Khovanov homology via derived categories of coherent sheaves

(following Cautis & Kamnitzer)

Loose idea: repeat the symplectic construction on the B side.

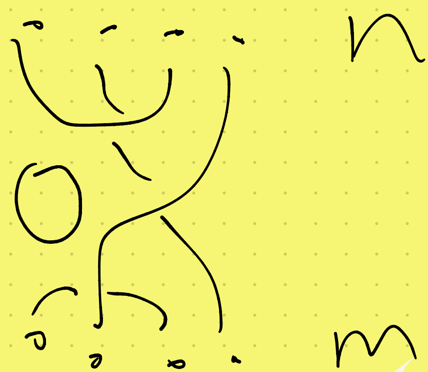
(mostly sl_2, some sl_n later)

Reshetikhin-Turaev invariants

V = standard rep / $U_q(sl_2)$

RT-invariants

$$\left\{\begin{matrix}(n,m)\\ \text{tangles}\end{matrix}\right\}$$



$$\Big\downarrow \Psi$$

$$\mathrm{Hom}_{U_q(sl_2)}\left(V^{\otimes n}, V^{\otimes m}\right)$$

recipe: project onto plane

flow down the diagram

caps $\mathbb{C}[q, q^{-1}] \to V \otimes V$

cups $V \otimes V \to \mathbb{C}[q, q^{-1}]$

braiding $V \otimes V \to V \otimes V$ 

defined using the R-matrix

RT prove that this doesn't depend on the planar projection

Links are (0,0)-tangle

So QT give you a link invt

$$\mathbb{C}[q,q^{-1}] \to \mathbb{C}[q,q^{-1}]$$
$$1 \longmapsto \text{Jones Polynomial}$$

Khovanov: categorify this

a weak categorification is a collection D_n of graded triangulated categories & a map

$$\left\{ \begin{matrix} (n,m) \\ \text{tangles} \end{matrix} \right\} \xrightarrow{\psi} \left\{ \begin{matrix} \text{isoclasses of exact} \\ \text{graded functors } D_n \to D_m \end{matrix} \right\}$$

$$D \xrightarrow{\{1\}} D$$

Satisfying

- $D_0 = D(\mathrm{grVect}_{\mathbb{C}})$

- $K_0(D_n) \cong V^{\otimes n}$ as $\mathbb{C}[q, q^{-1}]$-modules

- $K_0(\Psi(T)) = \psi(T)$ as morphisms $V^{\otimes n} \to V^{\otimes m}$

Link invts

T a link, get

$\Psi(T) : D(\mathrm{grVect}_{\mathbb{I}}) \circlearrowleft$

get a bigraded v.s.

$$H^{i,j}(T) := H^{i,j}(\Psi(T)(\mathbb{I}))$$

whose graded χ is

the Jones polynomials

Bernstein-Frenkel-Khovanov, Stroppel:

$\exists$ a categorification with D_n built out of 'parabolic category $\mathcal{O}$' for gl_n which recovers **Khovanov homology**

Today

want something geometric

Take D_n to be

$$D^b(\mathrm{Coh}_{\mathbb{C}^\times}(Y_n))$$

smooth proj. variety with a $\mathbb{C}^\times$-action

Given a tangle, we need functors

$$D(Y_n) \longrightarrow D(Y_m)$$

we'll get these as

Fourier-Mukai transforms

Main theorems

1) Ψ is a tangle inv^t

2) $K_0 \Psi = \psi$

i.e. Ψ is a weak categorification

3) Ψ is functorial w.r.t tangle cobordisms

4) for links,

$H^{i,j} \simeq Kh^{i+j,j}$

Building the Y_n

for $N >> 0$, put $N \geq 2n$

$\mathbb{C}^{2N} = \operatorname{span}\{e_1 - e_N, f_1 - f_N\}$

& let $z : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$

be $zf_1 = 0 = ze_1$

$$\left.\begin{aligned} ze_i &= e_{i-1} \\ zf_i &= f_{i-1} \end{aligned}\right\} i > 0$$

'nilpotent of Jordan type (N,N)'

Let Y_n be

$$\left\{ L_1 \subseteq \cdots L_i \subseteq L_n \;:\; zL_i \subseteq L_{i-1} \right\}$$

