Deformation Theory

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What is deformation theory?

- The study of infinitesimal deformations of geometric objects (varieties, schemes, stacks...)
- Deformations of a point $x \in X$ can be used to get local information about X
- Applications to moduli problems

Setup

- We'll work over an algebraically closed field k
- The ring of dual numbers is the ring $k[\varepsilon] := k[x]/(x^2)$
- The element ε behaves like an infintesimal since $\varepsilon^2 = 0$
- In this talk we'll be interested in **first-order deformations**; i.e. deformations over the dual numbers. They behave in a 'linearised' way

The basic objects of study

Definition

A **variety**[1] over k is a subset of k^n given by the zero locus of a finite set of polynomials $P_1, \dots, P_r \in k[x_1 \dots, x_n]$. The **coordinate ring** of a variety $X = V(P_1, \dots, P_r)$ is the ring $C(X) := k[x_1, \dots, x_n]/\mathrm{rad}(P_1, \dots, P_r)$. Elements of the coordinate ring are polynomial functions on X.

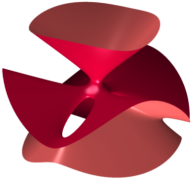
Remark

The radical shows up because $P_1, \dots, P_r$ and $P_1^{m_1}, \dots, P_r^{m_r}$ define the same variety.

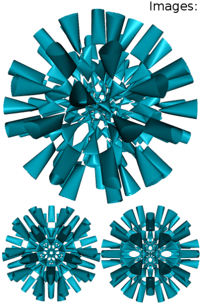
[1] More precisely an affine variety

Examples of varieties

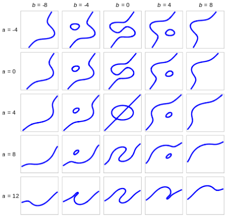
Images: Wikipedia



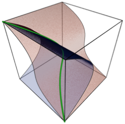
Clebsch cubic



Sarti surface



Cubic plane curves



Twisted cubic

Extending varieties

- We'd like a geometric analogue of the dual numbers $k[\varepsilon]$
- In particular we'd like to think of $k[\varepsilon]$ as the coordinate ring of some variety
- This can't happen since (x^2) is not radical in $k[x]$
- We need to use more general geometric objects

Schemes

- An affine scheme is a space $\mathrm{Spec}(R)$ built from a commutative ring R in such a way that[2] $C(\mathrm{Spec}(R)) = R$
- A scheme is a space glued together from affine schemes, just like a manifold is glued together from $\mathbb{R}^n$s
- If $P_1, \ldots, P_n$ are polynomials and $X = V(P_1, \ldots, P_n)$ then $X = \mathrm{Spec}(C(X))$
- Nice[3] schemes are glued together from varieties

[2] By $C(X)$ I really mean $\Gamma(X, \mathcal{O}_X)$

[3] Integral, separated, locally of finite type

Fat points

We can think of Spec($k[\varepsilon]$) as a point together with an infintesimal tangent direction. Intuitively[4], giving a map Spec($k[\varepsilon]$) $\to X$ is the same as giving a point $x \in X$ together with a tangent vector in $T_x X$.

Definition

A **fat point** is a point together with some data about an infintesimal neighbourhood.[5]If m is an integer then Spec($k[x]/(x^m)$) is a fat point containing 'm^{th}-order data' about an infintesimal neighbourhood.

