TC seminar

Matt Booth

December 10, 2025

These notes are available at `http://mattbooth.info/TCseminar.pdf`. They are currently incomplete, so please don't share too widely!

Contents

1 Introduction

1.1 HC and trace methods

To be filled in later...

1.2 From HC to TC

If A is a $\mathbb{Q}$-algebra then trace methods give us a good handle on the rational K-theory of A. What about at other primes? Relatedly - what happens if we try to run the Hochschild, cyclic etc. constructions with a ring which is not flat over $\mathbb{Z}$?

Remark 1.1. The **arithmetic fracture square** says that if X is a complex of $\mathbb{Z}$-modules, then there is a homotopy Cartesian square

$$\begin{array}{ccc} X & \longrightarrow & X_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ \prod_p \hat{X}_p & \longrightarrow & \left(\prod_p \hat{X}_p\right)_{\mathbb{Q}} \end{array}$$

In other words, X can be recovered from its rationalisation $X_{\mathbb{Q}}$ along with all its derived p-completions $\hat{X}_p$ at all primes, along with some extra gluing information. Given this we will often focus on the p-complete information and leave the gluing until later.

One answer is to use **spectra** instead of chain complexes of abelian groups (a.k.a. dg abelian groups.) A classical way to think is to view a spectrum as a kind of stabilised topological space - i.e. a simplified version of a topological space where we have only remembered the information that is 'stable' under applications of the suspension functor.

A more modern way to think is to view a spectrum as a 'derived analogue' of an abelian group. Indeed, there's a notion of **ring spectrum** - the spectral analogue of a dg algebra - and the sphere spectrum $\mathbb{S}$ is the initial ring spectrum. In fact, spectra are no more than modules over the 'ring' $\mathbb{S}$.

One then tries to perform the above constructions in the world of spectra and see what comes out. In particular one constructs a **topological Hochschild homology** spectrum THH, morally by replacing the $\mathbb{Z}$s in the definition of HH by $\mathbb{S}$s. One then notices that THH has some special S^1-equivariant structure, namely that of a **cyclotomic spectrum**. Loosely, a cyclotomic spectrum is a spectrum X with an S^1-action together with various S^1-equivariant fixed point maps $X^{C_n} \to X$ that commute with the restriction maps between the C_n. One then uses this cyclotomic structure on THH to construct the **topological cyclic homology** spectrum TC. Finally one constructs a trace map and proves the **Dundas–Goodwillie–McCarthy theorem**, the desired 'topological' version of Goodwillie's theorem.

Warning 1.2. TC is not simply the homotopy orbits/fixed points/Tate construction of an S^1-action on THH. More accurately, TC is to be thought of as a fancier version of TC^-, constructed using the cyclotomic structure on THH. The historically first definition goes through the intermediate **topological restriction homology** TR, which is supposed to behave like the de Rham–Witt complex (very roughly, this is given by taking the Witt vectors on the de Rham complex, and is the thing that computes crystalline cohomology). Roughly, TR is constructed as the inverse limit of the fixed point maps appearing in the cyclotomic structure.

1.3 Notation and conventions

Notational warning: Complexes/spectra are denoted without ornamentation. Homology/homotopy groups will be denoted with a $*$. So for example, $\mathrm{HH}(A)$ means the Hochschild homology **complex** of a ring A, whereas $\mathrm{HH}_*(A)$ denotes the graded $\mathbb{Z}$-module given by taking its homology.

Convention: rings are associative and unital but not necessarily commutative. Zero is a natural number.

2 Classical Hochschild and cyclic theory

A good reference is [Lod98].

2.1 Simplicial objects and totalisation

A **simplicial object** in a category $\mathcal{C}$ is a collection of objects $X_{n n\in\mathbb{N}}$ together with **face maps** $d_n^0, \dots d_n^n : X_n \to X_{n-1}$ and **degeneracy maps** $s_n^0, \dots, s_n^n : X_n \to X_{n+1}$. These satisfy some identities which I don't want to write down but you can easily look up.

Proposition 2.1. *The* ***simplex category*** *is the category Δ of finite ordinals and order-preserving maps. A simplicial object in $\mathcal{C}$ is the same thing as a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$.*

If $\mathcal{C}$ is abelian and $X_\bullet$ is a simplicial object in $\mathcal{C}$, its **totalisation** (or more accurately **realisation**) $|X_\bullet|$ is a chain complex in $\mathcal{C}$: at level n it is X_n and the differential is given by the alternating sum of the d_n^i. Sometimes $X_\bullet$ will be a simplicial object in chain complexes: each level X_n is a chain complex and all of the face and degeneracy maps are chain maps. In this case $|X_\bullet|$ is naturally a double complex and I will often implicitly totalise this double complex to view $|X_\bullet|$ as a plain chain complex. The boundedness ensures that it doesn't matter whether I totalise with sums or products.

Remark 2.2. A **semi-simplicial object** is like a simplicial object but only with the face maps. When totalising we are only using the semi-simplicial structure.

2.2 Hochschild theory

Fix a base commutative ring k and let A be a flat k-algebra. Unadorned tensor products will be taken over k. The **bar resolution** is the simplicial A-bimodule $\mathrm{Bar}_\bullet(A)$ which is $A^{\otimes n+2}$ at level n. The face maps compose adjacent tensorands and the degeneracy maps insert identities. The **enveloping algebra** of A is the algebra $A^e \coloneqq A \otimes_k A^{\mathrm{op}}$; modules over A^e are the same as A-bimodules.

Proposition 2.3. *The complex* $|\mathrm{Bar}_\bullet(A)|$ *is a bimodule resolution of* A.

Proof. Clearly the totalisation is a complex of free bimodules. One can write down an explicit contracting homotopy. □

We put $\mathrm{Hoch}_\bullet(A) \coloneqq \mathrm{Bar}_\bullet(A) \otimes_{A^e} A$. Since tensor products commute with totalisation, the realisation $\mathrm{Hoch}(A) \coloneqq |\mathrm{Hoch}_\bullet(A)|$ computes the **Hochschild homology** $\mathrm{HH}(A;k) \coloneqq A \otimes^{\mathbb{L}}_{A^e} A$. We call $\mathrm{Hoch}(A)$ the **Hochschild complex** or the **cyclic bar complex** of A.

Explicitly, one has $\mathrm{Hoch}_n(A) = (A^{\otimes n+2}) \otimes_{A^e} A \cong A^{\otimes n+1}$. Since the bimodule structure on the bar complex comes from the copies of A at either end, rather than the rightmost two copies, the final face map d_n^n of the Hochschild complex wraps around and sends a tensor $a_0 \otimes \cdots \otimes a_n$ to $a_n a_0 \otimes \cdots \otimes a_{n-1}$. This is, at a fundamental level, where the later-to-become-very-important *cyclic structure* or *circle action* on the Hochschild complex comes from.

Remark 2.4. More generally, one can make the same constructions when A is a dg algebra over k. This time, the Hochschild complex is a simplicial object in complexes.

