Dissections: How to cut things up

Matt Booth

PG Colloquium, University of Edinburgh

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What is a dissection?

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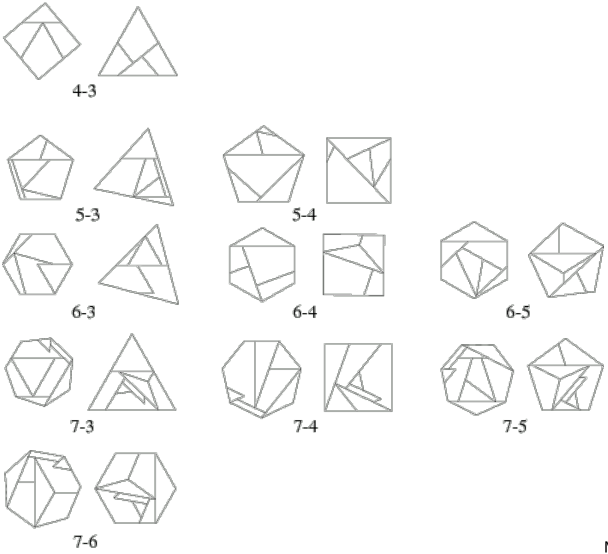

MathWorld

Scissors-congruence

Definition

Two polygons $P, Q \subseteq \mathbb{R}^2$ are **congruent**, written $P \simeq Q$, if Q can be obtained from P by translations and reflections.

Definition

Two polygons P, Q are **scissors-congruent**, written $P \sim Q$, if they decompose as disjoint[a] unions of polygons $P = \bigcup_{i=0}^{n} P_i$ and $Q = \bigcup_{i=0}^{n} Q_i$ with each $P_i \simeq Q_i$.

[a] up to boundary, i.e. area$(P_i \cap P_j) = 0$ for $i \neq j$

Scissors-congruence

- Note that we allow polygonal cuts: the definition is the same if we allow only straight-line cuts, but the minimal number of cuts needed to dissect one shape into another may change.

- Scissors-congruence is an equivalence relation! To see transitivity, superimpose cutting patterns.

The WBG theorem

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- Clearly if $P \sim Q$ then they have the same area.
- The **Wallace-Bolyai-Gerwien theorem** (1807, 1833, 1835) says that the converse is also true: if two polygons have the same area, then they're scissors-congruent.

The WBG theorem

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Any polygon P is scissors-congruent to a square of the same area.

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First cut P up into triangles. Then we'll show that:

- A triangle is scissors-congruent to a parallelogram with the same base (and half the height)
- Two parallelograms of the same base and height are scissors-congruent
- Two squares are scissors-congruent to one big square

It's possible to give 'Euclid-style' proofs of the above, but we'll give a more modern proof using group actions on $\mathbb{R}^2$.

Pak's proof

Definition

Let G be a group acting on $\mathbb{R}^2$. A **fundamental domain** for G is a set $X \subseteq \mathbb{R}^2$ containing exactly one element from every orbit of G.

Lemma

Let G be a discrete group of isometries of $\mathbb{R}^2$, and suppose that P, Q are polygons that are (the closures of) fundamental domains for G. Then P and Q are scissors-congruent.

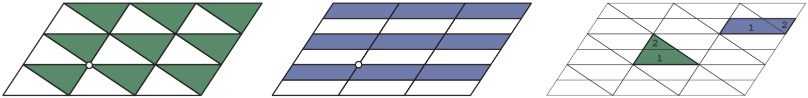
Pak's proof

Proof of the Lemma

Since Q is a fundamental domain, the translates $\{gQ : g \in G\}$ tile the plane. Since G is discrete, only finitely many of the gQ intersect P nonemptily; write $P_i = P \cap g_i Q$ for these, so that $P = \bigcup_i P_i$. Set $Q_i := g_i^{-1} P_i$. Clearly $P_i \simeq Q_i$, and since P is a fundamental domain, $Q = \bigcup_i Q_i$ and the Q_i are pairwise disjoint. Hence $P \sim Q$.

