Threefold Flops and the Contraction Algebra

Matt Booth

March 2017

April 2018: added some references, corrected a typo, made some minor rearrangements.

1 Basic birational geometry

I'll work with irreducible complex varieties.

Recall that a **rational map** $X \dashrightarrow Y$ is a morphism $U \to Y$, where U is an open subset of X, and that a **birational map** is a rational map with rational inverse. So two varieties are birational if and only if they're isomorphic outside of lower-dimensional subsets.

Recall that the **function field** $K(X)$ of a variety X is the local ring of its generic point; equivalently $K(X)$ is the field of fractions of $\mathcal{O}_X(U)$ for any open affine subset U of X. It's clear that the function field is a birational invariant, and moreover two varieties are birational if and only if they have the same function field.

Example 1.1. Projective n-space $\mathbb{P}^n$ is birational to $\mathbb{A}^n$ via the usual coordinate map $\mathbb{P}^n \dashrightarrow \mathbb{A}^n$ sending $[x_0 : x_1 : \cdots : x_n]$ to $(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$.

Proposition 1.2. *Every variety is birational to a projective variety.*

Proof. In fact, every n-dimensional variety is birational to a hypersurface in $\mathbb{P}^{n+1}$. See [Har77], I.4.9. □

Let X be a variety and Z a subvariety of codimension $c > 0$. The **blowup** $\mathrm{Bl}_Z(X)$ of Z in X separates all of the lines pointing in different directions out of Z. More formally, if $\mathcal{I}$ is the ideal sheaf of Z, then the blowup is $\mathrm{Proj}(\oplus_n \mathcal{I}^n)$. The map $\pi : \mathrm{Bl}_Z(X) \to X$ is birational since it's an isomorphism outside of the exceptional divisor $E := \pi^{-1}(Z)$, which has positive codimension.

Example 1.3. If $X = \mathbb{A}^2$ and Z is the origin, then the blowup is the subvariety of $\mathbb{A}^2 \times \mathbb{P}^1$ given by the points $((x, y), [u : v])$ where $xu - yv = 0$, and the map to X is simply the projection onto the first factor. Topologically the blowup is $\mathbb{P}^2 \# \mathbb{P}^2$. More generally, if Z is a smooth point of X, then the blowup is just X but with Z replaced by a copy of the projectivised tangent space $E = \mathbb{P}T_Z X$.

A **monoidal transformation** is the operation of blowing up a single point. Monoidal transformations are useful for resolving singularities:

Example 1.4. Let X be the variety $\mathrm{Spec}(k[x, y, z]/(xy - z^2)) \subseteq \mathbb{A}^3$. One can think of X in a few different ways: as the affine cone over the smooth projective curve $xy - z^2 = 0$ in $\mathbb{P}^2$, or as the quotient of $\mathbb{A}^2$ by $\mathbb{Z}/2\mathbb{Z} \subseteq SL_2$, or simply as a singular surface in $\mathbb{A}^3$. The variety X has a unique singular point at the origin, and if we blow it up[1] we get a smooth variety $\tilde{X}$ with a map $\tilde{X} \to X$.

[1] Okay, really we need to take the strict transform of X inside the blowup $\mathrm{Bl}_0(\mathbb{A}^3)$. I'll be lax about these differences.

More generally, a proper birational map $\tilde{X} \to X$ is a **resolution** if $\tilde{X}$ is smooth. In characteristic zero, resolutions always exist: this is a famous theorem of Hironaka [Hir64]. A resolution $\tilde{X} \to X$ is a **minimal resolution** if any other resolution factors through it. Curves and surfaces always have minimal resolutions, but higher-dimensional varieties may not - this is closely related to the existence of flops, which we'll see more about later.

2 The Minimal Model Program

Goal. Classify all varieties up to isomorphism.

Problem. This is far too difficult.

New goal. Find nice varieties in each birational equivalence class.