Remark 2.5 (Shukla homology). Let k be a commutative ring and let A be a dg-k-algebra. The **derived enveloping algebra** is the dg-k-algebra $A \otimes^{\mathbb{L}}_k A^{\mathrm{op}}$. One can then form the **Shukla homology** analogously to the Hochschild homology, but using the derived enveloping algebra instead. When A is flat over k then the Shukla homology agrees with the Hochschild homology and one can compute it using the bar complex. In general, one should first resolve A as a dg algebra. Hochschild and Shukla homology may differ: think about the derived vs. underived enveloping algebras of the $\mathbb{Z}$-algebra $\mathbb{Z}/p$.

2.3 Connes' B-operator

There is a **rotation operator** $T : \mathrm{Hoch}_n(A) \to \mathrm{Hoch}_n(A)$ which sends $a_0 \otimes \cdots \otimes a_n$ to $(-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$. The B-operator is defined to be the composition $B := (1-T)s_n^0(1+T+T^2+\cdots+T^n)$, which is a map $\mathrm{Hoch}_n(A) \to \mathrm{Hoch}_{n+1}(A)$. Note that the map s_n^0 sends a to $1 \otimes a$.

Lemma 2.6 (Connes)**.** *The B operator intertwines with the Hochschild differential. Moreover $B^2 = 0$ and hence the diagram* $\mathrm{Hoch}(A) \xleftarrow{B} \mathrm{Hoch}(A)[1]$ *is a double complex.*

In this business, one often refers to the Hochschild differential as b, so that the content of the previous lemma is the assertion that $B^2 = 0$ and $bB + Bb = 0$.

Definition 2.7.

1. The **cyclic complex** of A, denoted by $\mathrm{HC}(A)$, is the totalisation (via sums or products - it makes no difference) of the double complex

   $$\mathrm{Hoch}(A) \xleftarrow{B} \mathrm{Hoch}(A)[1] \xleftarrow{B} \mathrm{Hoch}(A)[2] \xleftarrow{B} \cdots$$

   The **cyclic homology** is the homology of the cyclic complex.

2. The **negative cyclic complex** of A, denoted by $\mathrm{HC}^-(A)$, is the direct product totalisation of the double complex

   $$\cdots \xleftarrow{B} \mathrm{Hoch}(A)[-2] \xleftarrow{B} \mathrm{Hoch}(A)[-1] \xleftarrow{B} \mathrm{Hoch}(A)$$

   The **negative cyclic homology** is the homology of the negative cyclic complex, with a sign flip in the index: $\mathrm{HC}_n^-(A) := H_{-n}\,\mathrm{HC}^-(A)$.

3. The **periodic cyclic complex** of A, denoted by $\mathrm{HP}(A)$, is the direct product totalisation of the double complex

   $$\cdots \xleftarrow{B} \mathrm{Hoch}(A)[-1] \xleftarrow{B} \mathrm{Hoch}(A) \xleftarrow{B} \mathrm{Hoch}(A)[1] \xleftarrow{B} \mathrm{Hoch}(A)[2] \xleftarrow{B} \cdots$$

   The **periodic cyclic homology** is the homology of the periodic cyclic complex.

Remark 2.8. The periodic cyclic homology is indeed periodic! The periodic cyclic complex looks like

$$\cdots \xleftarrow{B-b} \prod_i \mathrm{Hoch}_{2i+1}(A) \xleftarrow{B-b} \prod_i \mathrm{Hoch}_{2i}(A) \xleftarrow{B-b} \prod_i \mathrm{Hoch}_{2i+1}(A) \xleftarrow{B-b} \cdots$$

which can also be viewed as the 'unrolling' of a $\mathbb{Z}/2$-graded chain complex.

Proposition 2.9. *There is a commutative diagram of long exact sequences*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{HC}_n^-(A) & \longrightarrow & \mathrm{HP}_n(A) & \longrightarrow & \mathrm{HC}_{n-2}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \mathrm{id} & & \\
\cdots & \longrightarrow & \mathrm{HH}_n(A) & \longrightarrow & \mathrm{HC}_n(A) & \longrightarrow & \mathrm{HC}_{n-2}(A) & \longrightarrow & \cdots
\end{array}$$

The bottom row is known as Connes' ***periodicity sequence****.*

Example 2.10. Let R be a smooth commutative k-algebra, where k is a field of characteristic zero. The HKR theorem identifies $\mathrm{HH}_n(R)$ with the Kähler differentials $\Omega^n(R/k)$. The B operator gets identified with the de Rham differential. So we have

- $\mathrm{HC}_n(R) \cong \Omega^n/d\Omega^{n-1} \;\oplus\; \bigoplus_{i\geq 1} \mathrm{H}_{\mathrm{dR}}^{n-2i}(R)$
- $\mathrm{HC}_n^-(R) \cong \ker(d : \Omega^n \to \Omega^{n+1}) \;\times\; \prod_{i\geq 1} \mathrm{H}_{\mathrm{dR}}^{n+2i}(R)$
- $\mathrm{HP}^-\, n(R) \cong \prod_{i\in\mathbb{Z}} \mathrm{H}_{\mathrm{dR}}^{n+2i}(R)$

where the extra summands come from edge effects in the double complex. Note that this breaks in positive characteristic since the HKR theorem fails to hold; see [AV17] for further discussion.

2.4 The cyclic category

A **cyclic object** in a category $\mathcal{C}$ is a simplicial object $X_\bullet$ together with **cyclic operators** $\tau_n : X_n \to X_n$ satisfying some compatibilites with the face and degeneracy maps, and $\tau_n^{n+1} = \mathrm{id}$.

The **cyclic category** Λ is defined similarly to the simplex category Δ but with the addition of extra cyclic operators t_n; by construction a cyclic object in $\mathcal{C}$ is the same thing as a $\mathcal{C}$-valued presheaf on Λ. The inclusion $\Delta \hookrightarrow \Lambda$ gives the forgetful functor from cyclic to simplicial objects.

Remark 2.11. The category Λ is self-opposite.

Suppose that $X_\bullet$ is a cyclic object in an abelian category $\mathcal{C}$. We can define new operators $b : X_n \to X_{n-1}$ and $B : X_n \to X_{n+1}$ which straightforwardly generalise the Hochschild and Connes differential respectively. The B operator makes the diagram $|X_\bullet| \overset{B}{\longleftarrow} |X_\bullet|[1]$ into a double complex, so exactly as before we may define the cyclic, negative cyclic, or periodic cyclic homology of X.

Proposition 2.12. *The rotation operators T make $\mathrm{Hoch}_\bullet(A)$ into a cyclic object. The corresponding notions of homology are the usual ones.*

Remark 2.13. There is a more abstract characterisation of the homology of a cyclic object in terms of functor homology.

2.5 Mixed complexes

Fix a field k and let R be the dg algebra $k[\varepsilon]/\varepsilon^2$, where we put ε in homological degree 1. A **mixed complex** is a dg module over R; more concretely a mixed complex is a triple (V, b, B) where (V, b) is a chain complex of k-vector spaces, (V, B) is a cochain complex, and $bB + Bb = 0$.