$\dim L_i = i$

$L_i \subseteq \mathbb{C}^{2N}$

It's a smooth proj var

$$\left.\begin{array}{c} Y_{n+1} \\ \Big\downarrow {\scriptstyle L_i \mapsto L_i} \\ Y_n \end{array}\right\} \text{ is a } \mathbb{P}^1 \text{ bundle. fibre is } \left(z^{-1}(L_n)/L_n\right)/\mathbb{C}^\times$$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^{2N}$

via

$$t \cdot e_i = t^{-2i} e_i$$

$$t \cdot f_i = t^{-2i} f_i$$

2 avoids having to half things later

Induces $\mathbb{C}^\times \curvearrowright Y_n$ by

$$t(L_1 - L_n) = (t \cdot L_1 - t L_n)$$

A subvariety

$$X_n^i = \left\{ L_1 - L_n \in Y_n : L_{i+1} = z^{-1} L_i \right\}$$

It's a $\mathbb{C}^\times$-equivt subvariety of Y_n

&

$$\left.\begin{array}{c} X_n^i \\ \Big\downarrow q : \text{replace } L_i, L_{i+1} \text{ by } zL_{i+2} \\ Y_{n-2} \end{array}\right\} \text{is an equivt } \mathbb{P}^1 \text{ bundle}$$

Vector bundles

there's a k-dim VB L_k on Y_n with fibres L_k

Put $\mathcal{E}_k = L_k / L_{k-1}$

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the X_n^i & $\mathcal{E}_i$ will be useful later when we define our Fourier Mukai transforms

Aside

non-equivariantly,

$Y_n \simeq (\mathbb{P}^1)^{\times n}$

via

choose lines M_i st

$L_k = M_1 \oplus \cdots \oplus M_k$ with

$L_{k-1} \perp M_k$

then the map sends

$L_1 \cdots L_n \longmapsto (CM_1 \cdots CM_n)$

$C: \mathbb{C}^{2N} \longrightarrow \mathbb{C}^2$

$e_i \longmapsto e$

$f_i \longmapsto f$

Fourier-Mukai transforms

X, Y, smooth projective $\mathbb{C}^\times$-varieties

$D(X) = D^b(\mathrm{Coh}_{\mathbb{C}^\times} X)$

$K \in D(X \times Y)$

Kernels

We get

$\Phi_K : D(X) \to D(Y)$

$\mathcal{F} \longmapsto \pi_*(\pi^*(\mathcal{F}) \otimes K)$

Fourier-Mukai transform

Examples

$\mathcal{O}_\Delta \in D(X \times X)$

gives id

given $f : X \to Y$

I get $\Gamma_f \in D(X \times Y)$

$\phi_{\Gamma_f} : D(X) \to D(Y)$ is f_*

$\phi_{\Gamma_f} : D(Y) \to D(X)$ is f^*

Aside

π_* is like $\int dx$

π^* is like
$f(x) \mapsto \big((x,y) \mapsto f(x)\big)$

$\otimes K$ is like multiplying by $K(x,y)$

So Φ_K is like the integral transform

$f \mapsto \left(y \mapsto \int K(x,y) f(x)\, dx\right)$

Properties

- composition of FM transforms is FM

proof uses a big diagram

$$\begin{tikzcd}
 & XYZ \arrow[dl] \arrow[d] \arrow[dr] & \\
XY \arrow[d] \arrow[dr] & XZ \arrow[dl] \arrow[dr] & YZ \arrow[dl] \arrow[d] \\
X & Y & Z
\end{tikzcd}$$

- adjoints of FM transforms are FM

proof uses Grothendieck duality $\pi_* \dashv \pi^!$

$$\pi^! \mathcal{F} = \pi^* \mathcal{F} \otimes \omega_\pi [\dim \pi]$$

Th^{m} (Orlov)

any equivalence between $D^b(\mathrm{Coh}(\text{smooth proj}))$ is a FM functor

Our functors

We need:

- caps

$$D(Y_{n-2}) \xrightarrow{G^i} D(Y_n)$$

- cups

$$D(Y_n) \xrightarrow{F^i} D(Y_{n-2})$$

- crossings

$$D(Y_n) \longrightarrow D(Y_n)$$

Caps

recall $X_n^i \subseteq Y_n$

$$\begin{array}{c} X_n^i \\ \downarrow \\ Y_{n-2} \end{array}$$

$\mathcal{G}^i$ is the sheaf

$$\mathcal{O}_{X_n^i} \otimes \pi_2^* \mathcal{E}_i \{1-i\}$$

& $G_i = \Phi_{\mathcal{G}^i}$

grading shift

Cups are similar & use $\mathcal{E}^{\vee}_{i+1}$

Crossings are more involved (4 types!) & use a subvariety $Z_n \subseteq Y_n \times Y_n$

- Prove ψ is a tangle inv't by checking invariance under tangle Reidemeister moves
geometry is involved but reasonably elementary