This is hopefully easier. In view of 1.2 and Hironaka's theorem, we can restrict our search to smooth projective varieties. (In fact, this is too narrow a class of varieties to consider, but will be enough for curves and surfaces.)

For curves we already have an answer: two smooth projective curves are birational if and only if they are isomorphic. So the set of birational equivalence classes of curves is in bijection with the set of isomorphism classes of smooth projective curves.

Now let's think about surfaces:

Proposition 2.1. *Any birational map between surfaces factors as a finite zig-zag of monoidal transformations.*

Proof. Like other proofs in this section, can be found in [Har77], V.5. □

Remark 2.2. A version of this statement is in fact true in all dimensions; this result is known as the **weak factorisation theorem**. See [HM10], Theorem 1.11.

So one can try to construct smooth projective surfaces that are 'minimal' in a birational equivalence class by contracting curves whilst remaining smooth.

Definition 2.3. A (-1)**-curve** C in a surface X is one whose self-intersection number C^2 is -1.

I learned both of the following from the same excellent Stack Exchange question:

Remark 2.4 ([Ele]). What does it mean for a curve to have negative self-intersection? Intuitively, to define C^2, we move C to a curve C' in general position and define $C^2 = C \cdot C'$, which makes sense and yields a finite number. So if $C^2 = -1$, then such a curve C cannot be moved. This is because the degree of $\mathcal{N}_{C/X}$ is -1, hence it has no global sections, and hence C is rigid in X.

Example 2.5 ([Bra]). If $\pi : X \to \mathbb{P}^2$ is the blowup at a point p, then the exceptional divisor $E \cong \mathbb{P}^1$ is a (-1)-curve. To see this, compute $\pi^*\mathcal{O}(1) \cdot \pi^*\mathcal{O}(1)$ in two different ways. First, find distinct lines l_1 and l_2 in $\mathbb{P}^2$, not passing through p, both representing $\mathcal{O}(1)$. Since π is an isomorphism away from p we have $\pi^*\mathcal{O}(1) \cdot \pi^*\mathcal{O}(1) = l_1 \cdot l_2 = 1$. Alternately, find distinct lines L_1 and L_2 in $\mathbb{P}^2$, both passing through p, representing $\mathcal{O}(1)$. Then $\pi^*\mathcal{O}(1)$ is represented by both $L_1 + E$ and $L_2 + E$, where I use the same notation for the L_i and their strict transforms. So we have $1 = (L_1 + E) \cdot (L_2 + E) = 2 + E^2$, using the fact that L_1 and L_2 do not meet in X.

This example holds much more generally: in fact, the only way to get a (-1)-curve in a smooth surface is from a blowup.

Proposition 2.6 (Castelnuovo). *If $C \cong \mathbb{P}^1$ is a (-1)-curve in a smooth surface X, then it can be smoothly blown down: in other words there exists a smooth surface X_0 and a birational map $\pi : X \to X_0$ that contracts C to a point and is an isomorphism outside of C.*

Definition 2.7. A **minimal surface** is a smooth surface with no (-1)-curves.

Proposition 2.8. *Every surface X is birational to a minimal surface.*

Proof. By resolving if necessary, we can assume that X is smooth. Now if X has no (-1)-curves, we're done. If not, contract any (-1)-curve and repeat the procedure. We obtain a sequence $X = X_0 \to X_1 \to X_2 \to \cdots$ of smooth varieties, where each map is a contraction of a (-1)-curve. The Picard number[2] drops at each step, so this process must stop. □

Remark 2.9. If X is not rational or ruled, then the minimal surface of Proposition 2.8 is unique; this is a theorem of Zariski. Observe that $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are two minimal surfaces that are birational: if Y is the blowup of two points on $\mathbb{P}^2$, then blowing down the strict transform of the line joining them gets us $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 2.10. A non-minimal surface can have infinitely many (-1)-curves; if one takes two generic smooth cubics in $\mathbb{P}^2$ and blows up their intersection points, the resulting surface has infinitely many (-1)-curves. This is Remark 1.5 of [CCJ+05].