The functor that assigns a cyclic object its corresponding negative/periodic/cyclic homology complex obviously factors through the category of mixed complexes.

Proposition 2.14. *Let M be a cyclic module and KM the corresponding mixed complex. Then:*

1. *$k \otimes^{\mathbb{L}}_R KM$ is the classical cyclic complex for M.*

2. *$\mathbb{R}\mathrm{Hom}_R(k, K(M))$ is the classical negative cyclic complex for M.*

3. *The map $k[1] \to R$ induces a functorial **norm map***

$$\nu : k[1] \otimes^{\mathbb{L}}_R - \longrightarrow \mathbb{R}\mathrm{Hom}_R(k, -)$$

Across the above quasi-isomorphisms, the norm map is identified with B. Hence $\mathrm{cone}(\nu)(KM)$ is the classical periodic cyclic complex for M.

Proof. Compute the derived functors using the resolution $\cdots \xrightarrow{\varepsilon} R \xrightarrow{\varepsilon} R$. □

2.6 The Goodwillie trace

Theorem 2.15 (Goodwillie [Goo86])**.** *There is a functorial **Jones–Goodwillie trace** $K(A) \to \mathrm{HC}^-(A;\mathbb{Z})$, which refines the classical **Hattori–Stallings trace** $K(A) \to HH(A;\mathbb{Z})$. Let $A \twoheadrightarrow B$ be a nilpotent thickening of $\mathbb{Q}$-algebras. Then the commutative square*

$$\begin{array}{ccc} K(A)_{\mathbb{Q}} & \longrightarrow & \mathrm{HC}^-(A;\mathbb{Z}) \\ \downarrow & & \downarrow \\ K(B)_{\mathbb{Q}} & \longrightarrow & \mathrm{HC}^-(B;\mathbb{Z}) \end{array}$$

is homotopy Cartesian.

The subscript $\mathbb{Q}$s denote rationalisation (of spectra...). Goodwillie's original formulation is stronger but requires one to rationalise HC^- too. When A and B are $\mathbb{Q}$-algebras, the version given above is equivalent to Goodwillie's.

As a consequence, we get a long exact sequence

$$\cdots \to K_n(A)_{\mathbb{Q}} \to \mathrm{HC}^-_n(A) \oplus K_n(B)_{\mathbb{Q}} \to \mathrm{HC}^-_n(B) \to \cdots$$

where again the subscript $\mathbb{Q}$ denotes rationalisation, this time $- \otimes_{\mathbb{Z}} \mathbb{Q}$.

3 A topological viewpoint on cyclic theory

3.1 Intuition on model categories

A good introduction to model categories is [DS95]. The point of a model structure on a category $\mathcal{C}$ is to axiomatise a notion of **resolution** in order to get a good handle on a localisation of $\mathcal{C}$. For example, one can compute morphisms in $D^b(A)$ as chain homotopy classes of morphisms between projective resolutions.

A **model category** is a complete and cocomplete category $\mathcal{C}$ equipped with three classes of morphisms:

1. Weak equivalences, denoted $\mathcal{W}$

2. Fibrations, denoted Fib

3. Cofibrations, denoted Cof

There are various axioms we will not spell out here. We call morphisms from $\mathcal{W}$ **acyclic**. An easy consequence of the axioms is that, once the class $\mathcal{W}$ is fixed, the classes Fib and Cof determine each other; cofibrations are precisely the morphisms that have the left lifting property with respect to acyclic fibrations, and similarly for fibrations. Loosely, fibrations are supposed to behave like 'relative injective resolutions' and cofibrations like 'relative projective resolutions'.

Remark 3.1. If $\mathcal{C}$ is a model category then so is $\mathcal{C}^{\mathrm{op}}$: the weak equivalences remain the same and the fibrations and cofibrations change places. In this sense, cofibrations and fibrations are formally dual: any theorem we prove about cofibrations has a counterpart for fibrations.

An object x is **fibrant** if the unique map $x \to *$ is a fibration. Similarly, x is **cofibrant** if $0 \to x$ is a cofibration. An object x is **bifibrant** if it is both fibrant and cofibrant. Every object has a weakly equivalent co/fibrant object; we call these its **co/fibrant resolutions**. They are unique up to an intrinsic notion of homotopy constructed analogously to homotopy equivalences of topological spaces.

The **homotopy category** of a model category is the category $\mathrm{Ho}(\mathcal{C})$ where the objects are the bifibrant objects of $\mathcal{C}$, and the morphisms $x \to y$ are the homotopy classes of maps $x \to y$. Alternately, one could define $\mathrm{Ho}(\mathcal{C})$ to have the same objects as $\mathcal{C}$, and the morphisms $x \to y$ are given by the homotopy classes of maps $\tilde{x} \to \tilde{y}$, where $\tilde{x} \to x$ is a cofibrant resolution and $y \to \tilde{y}$ is a fibrant resolution.

Theorem 3.2 (Quillen)**.** *There is an equivalence* $\mathrm{Ho}(\mathcal{C}) \simeq \mathcal{C}[\mathcal{W}^{-1}]$.

In particular, the homotopy category is a well-behaved presentation of the localisation of $\mathcal{C}$ at the weak equivalences, which is a priori difficult to understand (and may not even exist, for set-theoretic reasons).

Example 3.3. We list some examples. Warning: **it is typically hard to prove directly that a candidate model structure satisfies the axioms**.

1. The category **Top** has a model structure, the **Quillen–Serre model structure**, where the weak equivalences are the weak homotopy equivalences, the cofibrations are the retracts of cell attachments, and the fibrations are the **Serre fibrations**. The cofibrant objects are the retracts of CW complexes, and all objects are fibrant. The homotopy category is the usual homotopy category of CW complexes.

2. The category **Top** has a model structure, the **Strøm model structure**, where the weak equivalences are the homotopy equivalences and the co/fibrations are the Hurewicz co/fibrations. Every object is bifibrant. The homotopy category is the usual homotopy category of topological spaces.

3. The category $\mathbf{Ch}(A)$ of chain complexes of A-modules has a model structure, the **projective model structure**, where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections. Every cofibrant object is a complex of projectives, and the converse is true for a bounded above complex. All objects are fibrant. The notion of homotopy is the usual notion of chain homotopy. The homotopy category is the derived category $D(A)$.

4. The category $\mathbf{Ch}(A)$ has a model structure, the **injective model structure**, where the weak equivalences are the quasi-isomorphisms and the cofibrations are the injections. Every fibrant object is a complex of injectives, and the converse is true for a bounded below complex. All objects are cofibrant. The notion of homotopy is the usual notion of chain homotopy. The homotopy category is the derived category $D(A)$.

5. The category **sSet** has a model structure, the **Kan–Quillen model structure**, where the weak equivalences are the simplicial weak homotopy equivalences, the cofibrations are the injections, and the fibrations are the **Kan fibrations**. The fibrant objects are known as **Kan complexes**, and all objects are cofibrant.

