Topological Hochschild Cohomology for schemes

LAGOON

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Plan

1) Recollections on HH^* incl. deformation theory

2) Spectral algebra and THH^*

3) THH^* for schemes & maybe other objects

1) Recollections on HH^*

- K a base comm. ring, A a K-algebra
- The enveloping algebra is $A^e := A \otimes_K A^{op}$

then: $A\text{-bimod} \simeq A^e\text{-mod}$

- The Hochschild cohomology is

$$HH_K^*(A) := \mathrm{Ext}_{A^e}^*(A, A)$$

Remark: can compute HH^* using the bar complex

$$\begin{array}{c} A \\ \Big\downarrow {\scriptstyle a \mapsto [-,a]} \\ \mathrm{Hom}_k(A,A) \\ \Big\downarrow {\scriptstyle f \mapsto \left\{ a\otimes b \mapsto af(b) - f(ab) + f(a)b \right\}} \\ \mathrm{Hom}_k(A \otimes_k A, A) \\ \Big\downarrow \\ \vdots \end{array}$$

Deformation theory

A square-zero extension of A is a surjection

$\pi : \tilde{A} \twoheadrightarrow A$ with

$\ker(\pi)^2 = 0$

Example $A = K$

$\tilde{A} = K[\varepsilon]/\varepsilon^2$ dual numbers

$\ker(\pi) = (\varepsilon) \subseteq \tilde{A}$

Example

$\mathbb{Z}/p^2 \xrightarrow[\text{mod } p]{} \mathbb{Z}/p$

- A square-zero ext^n is **K-split** if

$$0 \to \ker \pi \to \tilde{A} \to A \to 0$$

is a split s.e.s. of K-modules

<u>Ex:</u> $K[\varepsilon]/\varepsilon^2 \to K$ is K-split

<u>Ex:</u> $\mathbb{Z}/p^2 \to \mathbb{Z}/p$ is **not** $\mathbb{Z}/p$-split

(or $\mathbb{Z}$-split)

<u>Th</u>

$\underline{\text{Th}^{m}}$ $\exists$ a bijection.

$$HH^2_K(A) \simeq \left\{ \begin{array}{l} K\text{-split} \\ \text{extensions of } A \\ \text{by } A \text{ up to} \\ \text{iso.} \end{array} \right\}$$

- Higher HH^* groups have interpretations

- what about <u>all</u> extensions?

- There's a 'non-linear' version of Hochschild cohom. called Mac Lane cohomology & one has a bijection

$$HML^2(A) \simeq \left\{ \begin{array}{l} \text{square-zero} \\ \text{extensions} \\ \text{of } A \\ \text{by } A \text{ up to} \\ \text{iso} \end{array} \right\}.$$

- Higher HML^* have similar interpretations

Topology:

- **Th^m** (B.-Kaledin-Lowen)

for A a ring,

$$HML^*(A) \simeq THH^*(A)$$

- On homology:

Pirashvili-Waldhausen '92

Horel-Ramzi '20

2) Spectral algebra

- Loosely: a spectrum is like a topological space but stabilised
- Alternately: a spectrum is like a chain complex but nonlinear

Spectra : higher algebra

Abelian groups : algebra

- **Def**: a spectrum is a sequence of spaces X_i, with structure maps

$$\Sigma X_i \to X_{i+1}$$

- Spectra have homotopy groups

$$\pi_* X := \varinjlim_r \pi_{*+r} X_r$$

Example X a space

Get a spectrum $\Sigma^\infty X$

with $(\Sigma^\infty X)_i = \Sigma^i X$

& $\pi_* \Sigma^\infty X = \pi_*^s X$

stable homotopy groups

Sub-example

$\mathbb{S} = \Sigma^\infty S^0$

sphere spectrum

$\mathbb{S}_i \simeq S^i$

$\pi_* \mathbb{S}$ are hopeless

Example

A a chain complex of abelian groups.

$\exists$ an Eilenberg-Mac Lane spectrum HA

with

$$\pi_* HA \simeq H_* A$$

Products

- The category of spectra admits a symmetric monoidal product, the smash product $\wedge$!

- a ring spectrum is a monoid for $\wedge$

- Can talk about modules over ring spectra

Example

$S^n \wedge S^m \simeq S^{n+m}$

& so $\mathbb{S} \wedge \mathbb{S} \simeq \mathbb{S}$

Turns $\mathbb{S}$ into a ring spectrum

Facts

- $\mathbb{S}$ is the initial ring spectrum (cf. $\mathbb{Z}$)
- $\mathbb{S}$-modules = spectra
- $\mathbb{S}$ is the unit for $\wedge$

Example

A a ring (or dga)

HA is a ring spectrum

- HA-mod $\sim$ dg A-modules

Fact

X a space

$$H^*(X, A) \simeq [\Sigma^\infty X, HA]$$

cup products $\sim$ algebra structure on HA

Now we have rings & modules, we can do homological algebra.

