GLOBAL CFT

§1. The function field analogy.

WARNING: this is no more than an analogy!
There are (as of yet) no theorems/formalisation to view the analogy as a special case of a 'global' statement.

Basic idea

$\mathbb{Z}$ looks a lot like $\mathbb{F}_p[x]$: they're PIDs with infinitely many primes & finitely many units.
So field extensions of $\mathbb{Q} = \mathrm{Frac}\,\mathbb{Z}$ should behave like field extensions of $\mathbb{F}_p(x) = \mathrm{Frac}\,\mathbb{F}_p[x]$.
Fin dim extensions of $\mathbb{Q}$ are number fields.
Fin dim extensions of $\mathbb{F}_p(x)$ are function fields of curves (recall $K(X) = \mathrm{Frac}\,\Gamma(X)$)

Analogy goes much deeper:
eg. • number fields have a genus, like curves
• curves have Weil zeta functions, like number fields have Dedekind zeta functions (eg. Riemann zeta!)

Aside: one possible attempt to formalise this is using $\mathbb{F}_1$, the 'field with one element'. Number fields then become function fields of curves over $\mathbb{F}_1$ & one ideally can eg. prove RH by doing intersection theory on the 'surface' $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$.

Instead of working over $\mathbb{F}_p$, one can try to look at curves over $\mathbb{C}$ instead & use complex geometry.

Dictionary

Number fields	Arithmetic curves	Complex curves
$\mathbb{Z}$	$\mathbb{F}_p[x]$	$\mathcal{O}_{\mathbb{C}}$ (holomorphic functions)
$\mathbb{Q}$	$\mathbb{F}_p(x)$	meromorphic functions
p prime	$y \in \mathbb{F}_p$	$y \in \mathbb{C}$
Spec $\mathbb{Z}$	$\mathbb{A}^1_{\mathbb{F}_p}$	$\mathbb{A}^1_{\mathbb{C}}$
K number field	function field of curve C	function field of curve C
$\mathcal{O}_K$	$\mathcal{O}_C$	$\mathcal{O}_C$
~~Spec $\mathcal{O}_K \to$ Spec $\mathbb{Z}$~~	—	

<u>Further remarks</u> The place at ∞ of $\mathbb{Z}$ is like the point at ∞ of $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ & the morphism Spec $\mathcal{O}_K \to$ Spec $\mathbb{Z}$ is like Spec $\mathcal{O}_C \to \mathbb{CP}^1$ given by the Riemann existence theorem: every complex curve is a branched cover of the Riemann sphere.
(So rings of integers of n° fields are like branched covers of $\mathbb{Z}$, & ramification in the number-theoretic sense is like geometric ramification)

How to ~~generalise~~ interpret CFT in the setting of complex curves?

§2. Local systems

recall that Artin reciprocity says that for an abelian extension $\mathbb{Q} \to K$, there's an isomorphism

$$\mathrm{Frob}: \underbrace{\{\text{ray class gp. of } K\}}_{\text{congruence conditions}} \xrightarrow{\sim} \underbrace{\mathrm{Gal}(K/\mathbb{Q})}_{\substack{\text{splitting behavior} \\ \text{of primes}}}$$

Let's start by focusing on the RHS.

Schemes have étale fundamental groups $\pi_1^{\acute{e}t} X$, defined via étale covers.

If K is a (number) field then $\pi_1^{\acute{e}t} \operatorname{Spec} K \simeq \mathrm{Gal}(K^{sep}/K)$ the absolute Galois group of K.

[Rmk for reasonable X/K there's an exact seq.

$$1 \to \pi_1^{\acute{e}t}(X^{sep}) \to \pi_1^{\acute{e}t}(X) \to \mathrm{Gal}(K^{sep}/K) \to 1 \;]$$

If X is a variety over $\mathbb{C}$, one can regard it as a topological space using the analytic topology & compute its usual fundamental group $\pi_1 X$.

Then $\pi_1^{\acute{e}t} X$ is the profinite completion of $\pi_1 X$.

<u>Takeaway</u> across the FF analogy, Galois groups behave like (analytic) fundamental groups.

If Σ is a Riemann surface, a <u>local system</u> on Σ is a locally constant (in analytic topology) sheaf of vector spaces L

<u>Fact</u> ~~the~~ dim(fibres of L) is constant; call this the <u>rank</u> of L.

<u>Prop</u> $\left\{ \begin{array}{c} \text{reps of } \pi_1 X \\ \text{on } GL_n \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{rank } n \text{ local} \\ \text{systems} \end{array} \right\}$

<u>Proof idea</u>

if L is a local system with fibre V, fix a point $x \in X$. Given a loop γ based at x we get a local system on $[0,1]$, which is constant, & hence an isomorphism $V \simeq V_0 \simeq V_1 \simeq V$. This gives, after picking a local trivialisation, an element $\rho\gamma \in GL(V)$.

Local systems on X of rank n fit together into a stack $\mathrm{Loc}_n X$, the character variety of maps $\pi_1 X \to GL_n$

A module over this stack should be something like a representation of $\pi_1 X$.

§3. Bundles

Now let's think about how to encode ray class groups geometrically.

If Σ is a curve, a rank n ~~vec~~ bundle is a holomorphic complex vector bundle of rank n.

Line bundles are rank 1 bundles

There's a classifying stack $Bun_n \Sigma$ which classifies rank n bundles.

For $n=1$, $Bun_1 \Sigma \simeq Pic\, \Sigma$ the Picard stack.

Black box

The geometric analogue of representations of the ray class group on GL_n should be the category of D-modules on Bun_n

Can be made more precise using adeles: both are loosely double coset spaces. (Weil uniformisation theorem)
End up with something like ℓ-adic perverse sheaves on a curve/$\mathbb{F}_p$, and perverse sheaves on a ~~Riemann surf~~ smooth complex variety are the same as D-modules by the Riemann-Hilbert correspondence.

A D-module is a device which encodes differential equations on a variety.

More precisely, there's a sheaf of differential operators Diff X & a D-Module is a module over this algebra.

Locally, $\mathrm{Diff}\,\mathbb{C}^n \simeq \mathbb{C}\langle x_1 - x_n, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_n} \rangle$ the nth Weyl algebra.

So D-modules on $\mathbb{C}^n$ are the same as modules over ~~Diff~~ W_n.

§4 Geometric CFT

Th^m (Laumon, Rothstein) if Σ is a ~~Riemann~~ ~~surface~~ complex curve then there's an equivalence

$$\mathrm{Mod}(\mathrm{Loc}_1 \Sigma) \simeq \mathrm{DMod}(\mathrm{Pic}\,\Sigma)$$

of derived categories.

Even better, this takes skyscraper sheaves on the LHS (ie. local systems on Σ) to "Hecke eigensheaves"

More generally we have

$$\mathrm{Mod}(\mathrm{Loc}_n \Sigma) \simeq \mathrm{DMod}(\mathrm{Bun}_n \Sigma)$$

this is "geometric Langlands for GL_n"