A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is **left Quillen** if it preserves cofibrations and acyclic cofibrations. In this setting it has a **total derived functor** $\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$. There is a dual notion of right Quillen. If $F \dashv G$ is an adjunction, then F is left Quillen if and only if G is right Quillen, and in this case we call $F \dashv G$ a **Quillen adjunction**. We obtain an adjunction

$$\mathbb{L}F \dashv \mathbb{R}G$$

of total derived functors. Say that a Quillen adjunction is a **Quillen equivalence** if its total derived adjunction is an equivalence. One can give equivalent definitions that refer solely to F and G.

Example 3.4. The nerve-realisation adjunction

$$\mathbf{Top} \longleftrightarrow \mathbf{sSet}$$

is a Quillen equivalence: topological spaces have the same homotopy theory as simplicial sets.

3.2 Intuition on ∞-categories

The standard reference here is [Lur09].

Roughly, you should think of an ∞-category as a **topologically enriched category**; i.e. a collection of objects supplied with mapping spaces $\mathrm{Hom}(X, Y)$ and continuous composition and unit maps. Topological spaces in general are quite badly behaved, and one usually prefers to replace them with the more combinatorial simplicial sets. A **simplicially enriched category** is precisely a category enriched in simplicial sets. Such a category $\mathcal{C}$ has a homotopy category $\pi_0\mathcal{C}$, defined by taking π_0 of the appropriate hom-spaces. Bergner proved that the category $\mathbf{sSet}-\mathbf{Cat}$ of simplicially enriched categories has a model structure. The weak equivalences are given by the **Dwyer–Kan equivalences**; those functors $F:\mathcal{C}\to\mathcal{D}$ satisfying

1. Homotopy fully faithfulness: For every x, y, the maps $F_{xy}:\mathcal{C}(x,y)\to\mathcal{D}(Fx,Fy)$ are weak equivalences in **sSet**.
2. Homotopy essential surjectivity: The induced functor π_0F is essentially surjective. In the presence of (1), this is the same as asking that π_0F is an equivalence.

The fibrations are those maps F whose component maps F_{xy} are fibrations and π_0F is an isofibration. A simplicially enriched category is fibrant precisely when it is enriched in Kan complexes.

The 'standard' model of ∞-categories is given by **quasicategories**; these are simplicial sets satisfying a certain lifting property that e.g. allows one to fill in the diagram $x\to y\to z$ to a 2-simplex, thought of as a composition of the two displayed maps. There is no composition function: rather, there is a contractible Kan complex of compositions of composable morphisms.

A **sSet**-category can be glued together into a quasicategory via a nerve procedure, known as the **homotopy coherent nerve**. This has a left adjoint, the **ridigification** of a quasicategory into a simplicially enriched category, and these form a Quillen equivalence between simplicially enriched categories and quasicategories.

Remark 3.5. Other models are also available, e.g. complete Segal spaces.

Example 3.6. Any 1-category gives an ∞-category by viewing its homsets as discrete spaces. The corresponding quasicategory is known as the **nerve**; the n-simplices are given by the strings of n composable morphisms.

In an ∞-category, any two objects X, Y have a *space* of maps between them $\mathrm{Map}(X,Y)$. By truncating these spaces by applying π_0, one obtains an actual category; hence one can truncate ∞-categories to obtain usual 1-categories. How can we get our hands on these mapping spaces?

If $\mathcal{C}$ is a 1-category and $\mathcal{W}$ a reasonable set of morphisms in $\mathcal{C}$, then one can localise $\mathcal{C}$ to a quasicategory, the **simplicial localisation** or **hammock localisation** $\mathcal{C}_{\mathcal{W}}$. The simplices have a definition in terms of 'hammocks' between objects of $\mathcal{C}$. We have $\pi_0(\mathcal{C}_{\mathcal{W}})\simeq\mathcal{C}[\mathcal{W}^{-1}]$, the classical localisation.

From this perspective, we view model categories as *presentations* of ∞-categories. Given a model category $\mathcal{C}$, one can calculate the mapping spaces in its simplicial localisation in terms of 1-categorical data: given a fixed $x, y\in\mathcal{C}$, one takes certain co/simplicial resolutions known as **framings** $x^\bullet\simeq x$ and $y_\bullet\simeq y$, and one then has

$$\mathrm{Map}_{\mathcal{C}_{\mathcal{W}}}(x,y)\simeq\mathcal{C}(x^\bullet,y)\simeq\mathrm{diag}\mathcal{C}(x^\bullet,y_\bullet)\simeq\mathcal{C}(x,y_\bullet)$$

as long as x was cofibrant and y fibrant.

Even better, if $\mathcal{C}$ is a **simplicial model category**, i.e. a model category and a simplicially enriched category in a compatible way, then we can simply compute

$$\mathrm{Map}_{\mathcal{C}_{\mathcal{W}}}(x, y) \simeq \mathcal{C}(x, y)$$

as long as x is cofibrant and y fibrant. The easiest example of a simplicial model category is **sSet** itself.

A key example: if A is a ring then $\mathrm{Ch}(A)$ is a simplicial model category. The simplicial hom spaces are given by applying the Dold–Kan correspondence to the usual mapping complexes $\mathrm{Hom}_A(X, Y)$. In particular, we have

$$\pi_i \mathrm{Map}_{\mathcal{D}(A)}(M, N) \simeq \mathrm{Ext}^{-i}(M, N)$$

where $\mathcal{D}(A)$ denotes the ∞-categorical localisation of $\mathrm{Ch}(A)$ at the quasi-isomorphisms. The flip in the sign of i is because we change from homological to cohomological grading conventions. Note that the Dold–Kan correspondence only sees the connective cover of the morphism complexes, i.e. the parts in nonpositive cohomological degrees (or nonnegative homological degrees), so it is not immediate (at least, if we pretend that we don't know about the formula $\mathrm{Ext}^j(M, N) \simeq \mathrm{Ext}^0(M, N[j])$) how to recover the positive Ext groups in terms of ∞-categorical mapping spaces. In fact, what is really going on here is that $D(A)$ is not just an ∞-category, but a **stable** ∞-category (think: enhanced triangulated category) and hence canonically enriched in spectra. We can recover the positive Ext^i groups as π_{-i} of the corresponding **mapping spectra**. We will return to this matter later.

3.3 Orbits and fixed points

For references, see [NS17] or [Hoy15].

Let G be a topological group. Its **classifying space** is the ∞-category BG which has a single object whose endomorphisms are G. Note that this is an **∞-groupoid**, i.e. an ∞-category whose 1-morphisms are all invertible.

A **homotopy G-action** on an object X of an ∞-category $\mathcal{C}$ is an ∞-functor $J : BG \to \mathcal{C}$ which sends the unique object of BG to X. When $\mathcal{C}$ is nice enough, one can then take **homotopy orbits** X_{hG} and **homotopy fixed points** X^{hG}. They are defined to be the homotopy colimit (resp. limit) over the diagram J. When G is finite and $\mathcal{C}$ is the derived category $D(A)$ of a ring A, then X_{hG} computes $H_*(X, G)$ and X^{hG} computes $H^*(X, G)$.

