GLOBAL KOSZUL DUALITY

MATT BOOTH
LANCASTER UNIVERSITY

J/W ANDREY LAZAREV

TRANSPENNINE TOPOLOGY TRIANGLE
13.05.22

Plan

1. The bar-cobar adjunction

2. Model structures

3. The global setting

4. Potential application: deformation theory

1. THE BAR-COBAR ADJUNCTION

K a field.

A **(dg) algebra** is a monoid in the category $(\mathrm{dgVect}, \otimes)$

$$\mu : A \otimes A \longrightarrow A$$

$$\eta : K \longrightarrow A$$

A **(dg) coalgebra** is a comonoid in dgVect:

$$\Delta : C \longrightarrow C \otimes C$$

$$\eta : C \longrightarrow K$$

Example V a (dg) vector space. The tensor coalgebra on V is

$$T^c(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}$$

with comultiplication given by

$$\Delta(v_1 \cdots v_n) = \sum_i v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$$

Example If (C, Δ, η) is a coalgebra then the linear dual $(C^\vee, \Delta^\vee, \eta^\vee)$ is an algebra

$$V^\vee \otimes U^\vee \longrightarrow (V \otimes U)^\vee$$

Warning The linear dual of an algebra A is only a coalgebra when A is finite dimensional!

An algebra is augmented if the unit map $K \to A$ admits a retract

$$A \to K$$

The augmentation ideal is

$$\mathrm{Ker}(A \to K) =: \overline{A}$$

It's a nonunital algebra

In fact

$$\mathrm{aug.Alg} \simeq \mathrm{nonu.Alg}$$

Similarly a coalgebra C is **coaugmented** if $C \to k$ has a section. The quotient $\overline{C} := \operatorname{coker}(k \to C)$ is the **coaugmentation coideal** and is a noncounital coalgebra under the reduced coproduct $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$.

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$$

A coaugmented coalgebra C is conilpotent if $\forall c \in C$ $\exists p$, s.t.

$$\overline{\Delta}^{p}(c) = 0$$

Example T^cV is conilpotent since $\overline{\Delta}(v) = 0$

In fact T^cV is the cofree conilpotent coalgebra on V.

The bar construction

A an augmented algebra.
The **bar construction** on A is the dg coalgebra BA whose underlying graded algebra is

$T^c(\Sigma \bar{A})$. The differential is the **bar differential**:

$$\partial = d_1 + d_2$$

d_1 = usual diff^l on

$$d_2(a_1 \text{—} a_n) = \sum_i a_1 \text{—} a_i \cdot a_{i+1} \text{—} a_n$$

Similarly if C is a conilpotent coalgebra there's a cobar construction ΩC equal to

$$\left(T(\Sigma^{-1}\overline{C}),\ d_1+d_2\right)$$

Thm Bar and cobar are adjoints:

$$\mathrm{Alg}_*(\Omega C, A) \simeq \mathrm{conil.Cog}_*(C, BA)$$

Proof idea

Show that both are isomorphic to a third functor.

C a coalgebra

A an algebra

The space $\mathrm{Hom}(C, A)$ admits a convolution product: $f, g : C \to A$

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A$$

making it into an algebra, the convolution algebra

E any algebra, possibly nonunital. A **Maurer-Cartan element** in E is $x \in E'$ such that

$$dx + x^2 = 0.$$

The set of all MC elts is $MC(E)$.

Th$^{\mathrm{M}}$

'twisting morphisms'

$$\mathrm{Hom}(\Omega C, A) \simeq \mathrm{MCHom}(\overline{C}, \overline{A})$$
$$\simeq \mathrm{Hom}(C, BA)$$

Proof idea

A map $\Omega C \to A$ is the same as a linear map $\Sigma^{-1}\overline{C} \to A$ compatible with the differential.

Rmk There's an equivalence

$$\mathrm{Cog} \simeq (\text{pro-fdAlg})^{op}$$

$$\simeq \left\{\begin{matrix}\text{pseudocompact} \\ \text{algebras}\end{matrix}\right\}^{op}$$

Proof idea

$$C = \varinjlim_{\substack{C' \hookrightarrow C \\ \text{fd}}} C' \quad (\text{Sweedler})$$

$$\mathrm{Cog} \simeq \text{ind-fdCog}$$

$$\simeq \text{ind-}(\text{fdAlg}^{op})$$

$$\simeq (\text{pro-fdAlg})^{op}$$

2. MODEL STRUCTURES

Thm (Hinich) The category Alg of dgas admits a model structure with

- weak equivs: quasi-isos
- fibrations: surjections

Alg_* is the slice category Alg/k so inherits the model structure.