- Check $K_0 \psi = \psi$

Proof actually explicitly identifies $K_0(Y_n)$ & identifies $\mathcal{G}^i, F^i$ etc

rmK

non-equivariantly,

$$K_0(Y_n) \simeq K_0((\mathbb{P}^1)^{\times n})$$

$$\simeq K_0(\mathbb{P}^1)^{\otimes n}$$

- Check invariance under tangle cobordism

uses an explicit combinatorial description of tangle cobordisms

- check we get Khovanov homology

check satisfies same LESs & value on unlink

Mirror Symmetry

$\mathbb{C}^{2n} \subseteq \mathbb{C}^{2N}$ is the span of $e_1 - e_n, f_1 - f_n$

$F_n = \{ L_1 - L_{2n} \in Y_{2n} : L_{2n} = \mathbb{C}^{2n} \}$

F_n is the Springer fibre of $z|_{\mathbb{C}^{2n}}$

Let $P: \mathbb{C}^{2N} \twoheadrightarrow \mathbb{C}^{2n}$ be the projection

Put $U_n = \{ L_1 - L_{2n} : P(L_{2n}) = \mathbb{C}^{2n} \}$

Then U_n is an open nbhd of $F_n \subseteq Y_{2n}$

$N \geq 2n$

$$F_n \subseteq U_n \subseteq Y_{2n}$$

$$\simeq \widetilde{S_n \cap N} \simeq M_n$$

let $S_n \subseteq SL_{2n}\mathbb{C}$ be the matrices

$$\begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & 0 & I & \\ & & & \ddots & \\ * & * & \cdots & * & * \end{pmatrix}$$

this guy appears in the Seidel-Smith story

$$\mathcal{S}_n \cap \mathcal{N} := \{ x \in \mathcal{S}_n : x \text{ nilp} \}$$

$$\widetilde{\mathcal{S}_n \cap \mathcal{N}}$$ Grothendieck–Springer rsln

observation (Lusztig)

$\exists$ an iso

$$U_n \xrightarrow{\sim} \widetilde{\mathcal{S}_n \cap \mathcal{N}}$$

restricting to $\mathrm{id}_{\mathcal{F}_n}$

Recall

The Seidel-Smith construction defined using $\mathcal{M}_n$ which is diffeomorphic to $\widetilde{S_n \cap N}$ but has a different complex structure

Mirror symmetry suggests

$\mathrm{Fuk}(M_n) \simeq D^b\mathrm{Coh}(U_n)$

or something similar

Seidel-Smith invt:

$Br_{2n} \circlearrowright \mathrm{Fuk}(M_n)$ &

the invt is

some Lagrangian

$HF(L, \beta L)$

for $\overline{\beta}$ = our link

Can check

our unit is

$$\mathrm{Ext}_{D(Y_{2n})}(L, \beta L)$$

$$= \mathrm{Ext}_{D(U_n)}(L, \beta L)$$

where L is $\mathcal{O}$ of a component of F_n

(up to a twist)

How to do sl_n, $n>2$?

the geometric Satake correspondence

$\mathfrak{g}$ a ss Lie algebra

Label tangle strands by dominant weights of $\mathfrak{g}$

Get Reshetikhin-Turaev invts

$$\{(\underline{\lambda}, \underline{\mu})\text{-tangles}\} \to \mathrm{Hom}_{U_q(\mathfrak{g})}(V_{\underline{\lambda}}, V_{\underline{\mu}})$$

irrep w/ highest wt λ

sps all λ_i minuscule

$G^\vee$ = Langlands dual to G

$$Gr = \frac{G^\vee(\mathbb{C}((z)))}{G^\vee(\mathbb{C}[[z]])}$$

affine Grassmannian

its an ind-scheme $/\mathbb{C}$

stratified by orbits

$\{Gr_\lambda : \lambda \in \Lambda_+\}$

$G^\vee(\mathbb{C}((z))) \curvearrowright Gr \times Gr$

Orbits are also labelled by Λ_+

have a convolution product

$Gr_{\underline{\lambda}} = \{ L_1 - L_n \in Gr^{\times n} :$

(L_i, L_{i+1}) is in

orbit indexed by $\lambda_{i+1} \}$

have $Gr_{\underline{\lambda}} \xrightarrow{m} Gr$

$L_1 - L_n \mapsto L_n$

Th^m (geometric Satake)

{perverse sheaves on Gr} is equivalent to Rep($\mathfrak{g}$)

Under this equivalence,

$$IC(Gr_\lambda) \mapsto V_\lambda$$

more generally,

$$m_* IC(Gr_{\underline{\lambda}}) \mapsto V_{\underline{\lambda}}$$

w/ $V_{\underline{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$

the λ_i are minuscule

$\Rightarrow$ $Gr_{\underline{\lambda}}$ is smooth

Idea

Put

$$D_{\underline{\lambda}} = D^b(Coh_{\mathbb{C}^\times}(Gr_{\underline{\lambda}}))$$

some natural correspondences give you the FM transforms