We will be interested only in the case where G is a compact Lie group. In this setting there is a **norm map** $\mathrm{Nm}_G : X_{hG} \to X^{hG}$, which when G is finite lifts the usual norm map m to $\sum_g gm$. The **Tate fixed points** X^{tG} is defined to be the homotopy cofibre of the norm map, so that we have a fibre sequence

$$X_{hG} \xrightarrow{\mathrm{Nm}_G} X^{hG} \to X^{tG} \to$$

When G is finite, X^{tG} hence computes the usual Tate cohomology $\hat{H}^*(X, G)$.

Let Λ be Connes' cyclic category, and $\tilde{\Lambda}$ its ∞-groupoid completion, which comes with a functor $\Lambda \to \tilde{\Lambda}$. If $\mathrm{Psh}(\Lambda, \mathcal{C})$ denotes the category of ∞-functors

$\Lambda \to \mathcal{C}$, then the natural functor $\mathrm{Psh}(\tilde{\Lambda}, \mathcal{C}) \to \mathrm{Psh}(\Lambda, \mathcal{C})$ exhibits $\mathrm{Psh}(\tilde{\Lambda}, \mathcal{C})$ as the subcategory of $\mathrm{Psh}(\Lambda, \mathcal{C})$ on those functors which send all maps to isomorphisms. General presheaf yoga shows that $\mathrm{Psh}(\tilde{\Lambda}, \mathcal{C})$ is in fact a reflective subcategory, which yields a reflection $\mathrm{Psh}(\Lambda, \mathcal{C}) \to \mathrm{Psh}(\tilde{\Lambda}, \mathcal{C})$. Since Λ is connected, $\tilde{\Lambda} \simeq BG$, where G denotes the automorphism group of the object $[0] \in \tilde{\Lambda}$. In particular, we can conclude the existence of a functor

$$\mathrm{Psh}(\Lambda, \mathcal{C}) \to \mathrm{Psh}(BG, \mathcal{C})$$

Proposition 3.7 (Connes)**.** *$\tilde{\Lambda}$ is a $K(\mathbb{Z}, 2)$.*

This implies that G is a $K(\mathbb{Z}, 1) \simeq S^1$, and hence we obtain a functor

$$\mathrm{Psh}(\Lambda, \mathcal{C}) \to \mathrm{Psh}(BS^1, \mathcal{C})$$

which we denote by $|-|$. Slogan: *cyclic objects get an induced homotopy S^1-action.*

In particular we may define

- $\mathrm{HC}(A; k) \coloneqq |\,\mathrm{HH}(A; k)|_{hS^1}$
- $\mathrm{HC}^-(A; k) \coloneqq |\,\mathrm{HH}(A; k)|^{hS^1}$
- $\mathrm{HP}(A; k) \coloneqq |\,\mathrm{HH}(A; k)|^{tS^1}$

Theorem 3.8 (Hoyois)**.** *These agree with the usual notions of cyclic, negative cyclic, and periodic cyclic homology.*

Proof sketch. We want to prove that if M is a cyclic module, then $|M|_{hS^1}$ agrees with the traditional cyclic homology of M (and the same for HC^- and HP). Recall the definition of the dg algebra R whose modules are the mixed complexes. Its derived category $D(R)$ is equivalent to the category $\mathrm{Psh}(BS^1, D(k))$ of complexes with a homotopy S^1-action, roughly since $R \simeq H^*(S^1)$. Across this equivalence, homotopy orbits corresponds to $k \otimes_R^{\mathbb{L}} -$ and homotopy fixed points to $\mathbb{R}\mathrm{Hom}_R(k, -)$. Hoyois now shows that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Psh}(\Lambda, D(k)) & \xrightarrow{|-|} & \mathrm{Psh}(BS^1, D(k)) \\ & \searrow^{K} & \downarrow \simeq \\ & & D(R) \end{array}$$

This proves the statement for cyclic homology since we now have

$$|M|_{hS^1} \simeq k \otimes_R^{\mathbb{L}} |M| \simeq k \otimes_R^{\mathbb{L}} KM \simeq \mathrm{HC}(M)$$

Negative and periodic cyclic homology are similar.

□

4 Spectra

An early reference, but still a good one, is [Ada74]. For more modern references you could try Hatcher.

4.1 Sequential spectra

From now on, every space will be pointed. If X, Y are spaces then $[X, Y]$ denotes the set of homotopy classes of pointed maps from X to Y. Recall the **Freudenthal suspension theorem**: if X,Y are finite CW complexes then the tower

$$[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to \cdots$$

eventually stabilises (the exact point at which it stabilises depends on the connectivity of X and Y). Write $[\Sigma^\infty X, \Sigma^\infty Y]$ for the limiting value.

We have $\Sigma = - \wedge S^1$, and in particular $\Sigma^n S^m \simeq S^{n+m}$. For any space X there is a fold map $\Sigma X \to \Sigma X \vee \Sigma X$ that collapses the 'equator' to a point. This makes ΣX a **homotopy cogroup**, and hence $[\Sigma X, Z]$ is a group. Even better, $\Sigma^2 X$ is an abelian cogroup (by the Eckmann–Hilton argument) and hence $[\Sigma^X, Z]$ is an abelian group. Moreover, the transition maps in the above tower are maps of abelian groups, and hence $[\Sigma^\infty X, \Sigma^\infty Y]$ is itself an abelian group.

Definition 4.1. The **stable homotopy groups** of a space Y are the abelian groups $\pi_i^s(Y) \coloneqq [\Sigma^\infty S^i, \Sigma^\infty Y]$. Hence $\pi_i^s(Y)$ is the colimit of the diagram

$$\pi_i(Y) \to \pi_{i+1}(\Sigma Y) \to \pi_{i+2}(\Sigma^2 Y) \to \cdots$$

Remark 4.2. We have $[\Sigma X, Y] \cong [X, \Omega Y]$, and hence a natural isomorphism $\pi_{i+1}(Y) \cong \pi_i(\Omega Y)$ for all $i \geq 0$. Hence we have $\pi_i^s(Y) \cong \varinjlim_n \pi_i(\Omega^n \Sigma^n Y)$ for $i \geq 0$.

Definition 4.3. A **sequential spectrum** is an $\mathbb{N}$-indexed sequence of spaces $X_0, X_1, \ldots$ together with structure maps $\sigma_n : \Sigma X_n \to X_{n+1}$. We may equivalently write the structure maps in terms of the adjoint maps $\rho_n : X_n \to \Omega X_{n+1}$.

The **homotopy groups** of a spectrum X are the abelian groups

$$\pi_i(X) \coloneqq \varinjlim_n \pi_{i+n}(X_n)$$

The structure maps in the colimit are obtained as the composition

$$\pi_{i+n}(X_n) \xrightarrow{\Sigma} \pi_{i+n+1}(\Sigma X_n) \xrightarrow{\sigma_n} \pi_{i+n+1}(X_{n+1})$$

Warning: homotopy groups are allowed to be negative! For example, $\pi_{-1}(X)$ is the colimit $\varinjlim_n \pi_n(X_{n+1})$, which has no reason to vanish. When $i \geq 0$, we have an isomorphism $\pi_i(X) \cong \varinjlim_n \pi_i(\Omega^n X_n)$, and when $i \leq 0$, we have an isomorphism $\pi_i(X) \cong \varinjlim_n \pi_n(X_{n-i})$.

