# Cheatsheet for dg-objects

Matt Booth

December 20, 2016

I often find myself confused about sign conventions for dg-objects. For example, if $A$ and $B$ are dg-algebras, what sign should $(a \otimes b)(a' \otimes b') = \pm(aa' \otimes bb')$ take in the tensor product? In fact, essentially all sign conventions can be worked out using the **Koszul sign rule**: that when $x$ moves past $y$, then a sign change of $(-1)^{|x||y|}$ is required. I'm collecting them here as a (hopefully) useful reference, but of course 'in the field' really all that one needs to remember is the Koszul rule.

## Conventions

I'll take a (unital, noncommutative) base ring $R$, usually left implicit. When discussing dglas, I need to work over a field $k$. Whether modules are left or right modules is mostly up to taste, although this becomes important e.g. when taking tensor products. All complexes are indexed cohomologically, so that the differential of a dg-module has degree 1.

## Remarks

Most of the sign conventions and definitions should be clear simply by use of the Koszul rule or occasionally the product rule.

- Let $X, Y$ be dg-modules. To work out the differential on $\underline{\mathrm{Hom}}(X, Y)$, we can either imagine passing $d_Y$ through $f$, introducing a $-(-1)^{|f|}$ factor, or we can specify that the (degree zero) evaluation pairing $\underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$ should be a cochain map. Note that $\underline{\mathrm{Hom}}(X, Y)$ is a priori only a dg-abelian group, but if $R$ is commutative it is a dg-$R$-module.

- The definition of a commutative dga becomes inevitable once we specify that the tensor product of two cdgas should also be a cdga.

- When defining a map of dgas, note that any map should preserve the unit so is necessarily of degree zero. As we hence stand no chance of obtaining Hom complexes, it then makes sense to simply restrict to maps that commute with the differentials.

# Modules

| | |
|---|---|
| **Graded modules** | Grading by $\mathbb{Z}$. Maps $f : X \to Y$ have a degree $r$ satisfying $f(X_i) \subseteq Y_{i+r}$. |
| **dg-modules** | Graded modules with degree 1 differentials. Maps are maps of graded modules. |
| **Tensor product** | $d_{X\otimes Y}(x\otimes y) = d_X(x)\otimes y + (-1)^{\lvert x\rvert} x\otimes d_Y(y)$. |
| **Enriched Hom** | $d_{\underline{\mathrm{Hom}}(X,Y)} f = d_Y f - (-1)^{\lvert f\rvert} f d_X$. Degree 0 cocycles are exactly the cochain maps. |
| **Duals** | Dual $X^\vee$ has $(X^\vee)_i = \mathrm{Hom}(X_{-i}, R)$ and $d_{X^\vee} f = -(-1)^{\lvert f\rvert} f d_X$. |

# Algebras

| | |
|---|---|
| **Graded algebras** | $X_i X_j \subseteq X_{i+j}$. Maps respect multiplication, and in particular are degree zero. |
| **Tensor product** | $(x\otimes y)(x'\otimes y') = (-1)^{\lvert x'\rvert\lvert y\rvert}(xx'\otimes yy')$. |
| **Commutativity** | $xy = (-1)^{\lvert x\rvert\lvert y\rvert} yx$. |
| **dg-algebras** | $d(xy) = d(x)y + (-1)^{\lvert x\rvert} x d(y)$. We see that $d(1) = 0$. Maps are maps of graded algebras commuting with the differentials. |
| **Tensor product** | Differential from underlying dg-module & multiplication from underlying gr. alg. |

# Rings

| | |
|---|---|
| **Graded rings** | Graded rings are of course just graded $\mathbb{Z}$-algebras. |
| **Modules, I** | A module over a graded ring $A$ is a graded $A$-module $M$ such that $A_p M_q \subseteq M_{p+q}$. |
| **dg-rings** | dg-rings are dg-$\mathbb{Z}$-algebras. |
| **Modules, II** | A (right) module over a dg-ring $A$ is a dg-abelian group $M$ that's a module over the graded ring $A$, satisfying $d_M(m.a) = d_M(m).a + (-1)^{\lvert m\rvert} m.d_A(a)$. |

# Lie algebras

| | |
|---|---|
| **Gr. Lie algebras** | gr.-$k$-modules $L$ with bracket satisfying $[L_i, L_j] \subseteq L_{i+j}$ , $[x,y] = -(-1)^{\lvert x\rvert\lvert y\rvert}[y,x]$ & $[[x,y],z] = [x,[y,z]] - (-1)^{\lvert x\rvert\lvert y\rvert}[y,[x,z]]$. |
| **dglas** | The product rule says that $d[x,y] = [dx,y] + (-1)^{\lvert x\rvert}[x,dy]$. |
| **Commutators** | $[x,y] = xy - (-1)^{\lvert x\rvert\lvert y\rvert} yx$. If $A = \mathrm{End}(X)$ for some dg-module $X$, then $d_A f = [d_X, f]$. |
| **Tensor product** | If $L$ is a dgla and $A$ is a cdga over $k$, then the tensor product of $k$-modules $L\otimes A$ is a dgla over $k$ with bracket $[x\otimes a, y\otimes b] = (-1)^{\lvert a\rvert\lvert y\rvert}[x,y]\otimes ab$. |