[4]We should also require x to be a rational point

[5]Technically a fat point is Spec of a local Artinian commutative k-algebra with residue field k

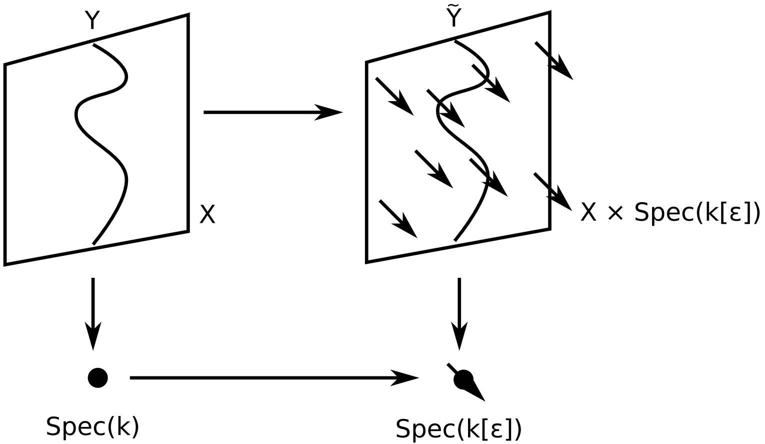
Deformations of a subscheme

Let X be a scheme and $Y \subset X$ a closed subscheme.

Definition

A **first-order deformation of** Y **in** X is a closed subscheme $\tilde{Y} \subset X \times \operatorname{Spec}(k[\varepsilon])$, such that the projection map $\pi : \tilde{Y} \to \operatorname{Spec}(k[\varepsilon])$ is flat, and Y is the preimage of the closed point of $\operatorname{Spec}(k[\varepsilon])$.

What does this mean geometrically?



Classifying deformations

- A general problem is to classify deformations in terms of other objects
- For example, the first order deformations of Y in X are in bijection with the cohomology group $H^0(X, \mathcal{N}_{Y/X})$
- If $x \in X$ is a point, then this cohomology group is the tangent space $T_x X$

Moduli problems

- Suppose we have some geometric objects we want to classify, together with a notion of equivalence between these objects
- We'd like to know if the set of such objects is itself some kind of space. In other words we'd like a **moduli space** $\mathcal{M}$ whose points are in bijection with isomorphism classes of the objects we want to classify

Baby example: conics in the projective plane

- **Projective** n**-space** $\mathbb{P}^n$ is k^{n+1} where we identify a vector x with all of its scalar multiples λx
- We can think of $\mathbb{P}^n$ as the set of lines through the origin in k^{n+1}. It's a scheme (it's a bunch of k^ns glued together)
- A **conic** in $\mathbb{P}^2$ is a curve in $\mathbb{P}^2$ cut out by a nonzero equation of the form $ax^2 + bxy + cxz + dy^2 + eyz + fz^2 = 0$

Baby example: conics in the projective plane

- A conic in $\mathbb{P}^2$ gives us a tuple $[a : b : c : d : e : f] \in \mathbb{P}^5$
- This map is a bijection, so we say that $\mathbb{P}^5$ is a moduli space for conics in $\mathbb{P}^2$
- Note that our conics are allowed to be highly singular e.g. $x^2 = 0$; the moduli space of **smooth** conics is an open subset $U \subseteq \mathbb{P}^5$

Examples of moduli problems

- Closed subschemes Y of a scheme X; moduli space is the **Hilbert scheme**[6]
- Curves in X; hard to classify
- Line bundles on X; moduli space is the **Picard scheme**[7]

[6] Ignoring some technical details
[7] This is only a coarse moduli space

Deformation theory and moduli spaces

- We've seen that deforming a point x of a scheme X is the same as picking a tangent vector to X at x
- So we can use deformation theory to find out local properties of spaces by deforming their points
- In particular, if X is a moduli space and x is a point representing some geometric object, then deforming x as a point of X should be the same as deforming the geometric object represented by x

Example: Hilbert Scheme

- Let X be a scheme and let H be the **Hilbert Scheme** of X; the points of H are in bijection with closed subschemes $Y \subset X$
- Deforming the point $Y \in H$ is the same as deforming Y as a subscheme of X. So we obtain an isomorphism $T_Y H \cong H^0(X, \mathcal{N}_{Y/X})$

Further reading

- Balázs Szendrői, *The unbearable lightness of deformation theory*
- Barbara Fantechi, *Elementary deformation theory*
- Robin Hartshorne, *Deformation theory*