Example 4.4. If X is a space, it has an associated spectrum $\Sigma^\infty X$, its **suspension spectrum**, which has $\Sigma^n X$ at level n. The structure maps are identities. We have the important isomorphism

$$\pi_i^s(X) \cong \pi_i(\Sigma^\infty X).$$

Example 4.5. If M is an abelian group, there is an associated **Eilenberg–Mac Lane spectrum** HM. The homotopy groups $\pi_*(HM)$ are concentrated in degree zero, where they are precisely M. More generally, a chain complex X of abelian groups has an associated **generalised EM spectrum** HX. This is obtained by use of the **stable Dold-Kan correspondence**: we have functors

$$\mathrm{Ch}(A) \simeq \mathbf{Sp}(\mathrm{Ch}_{\geq 0}(A)) \simeq \mathbf{Sp}(\mathbf{sMod}\text{-}A) \xrightarrow{U} \mathbf{Sp}(\mathbf{sSet}) \simeq \mathbf{Sp}$$

whose composition we define to be H. We have $\pi_i(HX) \simeq H_i X \simeq H^{-i} X$.

Definition 4.6. A spectrum X is an **Ω-spectrum** when all of the component maps ρ_n are weak homotopy equivalences. This implies that we have weak equivalences $X_i \simeq \Omega^n X_{n+i}$ for all n, i. Such spaces are known as **infinite loop spaces**. If X is an Ω-spectrum then one has $\pi_i X \cong \pi_i X_0$ for $i \geq 0$ and $\pi_i X \cong \pi_0 X_i$ for $i \leq 0$.

Remark 4.7. A spectrum is **connective** if it has no negative homotopy groups. Up to homotopy, connective spectra are the same thing as infinite loop spaces.

Say that a map of spectra is a **stable equivalence** if it induces an isomorphism on all π_i.

Lemma 4.8. *Every spectrum has a stably equivalent Ω-spectrum.*

Proof. If X is a spectrum, let QX be the spectrum with n^{th} level $\varinjlim_i \Omega^i X_{i+n}$. By design, QX is an Ω-spectrum which comes with a natural map from X. One computes

$$\pi_j(QX) \cong \varinjlim_n \varinjlim_i \pi_{j+n}(\Omega^i X_{i+n}) \cong \varinjlim_n \varinjlim_i \pi_{j+n+i}(X_{i+n}) \cong \varinjlim_k \pi_{j+k}(X_k) \cong \pi_j X$$

and moreover the map $X \to QX$ induces this isomorphism. □

Remark 4.9. The Q functor is a fibrant replacement functor for the Bousfield–Friedlander model structure on sequential spectra. A map f is a stable equivalence if and only if Qf is a levelwise weak equivalence.

Example 4.10. The **free infinite loop space** functor sends a space Z to the space $\Omega^\infty \Sigma^\infty Z := \varinjlim_n \Omega^n \Sigma^n Z$. One can check that $Q\Sigma^\infty Z$ has $\Omega^\infty \Sigma^\infty \Sigma^n Z$ in degree n.

If X is a spectrum, its **shift** $X[k]$ has $X[k]_n = X_{k+n}$ whenever this makes sense, and $*$ otherwise. Clearly one has $\pi_i(X[k]) \cong \pi_{i-k}(X)$. One can check that $[1]$ is both left and right adjoint to $[-1]$.

Remark 4.11. $X[-1][1]$ is precisely X, whereas $X[1][-1]$ has $*$ at level 0 and agrees with X elsewhere. They are clearly stably equivalent.

We can define suspension and loop functors on spectra by applying them levelwise. Moreover one still has $\Sigma \dashv \Omega$. There are natural maps

$$\Sigma X \to X[1]$$

which in degree n is σ_n and

$$X[-1] \to \Omega X$$

which in degree n is ρ_{n-1}.

Proposition 4.12. *The natural map $\Sigma X \to X[1]$ is a stable equivalence. If X is an Ω-spectrum, then the natural map $X[-1] \to \Omega X$ is a stable equivalence.*

Proof. The first statement follows from the second by a Yoneda argument. One can prove the latter directly. □

4.2 The stable model structure

From now on, instead of spectra in topological spaces, we work with spectra in simplicial sets instead.

Theorem 4.13 (Bousfield–Friedlander)**.** *The category of sequential spectra admits a model structure, with weak equivalences the stable equivalences. The fibrations are the maps $X \to Y$ such that the induced square*

$$\begin{array}{ccc} X & \longrightarrow & QX \\ \downarrow & & \downarrow \\ Y & \longrightarrow & QY \end{array}$$

is levelwise a homotopy pullback square of simplicial sets. The fibrant objects are precisely the Ω-spectra which are also levelwise Kan complexes. The cofibrant objects are those X for which the structure map $\Sigma X_n \to X_{n+1}$ is an injection.

Moreover, **SeqSp** is a simplicial model category - the mapping spaces are given by $\mathbf{SeqSp}(X \wedge \Delta^{\bullet}_{+}, Y)$, where the smash product of a spectrum X and a simplicial set K is defined levelwise. The subscript $+$ indicates that we add a disjoint point to Δ^n to make it an augmented simplicial set.

Corollary 4.14. *Σ and Ω are mutually inverse functors on* $\mathrm{Ho}(\mathbf{SeqSp})$.

Let X, Y be spectra. One can define a **function spectrum** $\mathrm{Fun}(X, Y)$ in the following way. For every $i \geq 0$, we have a simplicial set $\mathbf{SeqSp}_{\bullet}(X, \Sigma^i Y) := \mathbf{SeqSp}(X \wedge \Delta^{\bullet}_{+}, \Sigma^i Y)$. We hope that we can form this collection into a spectrum. Observe that there is an 'assembly map' $\Sigma \mathbf{SeqSp}_{\bullet}(X, \Sigma^i Y) \to \mathbf{SeqSp}_{\bullet}(X, \Sigma^{i+1} Y)$; viewing $\Sigma = - \wedge S^1$ this is the map that sends a pair $f \wedge t$ to the map which sends x to $f(x) \wedge t$. This assembly map is then the structure map of $\mathrm{Fun}(X, Y)$.

Remark 4.15. Warning: **SeqSp** is *not* symmetric monoidal for any sensible notion of the smash product of spectra, and there is no sense in which this function spectrum construction gives an enrichment (let alone a model enrichment) of **SeqSp** over itself.

4.3 Intuition on stable ∞-categories

The reference here is Lurie's *Higher Algebra*. Spectra fit together into an ∞-category which can be obtained via a general ∞-categorical stabilisation procedure starting from spaces or simplicial sets. If $\mathcal{C}$ is an ∞-category with a final object $*$, the **suspension** of an object X is the homotopy pushout of the diagram

$$* \leftarrow X \rightarrow *$$

When $\mathcal{C}$ arises from a model category, one can compute this homotopy pushout as an actual pushout on an appropriately resolved diagram. If every object is cofibrant, it turns out that we simply need to replace one of the morphisms by a cofibration.

