Topological Hochschild cohomology for schemes

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1) Recollections on HH^*, THH^* for rings

2) HH^* and THH^* for schemes

3) Computations

1) Recap on HH^*, THH^*

k a c.ring, $A \in k\text{-alg}$

The enveloping algebra is $A^e_k := A \otimes_k A^{op}$

A^e_k-modules $=$ A-bimodules

The Hochschild cohomology is

$$HH^n_k(A,M) := \mathrm{Ext}^n_{A^e_k}(A,M)$$

Can compute HH^* via the bar complex.

Thm

$$HH^2_k(A, M) \cong \left\{ \begin{array}{c} k\text{-split} \\ \text{square-zero} \\ \text{extensions} \\ \text{of } A \text{ by } M \end{array} \right\} \Big/ \text{iso.}$$

$M^2 = 0$

$$0 \to M \to \hat{A} \to A \to 0$$

Higher HH^* have interpretations.

Examples

K a field,

$$\left(\frac{K[\varepsilon]}{\varepsilon^2} \longrightarrow K\right) \in HH^2_K(K)$$

$$= 0$$

$$\mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p$$

is not $\mathbb{Z}/p$-linear

is not $\mathbb{Z}$-split.

$\rightsquigarrow$ not detected by HH^*.

Th^m

$THH^2(A, M)$ is in bijection with

$$\left\{ \begin{array}{l} \underline{\text{all}} \text{ square-zero} \\ \text{extensions of } A \\ \text{by } M \end{array} \right\} \Big/ \text{iso.}$$

if A is a ring, its topological enveloping algebra is $A^e_{\mathbb{S}} := A \otimes_{\mathbb{S}} A^{op}$.

Its topological Hochschild cohomology is

$$THH^n(A,M) := \mathrm{Ext}^n_{A^e_{\mathbb{S}}}(A,M)$$

Relative version

R a commutative ring spectrum, A an R-algebra

Can form A^e_R &

THH_R. Usual THH is $\mathrm{THH}_{\mathbb{S}}$.

When k is a field,

$\mathrm{THH}_k \simeq \mathrm{HH}_k$. Shukla

$\mathrm{THH}_k(A) \simeq \mathrm{Ext}_{A \otimes^{\mathbb{L}}_k A}(A, A)$

Sample computations

$$\mathbb{Q}^e_{\mathbb{S}} = \mathbb{Q} \otimes_{\mathbb{S}} \mathbb{Q} \simeq \mathbb{Q}$$

so if A is a $\mathbb{Q}$-alg.,

$$THH^*(A, M) \simeq HH^*_{\mathbb{Q}}(A, M)$$

Bökstedt periodicity:

k a perfect field, char p

$$\underline{THH^*(k)} \simeq \underline{k[u]}$$

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2) HH^* & THH^* for schemes

How to extend the definition of THH^* to schemes?

Convention All schemes are separated & noetherian over a field k

Defining HH^*

X a scheme.

1) define $HH^*_k(X)$ locally & glue

2) $HH^*_k(X) := \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$

3) $HH^*_k(X) := HH^*_k(\mathrm{per} X)$

We choose 3) as our model for THH^*.

On homology, done by Blumberg-Mandell.

__Idea__ Just like one can take HH_k^* of a k-linear dg category, one can take THH^* of a spectral category

(= category enriched in Spectra)

So one can go

$X \rightsquigarrow \mathrm{per}X$ k-linear dg cat.

$\rightsquigarrow \mathrm{per}X$ Hk-linear spectral cat.

$\rightsquigarrow \mathrm{per}X$ $\mathbb{S}$-linear spectral cat.

$\rightsquigarrow THH^*(\mathrm{per}X)$

$\underline{\text{Th}^{\text{m}}}$ B.-Kaledin-Lowen
(after Lowen-Van den Bergh)

THH^* satisfies strong invariance results for special categories.
In particular one has

$$\begin{aligned} THH^* X &\simeq \underline{THH^* \operatorname{per} X} \\ &\simeq \underline{THH^* D^b \operatorname{Coh} X} \\ &\simeq \underline{THH^* DQ\operatorname{Coh} X} \\ &\simeq \dots \end{aligned}$$

Proof uses the bar cplx.

<u>Cor.</u> If R is a commutative noetherian ring then

$$THH^*(\operatorname{Spec} R) \simeq THH^*(R)$$

<u>Rmk</u> If X is a $\mathbb{Q}$-scheme then

$$THH^*(X) \simeq HH^*_{\mathbb{Q}}(X).$$

3) Sample computations

X smooth proper / a field k.

Th^m (BKL)

$\exists$ a multiplicative, convergent upper half-plane spectral sequence with E^2 page

$$HH^p_k(X) \otimes_k THH^q(k)$$

$$\Longrightarrow THH^{p+q}(X)$$

In particular

if k is perfect of characteristic $p>0$ we get

$$HH^*_k(X)[u] \Longrightarrow THH^*(X)$$

This degenerates at E^2 when:

- X a curve
- X is $\mathbb{P}^2$ and $p>2$
- X a K3 surface & $p>2$

e.g $X = \mathbb{P}^1_{\mathbb{Z}/p}$.

Then $THH^* X$ is

$$\mathbb{Z}/p[a, b, c, u] \big/ \begin{matrix} a^2, b^2, c^2 \\ ab, bc, ac \end{matrix}$$

RmK $THH_s X$ is always an E_2-algebra so

$THH^* X$ is graded-commutative.

Where to go now?

- It's unclear in what sense we have

$$THH^2 X \simeq \left\{ \begin{matrix} \text{nonlinear} \\ \text{deformations} \\ \text{of } X \end{matrix} \right\}$$

when X is non-affine

- e.g. $\mathbb{P}^1_{\mathbb{Z}/p^2}$ ought to show up as an element of $THH^2(\mathbb{P}^1_{\mathbb{Z}/p}) \simeq \mathbb{Z}/p$

- Loose idea:
$THH^2 X$ parameterises non-additive deformations of $\mathrm{Coh}\, X$

- Precise relationship between THH^* & deformations of abelian categories is work in progress.

Thanks
for
listening!

A ring

M A-finite

$$THH^*(A, M)$$

$$\cong HML^*(A, M)$$