Aim Put a model structure on $\mathrm{conil.Cog}_*$ making

$$\Omega \dashv B$$

into a Quillen adjunction.

$\underline{\text{Th}^{\text{m}}}$ 1) the counit map $\Omega BA \to A$ is a cofibrant resolution of A

2) B preserves quasi-isomorphisms

3) Ω does not preserve quasi-isomorphisms

$$C = B(K \oplus K) \simeq K$$

ΩK acyclic

$$\Omega C \simeq K \oplus K$$

Upshot: quasi-isos are too coarse a notion of weak equivalence to get the desired adjunction.

Fix: Create the weak equivalences through Ω.

Th^m (Quillen, Hinich, Lefèvre-Hasegawa)

The category of conilpotent coalgebras admits a model structure with

- weak equivs: f s.t. Ωf is a quasi-iso.
- cofibrations: injections

Thm

1) Weak equivs are quasi-isomorphisms

2) Converse is true for coconnective coalgebras but false in general

3) $C \to B\Omega C$ is a fibrant resolution

$\underline{\text{Thm}}$ ("Koszul duality")

$$\Omega \dashv B$$

is a Quillen equivalence.

$$\mathrm{Ho}(\mathrm{conil.Cog}_*) \simeq \mathrm{Ho}(\mathrm{Alg}_*)$$

Rmk Can transfer the model structure on conilpotent coalgebras to one on

$$\mathrm{proArt}$$

For connective pro-Artinian algebras, recovers a model structure of Pridham.

3. THE GLOBAL SETTING

Q. Do the previous results have analogues when one drops conilpotency?

why global?

conilpotent dg coalgebras $\sim$ formal stacks

Hinich

The extended bar construction

there's a cofree coalgebra functor $\mathrm{Cog}: \mathrm{dgVect} \to \mathrm{Cog}_*$.

Even $\mathrm{Cog}(k)$ is complicated.

A an augmented alg.
its extended bar constr^n
is

$$\check{B}A := (\mathrm{Cog}(\Sigma\bar{A}), \partial)$$

∂ is like the bar diff^l.

Th^{m} (Anel–Joyal, ...)

there's an adjunction

$$\Omega : \mathrm{Cog}_* \longleftrightarrow \mathrm{Alg}_* : \check{B}$$

& moreover a similar interpretation for MC.

Problem $\check{B}$ doesn't preserve quasi-isomorphisms.

ΩC need not be cofibrant unless C was conilpotent.

MC equivalences

E a dgca.

There's a dg category $MC_{dg}(E)$ with

- Objects: $x \in MC(E)$
- Morphism spaces:

$$\mathrm{Hom}(x,y) = \mathrm{Hom}_E(E^{[x]}, E^{[y]})$$

Th^m (Chuang-Holstein-Lazarev)

isoclasses in $H^0 MC_{dg}(E)$ $\simeq$ $\dfrac{MC(E)}{(\text{htpy gauge equiv.})}$

In particular:

Can look at

$\mathcal{D} := MC_{dg}(Hom(\overline{C}, \overline{A}))$.

Put $\mathcal{MC}(C,A) := \pi_0 \mathcal{D} \cong H^0\mathcal{D}$ (isoclasses)

the MC moduli set

Thm (CHL)

$$\mathcal{MC}(C,A) \cong \frac{Hom(\Omega C, A)}{\text{3-homotopy equiv}}$$

$$\cong \frac{Hom(C, \check{B}A)}{\text{3-homotopy equiv.}}$$

Def A map $C \to C'$ is an MC equivalence if $\forall A$,

$$MC(C', A) \to MC(C, A)$$

is an isomorphism.