Example 4.16.

- When $\mathcal{C}$ is $\mathrm{Ch}(A)$ for a ring A, it is easiest to compute the above homotopy pushout using the injective model structure. Every object is cofibrant so it suffices to replace one of the maps in the diagram by a cofibration (i.e. an injection). One choice of resolved diagram is hence the diagram

  $$0 \leftarrow X \rightarrow \mathrm{cone}(\mathrm{id}_X)$$

  and taking the pushout, i.e. the cokernel of $X \rightarrow \mathrm{cone}(\mathrm{id}_X)$, yields $X[1]$.

- When $\mathcal{C} = \mathbf{sSet}$, again every object is cofibrant, so one choice of resolved diagram is

  $$* \leftarrow X \rightarrow CX$$

  where CX denotes the cone on X (equivalently, the mapping cone on id_{X}). Taking the cone and collapsing the end copy of X to a point yields precisely the usual (reduced) simplicial suspension ΣX.

Dually, when $\mathcal{C}$ has an initial object 0, the **loop space** of X is given by the homotopy pullback of

$$0 \rightarrow X \leftarrow 0.$$

Example 4.17.

- In $\mathrm{Ch}(A)$ we may compute the homotopy pullback using the projective model structure, where we replace one of the maps by a surjection. One choice of resolved diagram is

  $$0 \rightarrow X \leftarrow \mathrm{cocone}(\mathrm{id}_X)$$

  and so the homotopy pullback yields $X[-1]$.

- In **Top**$_*$ every object is fibrant, so we need to replace a map by a Serre fibration. One choice is the **path fibration** $X^{[0,1]} \to X$ where $X^{[0,1]}$ is the space of paths in X starting at $* \in X$, and the fibration sends a path p to its endpoint $p(1)$. The homotopy pullback is hence the set of paths $p \in X^{[0,1]}$ such that $p(1) = * \in X$. This is precisely the space ΩX of loops in X based at $*$.

Proposition 4.18. *Let $\mathcal{C}$ be a finitely bicomplete ∞-category with a zero object. Then Σ is left adjoint to Ω as endofunctors on $\mathcal{C}$.*

Proof. We have

$$\begin{aligned}\mathrm{Map}(\Sigma X, Y) &\simeq \mathrm{Map}(\mathrm{hocolim}(* \to X \leftarrow *), Y)\\ &\simeq \mathrm{holim}\,(* \to \mathrm{Map}(X, Y) \leftarrow *)\\ &\simeq \Omega\mathrm{Map}(X, Y)\\ &\simeq \mathrm{Map}(X, \Omega Y)\end{aligned}$$

as required. □

Say that $\mathcal{C}$ is **stable** if it is finitely bicomplete, has a zero object, and Σ is an autoequivalence of $\mathcal{C}$ (necessarily it then has inverse Ω). Note that this is a *property* of $\mathcal{C}$, rather than *structure*.

Proposition 4.19. *If $\mathcal{C}$ is stable then $h\mathcal{C}$ is triangulated.*

Proof. Since $\mathcal{C}$ is stable, we have

$$\pi_0\mathrm{Map}(X, Y) \cong \pi_0\mathrm{Map}(\Omega^2 X, \Omega^2 Y) \cong \pi_2\mathrm{Map}(\Omega^2 X, Y)$$

and hence $h\mathcal{C}$ is an additive category. The suspension functor on $h\mathcal{C}$ is given by Σ, so we just need to specify the exact triangles. Consider the diagrams in $\mathcal{C}$ of the form

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & W \end{array}$$

where both internal squares, and hence the outer square, are homotopy pushouts. The fact that the outer square is a homotopy pushout yields an isomorphism $W \simeq \Sigma X$, and the distinguished triangles in $h\mathcal{C}$ are defined to be the triangles we obtain by pushing the diagram $X \to Y \to Z \to \Sigma X$ down to $h\mathcal{C}$. □

So one perspective is to view stable ∞-categories as *enhanced triangulated categories*. Every triangulated category encountered in nature has an enhancement to a stable ∞-category.

Remark 4.20. Stable ∞-categories are the homotopy-theoretic analogue of abelian categories: indeed, they are (homotopy) additive, every morphism $X \to Y$ has a fibre (homotopy kernel) and a cofibre (homotopy cokernel), and a triple $X \to Y \to Z$ is a fibre sequence exactly when it is a cofibre sequence. This is in fact an alternate *characterisation* of stability.

Stable ∞-categories have **mapping spectra**: if X, Y are two objects, consider the sequence of spaces $\underline{\mathrm{Map}}_n(X, Y) \coloneqq \mathrm{Map}(X, \Sigma^n Y)$. We have isomorphisms

$$\Omega \underline{\mathrm{Map}}_{n+1}(X, Y) \simeq \underline{\mathrm{Map}}_n(X, Y)$$

which give these collection of spaces the structure of a spectrum. In particular we have $\pi_i \underline{\mathrm{Map}}(X, Y) \cong \pi_0 \mathrm{Map}(\Sigma^i X, Y)$ for $i \in \mathbb{Z}$.

Example 4.21. In the derived ∞-category $\mathcal{D}(A)$, we have $\pi_i \underline{\mathrm{Map}}(X, Y) \cong \mathrm{Ext}_A^{-i}(X, Y)$.

Example 4.22. In the ∞-category of spectra, mapping spectra are suitably derived versions of the function spectra we saw above.

Remark 4.23. If $\mathcal{C}$ is any ∞-category, its **stabilisation** is the homotopy limit (taken in the ∞-category of ∞-categories!)

$$\mathbf{Sp}(\mathcal{C}) \coloneqq \mathrm{holim}\left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}\right)$$

also known as the category of **spectrum objects in** $\mathcal{C}$. Stabilisation is a functor, adjoint to the inclusion of stable ∞-categories into all ∞-categories. We have $\mathbf{Sp}(\mathbf{sSet}) \simeq \mathbf{Sp}$ and $\mathbf{Sp}(\mathrm{Ch}_{\geq 0}(A)) \simeq \mathrm{Ch}(A)$.

4.4 Aside: cohomology theories

If A is a spectrum, the associated **cohomology theory** is the spectrum $E^*(X) \coloneqq \underline{\mathrm{Map}}(\Sigma^\infty X, E)$. We put $E^i(X) = \pi_{-i} E^*(X)$. Dually, the associated **homology theory** is $E_* X \coloneqq E \wedge X$. We put $E_i(X) \coloneqq \pi_i(E_* X)$.

Example 4.24. If A is a ring then the EM spectrum HA yields ordinary singular co/homology with coefficients in A.

Example 4.25. The cohomology theory associated to $\mathbb{S}$ is known as **stable cohomotopy**.

Brown representability says that every generalised co/homology theory is co/representable by a spectrum. In fact, even better, every map of theories lifts to a map of spectra. Be warned that this is **not** an equivalence of categories, since the passage from spectra to cohomology theory is not faithful, due to the existence of *phantom maps*.