- A a ring spectrum

$\Rightarrow \exists!\ \mathbb{S} \to A$

- topological enveloping alg.

is

$$A^{te} := A \wedge_{\mathbb{S}} A^{op}$$

Examples

$\mathbb{Q} \wedge_{\mathbb{S}} \mathbb{Q} \simeq \mathbb{Q}$

so if A is a $\mathbb{Q}$-alg,

$A \wedge_{\mathbb{S}} A^{op} \simeq A \otimes_{\mathbb{Q}} A^{op}$

But

$\mathbb{F}_p \wedge_{\mathbb{S}} \mathbb{F}_p$ is highly nontrivial

homotopy is dual Steenrod algebra

- A a ring. Its **topological Hochschild cohomology** is

$$THH^*(A) := \mathrm{Ext}_{A^{te}}(A, A)$$

- if $A \in \mathbb{Q}$-alg,

$$THH^*(A) \simeq HH^*_{\mathbb{Q}}(A)$$

Rmk Can talk about a relative version: if R commutative & A is an R-algebra then have $THH^*_R(A)$

Examples

- $THH^* \simeq THH^*_{\mathbb{S}}$

- k a c. ring, A a flat k-algebra:

$THH^*_k(A) \simeq HH^*_k(A)$

- THH^* is an interesting arithmetic invariant:

$$THH^*(\mathbb{F}_p) \simeq \mathbb{F}_p[u]$$

("Bökstedt periodicity")

- True more generally for perfect fields in char p.

3) THH^* for schemes

All schemes are noetherian & separated over a field K (e.g. varieties)

- <u>Th^m</u> (Lowen-Van den Bergh)

$\mathcal{A}$ dg category, $\mathcal{A} \hookrightarrow \text{mod}\,\mathcal{A}$

if $\mathcal{A} \hookrightarrow \mathcal{B} \hookrightarrow \text{mod}\,\mathcal{A}$ with all objs of $\mathcal{B}$ cofibrant

then $HH^*\mathcal{B} \xrightarrow{\sim} HH^*\mathcal{A}$

- **Cor**

$$
\begin{aligned}
HH^*(X) &\simeq HH^*(\mathrm{per}\,X) \\
&\simeq HH^*(D^b \mathrm{Coh}\,X) \\
&\simeq HH^*(D\,\mathrm{QCoh}\,X) \\
&\simeq \cdots
\end{aligned}
$$

Slogan: One can compute $HH^*(X)$ as the HH^* of any reasonable derived category of sheaves on X

Proof Use the bar complex for HH^* to get limited functoriality w.r.t. fully faithful morphisms of dg categories (Keller, uses gluings)

Then check that certain functors (viewed as bimodules) induce equivalences on HH^*

- **Thm** (B.–Kaledin–Lowen)

A, B as above. Then:

$$THH^*B \xrightarrow{\sim} THH^*A$$

- **Cor**

$$THH^*(\mathrm{per}X) \simeq THH^*(D^b\mathrm{Coh}X) \simeq THH^*(D\mathrm{QCoh}X) \simeq \dots$$

- Gives a 'noncommutative' definition of $THH^*(X)$

Proof idea

- Prove analogues of Lowen-Van den Bergh's results in the setting of spectral categories (uses bar complex for THH^*)

- Show that the bimodules inducing HH^*-equivalences also induce THH^*-equivalences

A sample computation

k a perfect field, char p

(e.g. k finite or $k=\bar{k}$)

<u>then</u> $THH^*(\mathbb{P}^1_k)$ is

a commutative k-alg.

generated by

a, b, c , u

generators of $HH^1_k \mathbb{P}^1_k \simeq H^0(\mathbb{P}^1, T)$ Bökstedt el^t

deg 1 deg 2

subject to $xy=0$

for $x, y \in \{a, b, c\}$

- Where does this come from?

Thm (B.-Kaledin-Lowen): if X smooth proper$/k$ $\exists$ a multiplicative spectral sequence

$$HH^{\ell}_{k}(X) \otimes_k THH^{q}(k) \Rightarrow THH^{\ell+q}(X)$$

This degenerates when

- $\dim X = 1$
- $\dim X = 2$ & $p > 2$

Higher structure $k = \mathbb{F}_p$

- THH^* is always commutative
- It has a Browder bracket (Gerstenhaber bracket)
- It admits E_2-Dyer-Lashof operations

$\left(p=2: \ Q_1 : THH^i \to THH^{2i-1} \right.$
satisfying compatibilities $\left. \right)$

Unanswered questions

- It's unclear in what sense we have

$$\mathrm{THH}^2 X \simeq \left\{ \begin{array}{c} \text{nonlinear} \\ \text{deformations} \\ \text{of } X \end{array} \right\}$$

when X is non-affine

- e.g. $\mathbb{P}^1_{\mathbb{Z}/p^2}$ ought to show up as an element of $\mathrm{THH}^2(\mathbb{P}^1_{\mathbb{Z}/p}) \simeq \mathbb{Z}/p$

- Loose idea:
$THH^2 X$ parameterises non-additive deformations of $\mathrm{Coh}\, X$

- Precise relationship between THH^* & deformations of abelian categories is work in progress.

- When A is a connective dga with bounded cohomology we have

$$\mathrm{THH}^2 A \simeq \left\{ \begin{array}{c} \text{first-order} \\ \text{deformations} \\ \text{of } A \end{array} \right\}$$

- What about connective dg categories or connective spectral categories?

Thanks
for
listing!