Curved talk

§1. Algebras

A a dga, $x \in A$. Observation: $[x,-]$ is a derivation.

Q. when is $(A, d+[x,-])$ a dga for $x \in A^1$?

Lem $(d+[x,-])^2 = [dx+x^2, -]$. The MC equation. Flat connections.

Def. a cAlg has (A,d,h) w/ $d^2 = [h,-]$ & $dh = 0$.

if (A,d) a dga & $h \in Z(A)^2 \rightsquigarrow (A,d,h)$ a cAlg.

Morphisms should involve a change of curvature!

~~ANA~~ A map $A \to B$ is (f,b) with f map of gr alg's, $b \in B^1$ st $f(da) = d(fa) + [b, fa]$

$$f(h_A) = h_B + db + b^2$$

$$(g,b)(f,a) = (gf, b+ga)$$

In particular if $h_A = 0 = h_B$, $A \to B$ is same as a pair (f,b) with $f: A \to B^b$ a dga morphism.

Rmk also $(f,b) = \underbrace{(id, b)}_{\text{change of curvature iso}} \underbrace{(f,0)}_{\text{uncurved}}$

Def, $x \in A^1$ is MC if $h + dx + x^2 = 0$.

In this case $d + [x,-]$ is a diff^l on A & A^x is a dga.

Moreover $(id, x): A^x \to A$ is an iso.

Lem $\text{dga} \simeq \text{cAlg}_{K/}$ fully faithful

Proof Have an obvious inclusion $i: \text{dgA} \hookrightarrow \text{cAlg}_{K/}$

observe that $MC(A) \simeq \text{Hom}_{\text{cAlg}}(K,A)$, so i is ess. surj. since if A has MC elts then it's $\simeq$ a dga.

§2. Modules

$(A,d,h) \in cAlg$. A dg-module is a gr A-module M with a differential $\partial : M \to M$ with $\partial^2 = h$

Rmk A need not be a module over itself!!

Extended example if time

$\mathbb{Z}/2$-graded things. R a commutative alg, $f \in R$

Then ~~we~~ get a cAlg $A = (R, 0, f)$

A dg module is a pair M_0, M_1 of R-mods with $M_0 \underset{d}{\overset{d}{\rightleftarrows}} M_1$ & $d^2 = f$ (matrix facⁿs...)

A twisted module is one that looks like $A \otimes V$ after forgetting the diffˡ. [diffˡs $= MC(A \otimes \mathrm{End} V)$]

$dgMod_A$ is a dg cat in the usual way.

$H^0 dgMod_A$ is triangulated.

Def $D_c^{II}(A) := Loc_{H^0 dgMod_A}(H^0 Tw_{fg} A)$

the compactly gen'd derived cat of 2nd kind

Cpct objects $= (H^0 Tw_{fg} A)^{\omega}$

Fact this ↗ is a Morita fibrant replacement of A
when A a sc,

Ex

Back to matrix facⁿs:

$HMF(R,f) \simeq H^0 Tw_{fg}(R, 0, f)$

Buchweitz–Orlov: when R is smooth, these are

$D_{sg}(\cancel{A}) \simeq \dfrac{D^b \mathrm{mod}(R/f)}{\mathrm{perf}(R/f)}$

If A is a cofibrant dga then

$DA \simeq D_c^{II} A$.

In general, if A is a dga then

DA is $D_c^{II} A[q.iso]^{-1}$

§3. Coalgebras

One can define a pseudocompact curved algebra in the same way & $\mathrm{cuCog} := \mathrm{pcAlg}^{op}$.

Def. A cCog (C, d, h) is a grcoalg C with a coderivation d & a functional $h: C \to k$ st

- $hd = 0$
- $d^2 x = h(x^1)x^2 - x^1 h(x^2)$ [Sweedler]

Morphisms defined as for cAlgs.

Similarly, $\mathrm{dgCog} \simeq \mathrm{cuCog}_{/k}$

§4. Comodules

a dg C-comodule is a gr-Comod M with $\partial: M \to M$ satisfying $\partial^2 = hp$

$\mathrm{Tw}\,C$ is a dg cat & $D^{II}C := H^0 \mathrm{Tw}\,C$

(Rmk: no Tw_{fg} + closure needed as every comod is filt. colim. of fd comods)

Rmk a curved coalgebra C has a cobar construction $\Omega C \in \mathrm{cAlg}$, & $D^{II}C \simeq D_c^{II}\Omega C$

When C is coaugmented then $\Omega C \in \mathrm{Alg}$ & $D^{II}C \simeq D(\Omega C)$

Conversely a $A \in \mathrm{Alg}$ has $BA \in$ conil. curved. Cog & $D^{II}BA \simeq DA$