5 Ring and module spectra

I like [HSS00, MMSS01, Hov01] for symmetric spectra. A good reference is [EKMM97].

5.1 Symmetric spectra

Just like one can take the smash product of a space with a spectrum (do it levelwise!) there are various possible definitions of a smash product of spectra. However, none of them make the category of sequential spectra into a symmetric monoidal category. Lewis in fact proved a no-go theorem: there is *no* symmetric monoidal smash product on spectra equipped with all of the properties one desires, such as $\mathbb{S}$ being the unit [Lew91].

Proof idea of Lewis' theorem. If $\mathbb{S}$ is the unit for the smash product, then $(\mathbb{S}, \wedge)$ is a commutative monoid in sequential spectra. If $\wedge$ interacts well with the wedge product for spaces, the multiplication map $\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ - which is an isomorphism - gives us an isomorphism $f : S^1 \wedge S^1 \to S^2$. Since the multiplication is commutative, this forces $f\tau = f$, where τ is the twist automorphism of $S^1 \wedge S^1$ which permutes the factors. Unfortunately, $\tau \neq \mathrm{id}$. □

On the other hand, it has been known for a long time that the stable homotopy category *does* admit a symmetric monoidal smash product. The issue is hence with lifting this smash product to the point-set level - in the homotopy category we simply have more freedom. One common modern answer is to attach extra bells and whistles to our spectra, often in the form of group actions at each level together with extra equivariancies on the structure maps. These objects are known as **highly structured spectra**. These structured spectra then admit a fully symmetric monoidal notion of smash product (often, this comes with some mild model-categorical complications).

Definition 5.1. A **symmetric spectrum** is a sequential spectrum X together with a symmetric group action $\mathcal{S}_n \curvearrowright X_n$ at each level. We demand that the induced maps $\Sigma^p \wedge X_n \to X_{n+p}$ are $\mathcal{S}_p \times \mathcal{S}_n$-equivariant: the action on the left hand side comes from the permutation action of $\mathcal{S}_p$ on $S^p \cong S^1 \wedge \cdots \wedge S^1$. A morphism of symmetric spectra should be equivariant at each level.

Remark 5.2. Since the symmetric groups are generated by transpositions, it is enough to assume that the maps $\Sigma^p \wedge X_n \to X_{n+p}$ are equivariant for $p = 1, 2$.

All of the constructions we did earlier have counterparts for symmetric spectra.

Smash products and a brief reminder on function spectra. Symmetric spectra form a monoidal model category, model enriched over itself.

5.2 Ring spectra

$\mathbb{S}$ as the initial ring spectrum. EM ring spectra. Maybe some remarks about cohomology operations.

5.3 Module spectra

Spectra as $\mathbb{S}$-modules. EM spectra and their modules. Ext and Tor.

6 THH

Definition. Bökstedt's computations of THH for $\mathbb{Z}$ and $\mathbb{Z}/p$, and Krause–Nikolaus's generalisation [KN22] via the spherical Witt vectors. The Dennis trace. Remark on the relative version and Shukla homology. Possibly some discussion of THH^*.

7 Cyclotomic spectra

For TC: the original reference is [BHM93]. A nice short overview is [May] and a longer, more detailed overview is [Mad94]. A third general reference is [HM97]. An excellent modern reference is [NS17]. They take a different route to the definition of TC, however. The introduction of [NS17] is especially good and I will mostly follow that for overview. For the definition of TC I will likely go to [Mad94] or [HM97].

A cyclotomic spectrum is loosely a spectrum X with an S^1-action together with various S^1-equivariant fixed point maps $X^{C_n} \to X$ that commute with the restriction maps between the C_n.

Theorem: THH upgrades to a cyclotomic spectrum (originally Hesselholt–Madsen).

8 TR

Roughly, TR is the homotopy limit over the diagram $\mathrm{THH}^{C_n} \to \mathrm{THH}^{C_m}$ for m dividing n (standard trick: index by $n!$ to turn this into an inverse limit).

9 TC

History: TC and the cyclotomic trace first appears in [BHM93]. I'll mostly follow [Mad94, HM97, NS17].

Definitions in terms of Frobenius on TR. Alternate definition from [May]. Various alternate characterisations from [NS17].

10 p-complete spectra

The point of this section: TC more or less splits into its p-completions.

p-completion for spectra, spherical Witt vectors, et cetera. Maybe a proof of Krause–Nikolaus's Bökstedt periodicity [KN22]. Discussion of p-completed TC, et cetera.

11 The cyclotomic trace

First appears in [BHM93]. The nilinvariance condition is the famous DGM theorem [DGM13].

12 Further afield

Possible further directions could include

- TC for categories [BM12].
- K-theory computations [HM97, HM03].
- motivic/number-theoretic applications [BMS19, Bou24]; the latter contains lots of good references.

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, Ill.-London, 1974.

[AV17] Benjamin Antieau and Gabriele Vezzosi. A remark on the Hochschild-Kostant-Rosenberg theorem in characteristic p. *arXiv e-prints*, page arXiv:1710.06039, October 2017.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[BM12] Andrew J. Blumberg and Michael A. Mandell. Localization theorems in topological Hochschild homology and topological cyclic homology. *Geom. Topol.*, 16(2):1053–1120, 2012.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[Bou24] Tess Bouis. Motivic cohomology of mixed characteristic schemes. *arXiv e-prints*, page arXiv:2412.06635, December 2024.

[DGM13] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2013.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May. *Rings, modules, and algebras in stable homotopy theory*, volume 47 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole.

[Goo86] Thomas G. Goodwillie. Relative algebraic K-theory and cyclic homology. *Ann. of Math. (2)*, 124(2):347–402, 1986.

[HM97] Lars Hesselholt and Ib Madsen. On the K-theory of finite algebras over Witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[HM03] Lars Hesselholt and Ib Madsen. On the K-theory of local fields. *Ann. of Math. (2)*, 158(1):1–113, 2003.

[Hov01] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[Hoy15] Marc Hoyois. The homotopy fixed points of the circle action on Hochschild homology. *arXiv e-prints*, page arXiv:1506.07123, June 2015.

[HSS00] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra. *J. Amer. Math. Soc.*, 13(1):149–208, 2000.

[KN22] Achim Krause and Thomas Nikolaus. Bökstedt periodicity and quotients of DVRs. *Compos. Math.*, 158(8):1683–1712, 2022.

[Lew91] L. Gaunce Lewis, Jr. Is there a convenient category of spectra? *J. Pure Appl. Algebra*, 73(3):233–246, 1991.

[Lod98] Jean-Louis Loday. *Cyclic homology*, volume 301 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mad94] Ib Madsen. Algebraic K-theory and traces. In *Current developments in mathematics, 1995 (Cambridge, MA)*, pages 191–321. Int. Press, Cambridge, MA, 1994.

[May] Peter May. Topological Hochschild and cyclic homology and algebraic K-theory. Available at `https://www.math.uchicago.edu/~may/TALKS/THHTC.pdf`.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[NS17] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *arXiv e-prints*, page arXiv:1707.01799, July 2017.