An Introduction to Topological Hochschild theory

Hodge Club, 13/11/20

Layout of talk

1) usual HH
2) stable homotopy theory
3) THH
4) Why you might care

1) usual HH

k a base commutative ring

A a k-algebra

$A^e = A \otimes_k A^{op}$ enveloping alg

$A^e\text{-Mod} =$ A-bimodules

Hochschild homology of A

$HH_*(A) = \mathrm{Tor}_*^{A^e}(A, A)$

Hochschild cohomology of A

$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$

Examples $A = k$

$A^e = k$ so $HH_* A \cong k \cong HH^* A$

$A = k[x]$, k a field

$A^e = k[x_1, x_2]$

Then $A = A^e / (\overbrace{x_1 - x_2}^{r})$

& a resolution of A is

$$\underset{0}{A^e} \xleftarrow{r} \underset{1}{A^e} \qquad A \xleftarrow{0} A$$

After tensoring w/ A, r vanishes

so $HH_0 A = A = HH_1 A$

More generally, if $A = K[x_1 - x_n]$ then

$$HH_*(A) = \Lambda^*_A(A^{\oplus n})$$

This is an example of the HKR theorem

Hochschild-Kostant-Rosenberg

A a commutative K-algebra, char 0

smooth of finite presentation, then $HH_*(A)$ is

$$\Omega^*(A/K)$$

The Bar complex

Bar(A) is the simplicial A-bimodule

$\Delta = \{\text{non } \emptyset \text{ fin sets}$

$\Delta^{op} \to \mathcal{C}$

n-simplices $A^{\otimes(n+2)}$

face maps $\partial_i(a_0 - a_{n+1}) =$

$$a_0 - a_i a_{i+1} - a_{n+1}$$

degeneracy maps

ith place

$$\sigma_i(a_0 - a_n) = a_0 - 1 - a_n$$

Can totalise to get a chain complex

Prop Bar A is an A-bimodule resolution of A

So can use to compute HH

Example

$$HH_0 A = \operatorname{coker}\left(A \otimes_k A \longrightarrow A\right)$$
$$a \otimes b \longmapsto ab - ba$$

$$= A/(\text{commutators}) \quad \text{cocentre}$$

$$HH^0 A = \operatorname{Ker}\left(A \longrightarrow \operatorname{Hom}_k(A, A)\right)$$
$$a \longmapsto [a, -]$$

$$= Z(A) \quad \text{centre}$$

rmk/ HH^* is not functorial!

Extensions

One can also define HH^* & HH_* for schemes & have HKR

$\wedge^i T$ ↦ { dg-algebras
dg-categories

have bar complexes

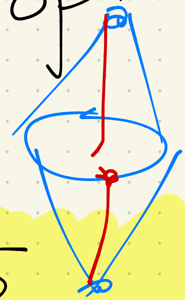
- $HH_n(X)$
- $\displaystyle\bigoplus_{p+q=n} H^p(X, \Omega^i)$

Loosely, $HH_* \sim$ cotangent

& $HH^* \sim$ tangent $\ulcorner$

2) Stable homotopy theory

goal: study stable phenomena in alg. top.



i.e. things invariant under suspension Σ

e.g. a reasonable space X has $\pi_n X \to \pi_{n+1} \Sigma X$

stable homotopy groups

$$\pi_n^s X = \varinjlim_r \pi_{n+r}(\Sigma^r X)$$

(Freudenthal)

Idea formally invert Σ

Def a (sequential) spectrum is a sequence $\{X_i\}_{i \in \mathbb{Z}}$ of (pointed) spaces with structure maps

$$\Sigma X_i \longrightarrow X_{i+1}$$

then $(\Sigma^j X)_i = X_{i+j}$

Can suspend & desuspend

Morphisms are tricky to define

Examples X a space

$$(\Sigma^\infty X)_i = \Sigma^i X$$

suspension spectrum

A an abelian group

$H(A,n)$ Eilenberg-Mac Lane space

$$\pi_i H(A,n) = \begin{cases} A & i=n \\ 0 & \text{else} \end{cases}$$

Fit together into an

EM Spectrum

$$(HA)_i = H(A,i)$$

Def X a spectrum

$$\pi_n X = \lim_{r \to} \pi_{n+r}(X_r)$$

e.g $\mathbb{S} = \Sigma^\infty S^0$

sphere spectrum

$\mathbb{S}_i \simeq S^i$

$\pi_* \mathbb{S} =$ stable homotopy groups of spheres

e.g $\pi_* HA \simeq A$

rmk

maps in homotopy category of spectra

$$\pi_* X \simeq [\mathbb{S}, X]$$

Products

X, Y, pointed spaces

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$ smash product

e.g. $\Sigma X = S^1 \wedge X$

Does not extend in a sym monoidal way to sequential spectra!

