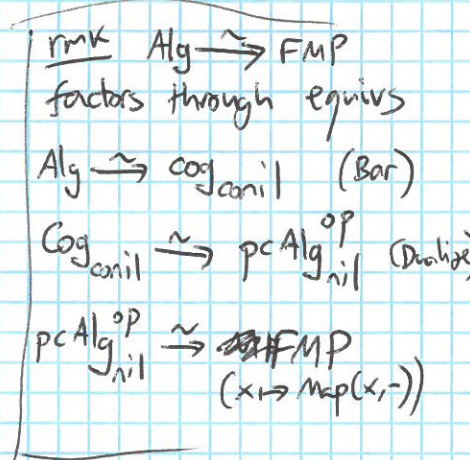
Bar & cobar are adjoints: $\Omega \rightarrow B$.

Not too hard to check that if Γ is ~~finite~~ Artinian

then $\Gamma^! \simeq \Omega(\Gamma^\vee)$, so that

$$\Psi(A)(\Gamma) = \text{Map}_{\text{Alg}}(\Gamma^!, A) \simeq \text{Map}_{\text{Alg}}(\Omega(\Gamma^\vee), A) \simeq \text{Map}_{\text{Cog}}(\Gamma^\vee, BA)$$

This is a coalgebraic object!



Moreover an algebra is pseudocompact if it's the dual of a coalgebra. There's a forgetful functor $\text{pc Alg} \rightarrow \text{Alg}$

Above formalism gives $\Psi(A)(\Gamma) \simeq \text{Map}_{\text{pc Alg}}(A^!, \Gamma)$

this is a prorepresentability statement.

Key ingredient conil. Cog & aug. Alg are model categories & $\Omega \rightarrow B$ is a Quillen equivalence.

aug. Alg : $\text{WE} = \text{quasi-iss}$ & $\text{Fib} = \text{surjections}$. (usual Hinich model structure)

conil. Cog : WEs are created by Ω & $\text{Cof} = \text{injections}$

weak equivs of coalgebras are γ -iss but converse is not true.

§3. GLOBAL KOSZUL DUALITY

There's an adjoint pair (Ω, \mathbb{B}) (extended Bar construction);
 $\Omega: \text{Coalg} \xleftarrow{\text{coalg}} \text{Alg} \xrightarrow{\mathbb{B}}$ uses cofree coalg. functor (nasty!)

Q. Can we put model str on each side making this a Quillen equivalence?

Such a result would give prorepresentability results in global deformation theory, where the small objects are all fin dim algs (not just Artinian ones)

neither \mathbb{B} nor Ω preserve q-isos so need a stronger notion of equivalence: MC-equivalence

if E is a dga have a dg category $\text{MC}_{\text{dg}}(E)$:
 • objects are MC elts of E
 • $\text{Hom}(x, y) := \text{Hom}_{E\text{-mod}}(E^x, E^y)$ } so this is a subcat of $\text{Tw}(E)$

if C a coalg & E an alg then $\text{Hom}(C, E)$ is a dga under

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

f.g

& $\text{MC Hom}(C, A) \simeq \text{Hom}(\Omega C, A) \simeq \text{Hom}(C, \mathbb{B}A)$

A map $A \rightarrow A'$ of algs is an MC equivalence

if $\forall C, \text{MC}_{\text{dg}} \text{Hom}(C, A^{\otimes}) \rightarrow \text{MC}_{\text{dg}} \text{Hom}(C, A')$ is a quasi-equivalence.

Similar defn for coalgs.

Cofibrations of coalgs are maps that $\mathcal{U}C_{dg} \text{Hom}(-, A)$ sends to fibrations of dg categories.

Similarly for fibrations of algebras.

Rmk There's a notion of strong homotopy between (co)alg maps.

Let D^n be a set be standard simplicial model of n -disc with 2 vertices, 2 edges, ... one n -cell & $I_n := \text{Ch}(D^n) \in \text{Cog}$.

Then I_n is an interval object for a notion of n -homotopy.

$n=1$: usual chain htpy \rightarrow

$n=2$: something weird

$n \geq 3$: notions are all equivalent.



Similar notion of n -homotopy for coalgebras using

$I_n^v \triangleq$ cochains on D^n

Fact $Ob \text{H}^0 \mathcal{U}C_{dg} \text{Hom}(C, A) \cong \text{Hom}(\mathcal{U}C, A) \cong \text{Hom}(C, \check{B}A)$
(3-htpy) (3-htpy)

Also = notion of ∞ -htpy defined using standard cellular model for S^∞ . It's equivalent to 3-htpy.

Technically a bit trickier since one has to think of I_∞^v as a pseudocompact algebra, use $\hat{\otimes}$, etc.