So every surface is birational to a (usually unique) minimal surface. What about higher dimensions? Unfortunately none of the above results really apply anymore. In particular we must give up hope that minimal models will be smooth. First, we want to find the correct definition of a minimal model in higher dimensions.

Definition 2.11. A $\mathbb{Q}$-Cartier divisor D on X is **nef** (numerically effective, or numerically eventually free) if $D \cdot C \geq 0$ for every curve C; i.e. the line bundle $\mathcal{O}(D)|_C$ on C has positive degree.

Remark 2.12. Beware that a nef divisor is not the same thing as a divisor numerically equivalent to an effective divisor.

Definition 2.13. A variety X is a **minimal model** if it has terminal $\mathbb{Q}$-factorial singularities[3] and the divisor K_X is nef.

Remark 2.14. A minimal model of a surface is a minimal surface. Conversely, a minimal surface is a minimal model as long as it's not rational or ruled - see [CCJ+05], 1.10.

Remark 2.15. Really, when talking about minimal models for a variety X, I should make the assumption that the Kodaira dimension of X is nonnegative. Note that the rational or ruled surfaces are precisely those surfaces X with $\kappa(X) = -\infty$. So we're already having some issues.

So to produce a minimal model of a variety, we'd like to repeatedly contract curves that have negative intersection number with the canonical divisor. This process is the basic idea of the **minimal model program** - one also has to make some modifications ('flips') at each step to ensure things do not get 'too singular'.

3 Threefold flops

Let's restrict to thinking about threefolds. In this case it's been known since the late 1980s that the MMP 'works', i.e. provides every threefold with a minimal model. The minimal models will not in general be unique. Is there a way to pass between them? In fact, any two minimal models are connected by a sequence of special birational operations called **flops** – this is true in all dimensions by a theorem of Kawamata [Kaw08]. Here's the definition, from [HM10]:

[2]The rank of the **Neŕon-Severi group**, the finitely generated abelian group $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.
[3]The $\mathbb{Q}$-factorial condition is there to ensure that K_X is a Cartier divisor.

Definition 3.1. A birational morphism $\pi : \tilde{X} \to X$ is **small** if the exceptional locus is of codimension at least two (in other words, π does not contract a divisor). Small morphisms exist - the exceptional locus need not be a divisor!

Definition 3.2. A **flop** is a commutative diagram

$$\begin{array}{ccc} X^- & \overset{\phi}{\dashrightarrow} & X^+ \\ & {\scriptstyle \pi^+}\searrow \quad \swarrow{\scriptstyle \pi^-} & \\ & Y & \end{array}$$

where the $\pi^{\pm}$ are small birational morphisms, plus some technical conditions[4]. It's also common to refer to the birational map ϕ as the flop.

Intuitively, a flop is a certain kind of codimension two surgery operation. Indeed, X^- and X^+ must have the same dimension, and ϕ is an isomorphism in codimension one. So for threefolds, a flop is essentially the process of cutting out a number of rational curves C_i and replacing them with others. Even if we replace the C_i by themselves, the induced birational map ϕ will not be an isomorphism.

Example 3.3 (the Atiyah flop). Let Y be the quadric cone $xy - zw = 0$ in $\mathbb{A}^4$. Blowing up the origin gives us a birational morphism $X \to Y$ with exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. One can contract either of the copies of $\mathbb{P}^1$ by a projection to end up with two varieties X^- and X^+ connected by a flop.

Remark 3.4. One can think of the existence of flops as controlling the nonuniqueness of minimal models, and hence the nonexistence of minimal resolutions. The Atiyah flop gives an example of a variety with no minimal resolution: X^- and X^+ are two different small resolutions of Y, neither of which factors through the other.