Problem there's a braiding $S^n \wedge S^m \cong S^m \wedge S^n$

need to remember this Σ_2-action

More generally

Products $S^{l_1} \wedge \cdots \wedge S^{l_n}$ have a Σ_n-action

Loose defn

a **symmetric spectrum** is a sequence X_q together with actions $\Sigma_q \curvearrowright X_q$ & structure maps

$$S^p \wedge X_q \longrightarrow X_{p+q}$$

which are $\Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{p+q}$ equivariant

Thm (Hovey-Shipley-Smith)

$\wedge$ extends to a sym. monoidal product on Sp^Σ

unit $= \mathbb{S}$

Rmk Can also use orthogonal spectra S-modules etc

Warning naive $\pi_* X$ may not agree with $[\mathbb{S}, X]$ $\leftarrow$ maps in $Ho(Sp^\Sigma)$

$$H^*(-,R) = [HR, -]$$

$K \qquad \pi_* \to \pi_{*+1} \Sigma$

E

$$H^*(-,R) \to H^*(-,R)$$

$$[HR, HR]$$

3 Spectral algebra

Def a ring spectrum is a monoid in $(Sp^\Sigma, \wedge)$

$$\begin{tikzcd} \mathbb{S} \wedge \mathbb{S} \arrow[r] \arrow[d, "\tau"] & \mathbb{S} \\ \mathbb{S} \wedge \mathbb{S} \arrow[ur] & \end{tikzcd}$$

Examples

$\mathbb{S}$ is
- unit for $\wedge$
- a commutative ring spectrum
- the initial ring spectrum

If A is a ring then HA is a ring spectrum, commutative if A was

I'll view

$$(\text{rings}) \subseteq (\text{ring spectra})$$

$$\exists!\ S \longrightarrow A$$

Once I have rings I can talk about

module spectra

Examples

$(\mathbb{S}\text{-mod}, \wedge) \simeq (Sp^{\Sigma}, \wedge)$

$(H\mathbb{Z}\text{-mod}, \wedge) \simeq (Ch(\mathbb{Z}), \otimes^{\mathbb{L}})$

Quillen equivalence

really I mean

$Ho(H\mathbb{Z}\text{-mod}) \simeq D(\mathbb{Z})$

as monoidal & Δ^{d} cats

Once I have modules,
I can do
homological algebra

e.g. homological algebra
over rings 'embeds'
in stable homotopy
theory

$$D(\mathbb{Z}) \xrightarrow{\simeq} Ho(H\mathbb{Z}\text{-mod})$$
$$\longrightarrow Ho(Sp^{\Sigma})$$

Def A a ring (spectrum)

$THH^*(A)$ is

$$\mathrm{Ext}^*_{A \wedge_{\mathbb{S}} A^{op}}(A, A)$$

$THH_*(A)$ is

$$\mathrm{Tor}_*^{A \wedge_{\mathbb{S}} A^{op}}(A, A)$$

example $THH_*(\mathbb{S})$ is $\pi_*(\mathbb{S}) \leftarrow$ hard to compute!

~~Thm~~ (Bökstedt) ← deg 2

$$THH^*(\mathbb{F}_p) \simeq \mathbb{F}_p[u]$$

Proof is not easy!

Rmk

$$\mathbb{F}_p^e = \mathbb{F}_p \wedge_{\mathbb{S}} \mathbb{F}_p$$ is

the dual Steenrod algebra

parametrising cohomology operations

$$H^*(-, \mathbb{F}_p) \to H^*(-, \mathbb{F}_p)$$

One more computation

A an algebra over $\mathbb{Q}$

then $\boxed{THH(A) \simeq HH(A)}$

because $\mathbb{Q} \wedge_S \mathbb{Q} \simeq \mathbb{Q}$

moral: over a <u>field</u>, THH only interesting in char p

('arithmetic' invariant)

Rmk There's a bar construction for THH too

Extensions

Can define THH for

schemes $\to$ "Blumberg-Mandell"

DGAs

DG categories

spectral categories

"$\mathbb{F}_1 = \mathbb{S}$"?

$\mathrm{THH}^*(\mathbb{Z})^\wedge_p$

$\mathbb{Z}/i$ deg $2i-1$

0 else

4 Why you might care

Homology : K-theory

A a ring, $HH_0 A = \dfrac{A}{[A,A]}$

'universal recipient of trace maps'

cf. Hattori–Stallings trace

there's a map

$$K_0(A) \longrightarrow HH_0 A$$

group completion of $(\mathrm{proj}\, A, \oplus)$

Upgrades to a map

$K_* A \longrightarrow HH_* A$

'chern character'

Upgrades to a map

$K_* A \longrightarrow THH_* A$

'Dennis trace'

Even better, upgrades to a map

$K_* A \longrightarrow TC_* A$

'cyclotomic trace'

Which is sometimes a good approximation

Cohomology: Deformation theory

A an alg / a field

$HH^2 A \simeq$ first-order deformations of A

More generally, HH^* 'controls' defs / A

$THH^* A$ controls non-additive deformations of A

cf. Mac Lane cohomology