& similarly for algebras

Examples

3-homotopy equivalence

$\Rightarrow$ MC equivalence

$\Rightarrow$ quasi-isomorphism

$$\left.\begin{array}{l} \bullet\ C \to \check{B}\Omega C \\[2em] \bullet\ \Omega\check{B}A \to A \end{array}\right\} \text{ are MC equivs.}$$

Thm* (B.-Lazarev)

1) Cog_* admits a cof. gen. model structure with
- weak equivs: MC equivs.
- cofibrations: injections

2) Alg_* admits a cof. gen model structure with
- weak equivs: MC equivs
- fibrations: surjections.

3) $\Omega \dashv \check{B}$ is a Quillen equivalence

Curvature.

Proof sketch (×2)

a) Jeff Smith's theorem requires Vopěnka's principle

b) More direct (w.i.p.)
(Hovey, May–Ponto II)

$$w \cap J\text{-inj} \subseteq I\text{-inj}$$

$C \hookrightarrow C'$ ind. fd. cofibs.

I 3-interval

wait

$$C' \sqcup_C C \otimes I \longrightarrow C' \otimes I$$

is an MC equivalence.

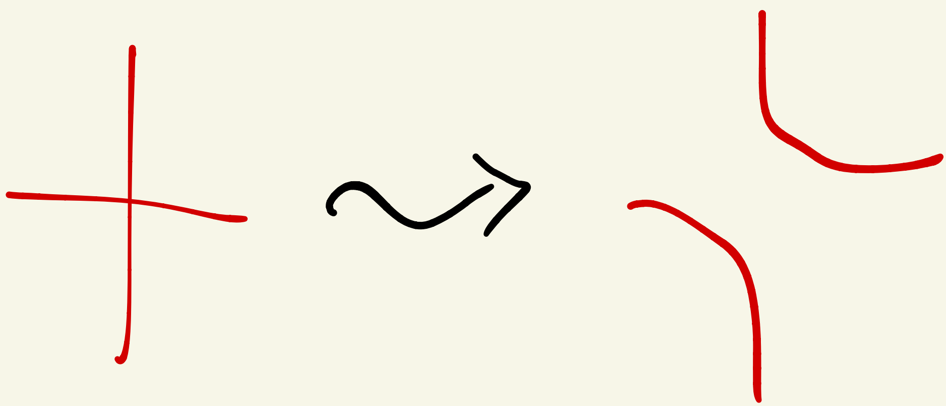
4. DEFORMATION THEORY

A deformation of a geometric object is an infinitesimal thickening

Ex. $\{xy=0\} \subseteq \mathbb{A}^2_k$

thickens to the family

$\{xy=t\}$ over $k[t]/t^2$



Philosophy

(Deligne, Hinich, Pridham, Lurie, ...)

In characteristic zero, commutative deformation problems are controlled by dg Lie algebras

This is Koszul duality!

Similarly, noncommutative deformation problems are controlled by noncommutative algebras (Lurie)

Can use nilpotent Koszul duality to get explicit prorepresenting objects for certain naturally occurring deformation problems (B.)

Hope: Global Koszul duality will give representability results for global deformation problems.

'global test algebras' $\longrightarrow$ sSet

$$\mathrm{Def}(M) \simeq \mathbb{R}\mathrm{Hom}\left(\check{B}(\mathrm{End}\,M)^{*}, -\right)$$

Thanks
for
listening!

$\check{B}(K \oplus K), \quad K$

$D^n \sim$ n-disc

chains $\rightsquigarrow$ D^n_*

$m, n \geqslant 3$

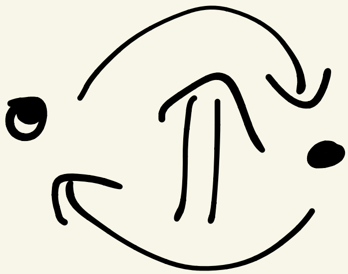
$$\Omega D^n_* \simeq \Omega D^m_*$$

$$\simeq$$

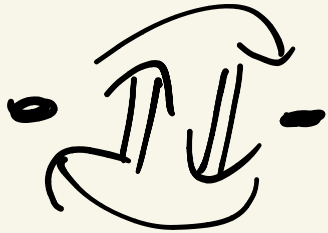
$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left, "\Delta"] \end{tikzcd}$$

$$\bullet \longrightarrow \bullet \quad D^1$$

D^2



D^3



+ extra

S^∞