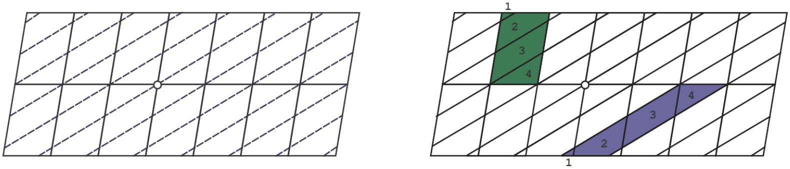
Triangles into parallelograms

$G = \mathbb{Z}^2$, acting by translations, extended by a copy of $\mathbb{Z}/2\mathbb{Z}$ acting by reflection in the origin; both the white triangles and the white parallelograms are fundamental domains.



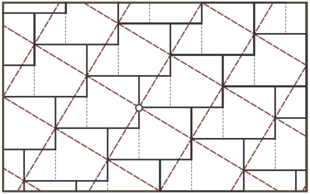
Parallelograms into parallelograms

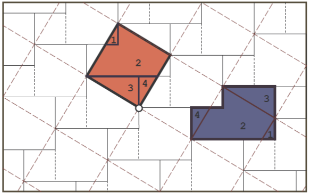
$G = \mathbb{Z}^2$, acting by translations.



Squares into squares

$G = \mathbb{Z}^2$, acting by translations along the red axes.





More general decompositions

Definition

Two sets $X, Y \subseteq \mathbb{R}^n$ are **equidecomposable** if they decompose as disjoint unions $X = \bigcup_{i=0}^{k} X_i$ and $Y = \bigcup_{i=0}^{k} Y_i$, and there exist isometries $f_1, \ldots, f_k$ of $\mathbb{R}^n$ such that $f_i(X_i) = Y_i$.

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- (Tarksi, 1924) For $n = 2$, any two polygons of equal area are equidecomposable.
- (Banach-Tarski, 1924) If $n \geq 3$, any two bounded sets with nonempty interior are equidecomposable (not true if $n \leq 2$).

Tarski's circle-squaring problem

- **Tarski's circle-squaring problem, 1925**: are a circle and a square of unit area equidecomposable?
- (Dubins, Hirsch, Karush, 1963) A unit circle and a unit square are not scissors-congruent. In fact, they're not equidecomposable if the pieces have Jordan curve boundary.

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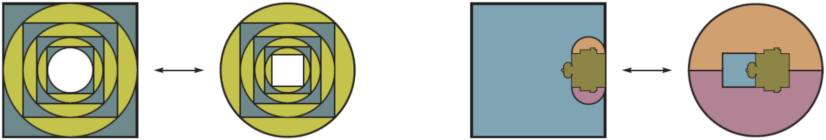
- (Dubins, Hirsch, Karush, 1963) A unit circle and a unit square are not scissors-congruent. In fact, they're not equidecomposable if the pieces have Jordan curve boundary.
- (Laczkovich 1990) The answer is yes! The proof uses about 10^{50} pieces, but they may not be (Lebesgue) measurable.
- (Grabowski, Máthé, Pikhurko, 2017) One can carry out the decomposition with measurable pieces.

Tarski's circle-squaring problem

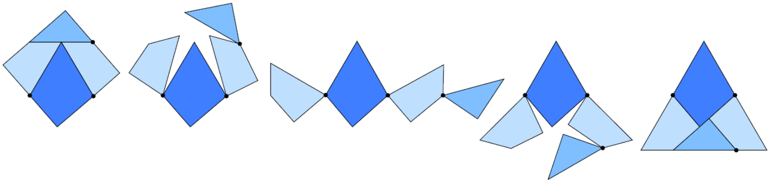
If one is allowed to use homotheties, there are much nicer solutions:



Pak

Hinged dissections

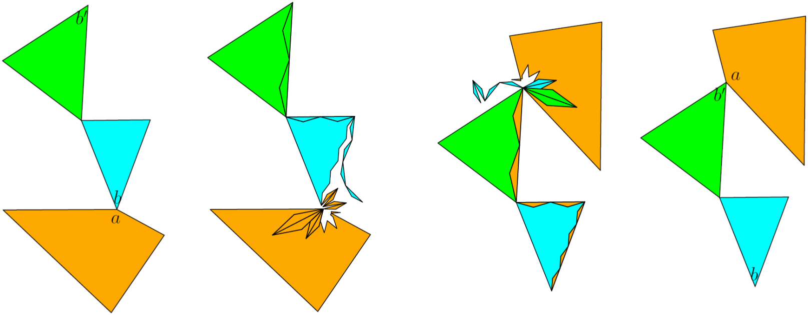
One can also consider dissections with hinges, such as Dudeney's famous 1902 dissection of a square into a triangle:



AACDDK

Hinged dissections

In fact any scissors-congruence dissection can be ‘hingified’ by adding chains of triangles to move pieces around (AACDDK 2012).



AACDDK

Hilbert's third problem

- Euclid knew that the volume of a tetrahedron is $\frac{1}{3}$(base) $\times$ (height), but all known proofs use (some form of) calculus. Is there a scissors-congruence between a tetrahedron of unit volume and a cube of unit volume?

Hilbert's third problem

- Euclid knew that the volume of a tetrahedron is $\frac{1}{3}(\text{base}) \times (\text{height})$, but all known proofs use (some form of) calculus. Is there a scissors-congruence between a tetrahedron of unit volume and a cube of unit volume?
- Unlike in the 2d case, the answer is no. This was proved by Max Dehn in 1900, and was the first of Hilbert's 23 problems to be solved. The idea of the proof is to define a new invariant of polyhedra.

Valuations

Definition

Let ϕ be a function from the set of convex polyhedra in $\mathbb{R}^3$ to some abelian group. Then ϕ is a **valuation** if it satisfies $\phi(P_1 \cup P_2) = \phi(P_1) + \phi(P_2)$ for disjoint (up to boundaries) P_1, P_2.

Definition

A valuation is **symmetric** if it's invariant under rigid motions (rotations and translations).

Volume is an example of a symmetric valuation (convex polyhedra) $\rightarrow \mathbb{R}$.

Valuations

Proposition

Let P_1, P_2 be two scissors-congruent convex polyhedra. Let ϕ be any symmetric valuation. Then $\phi(P_1) = \phi(P_2)$.

Proof

Decompose $P_1 = \bigcup_{i=0}^{m} \Delta_i^1$ and $P_2 = \bigcup_{i=0}^{m} \Delta_i^2$ into tetrahedra with $\Delta_i^1 \simeq \Delta_i^2$. Then $\phi(P_1) = \sum_{i=0}^{m} \phi(\Delta_i^1) = \sum_{i=0}^{m} \phi(\Delta_i^2) = \phi(P_2)$.

So we want to find a symmetric valuation ϕ such that $\phi(\text{unit cube}) \neq \phi(\text{unit tetrahedron})$.

Total mean curvature

Definition

Let P be a convex polyhedron. If e is an edge of P, let ℓ_e be the length of e, and let θ_e be the dihedral angle at e. The **total mean curvature** of P is $H(P) := \frac{1}{2}\sum_e \ell_e \theta_e \in \mathbb{R}$.

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- H is almost a symmetric valuation: we have $H(P \cup Q) = H(P) + H(Q) - H(P \cap Q)$. So if $P \cap Q$ is a polygon, we have $H(P \cap Q) = \sum_e \ell_e \pi$.
- Dehn's idea is to take $\theta_e \mod \pi$ to make H into a symmetric valuation.

The Dehn invariant

- The Dehn invariant is going to take values in the infinite-dimensional real vector space

$$\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi)$$

- If you don't like tensor products: take a $\mathbb{Q}$-basis $\{\pi\} \cup B$ of $\mathbb{R}$. Then

$$\mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi) \cong \mathbb{R}^B$$

  via the map $\mathbb{R}^B \to \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\mathbb{Q}\pi)$ that sends a basis vector b to $1 \otimes b$.

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- Hence, a cube and a regular tetrahedron don't have the same Dehn invariant, and are not scissors-congruent.
- A theorem of Sydler says that volume and Dehn invariant are enough to characterise scissors-congruence.

References

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