Example 3.5. After an affine change of coordinates, the quadric Y featuring in the Atiyah flop can be written as $xy - (z+w)(z-w) = 0$. Reid in [Rei83] generalised this example to the **pagoda flop**, where we now consider $xy - (z + w^n)(z - w^n) = 0$.

4 The Contraction Theorem

Our main reference from now on will be [DW15] and [DW16], although [Wem18] is good for the big picture. Throughout, the setup will be that of a threefold contraction: a projective birational map $f : X \to X_{\text{con}}$ between (not necessarily smooth) threefolds, with at most one-dimensional fibres, satisfying some extra conditions[5] (loosely, that X and X_{con} are not too singular).

Question. Is f a flopping contraction? That is, is there a flop $X \dashrightarrow X^+$ over X_{con} which is the composition of f with another birational map?

Let L be the locus in X_{con} over which f is not an isomorphism. Then f is a flopping contraction if and only if it contracts curves without contracting a divisor, which is the case if and only if L is zero-dimensional.

To try and characterise L locally around a closed point $p \in L \subseteq X_{\text{con}}$, a natural idea is to look at the deformations of the curves above p. The preimage $f^{-1}(p)$ is set-theoretically a union of $\mathbb{P}^1$s. Assume for now that $f^{-1}(p)$ is a single curve C (we'll look at the multiple curves case later on, since it requires a little more technology). Giving C the reduced scheme structure, we see that $C \cong \mathbb{P}^1$. What deformation-theoretic framework is the right one to use to detect deformations of C?

[4] We require that the $\pi^{\pm}$ are of relative Picard number 1, and the $\omega_{X^{\pm}}$ are trivial over Y.

[5] In [DW15] it's required that $\mathbb{R}f_*\mathcal{O}_X \cong \mathcal{O}_{X_{\text{con}}}$ whereas in [Wem18] it's required that f is crepant and some mild assumptions on singularities are made.

Theorem 4.1. *There is a noncommutative* $\mathbb{C}$*-algebra* A_{con} *(depending on* f *and* p*), that prorepresents the functor* ncDef_X^C *of* ***noncommutative deformations*** *of the sheaf* $\mathcal{O}_C(-1)$*, such that*

$$f \text{ is a flopping contraction } \iff \dim_{\mathbb{C}} A_{\text{con}} < \infty$$

Corollary 4.2. *The morphism* f *is a flopping contraction if and only if* ncDef_X^C *is representable.*

What is a noncommutative deformation of a sheaf? Essentially, we pass through a derived equivalence to a noncommutative ring A, and deform the image of the sheaf $\mathcal{O}_C(-1)$ as an A-module. We'll recover A_{con} as a certain quotient of A.

For technical reasons[6], we need to work complete locally. So let $\text{Spec}(R)$ be a complete local affine neighbourhood of p, and base change f to a morphism $U \to \text{Spec}(R)$.

Theorem 4.3 ([VdB04], Theorem A)**.** *There is a tilting bundle* $V = \mathcal{O}_U \oplus \mathcal{N}$ *on* U *inducing a derived equivalence*

$$D^b(\text{Coh}(U)) \xrightarrow{\mathbb{R}\text{Hom}(V,-)} D^b(\text{End}_U(V))$$

By [VdB04], Lemma 4.2.1 we have an isomorphism $\text{End}_U(V) \cong \text{End}_R(f_*V)$, so we can work on the base $\text{Spec}(R)$. Put $N = f_*\mathcal{N}$, so that we have $\text{End}_R(f_*V) \cong \text{End}_R(R \oplus N)$. Let A be the basic algebra Morita equivalent to $\text{End}_R(R \oplus N)$; one can write it as $A \cong \text{End}_R(R \oplus M)$ for a certain module M. Define A_{con} to be $A/[R]$, where $[\mathcal{F}]$ denotes the ideal of maps factoring through sums of summands of $\mathcal{F}$. A priori, A_{con} may depend on the tilting bundle: Van den Bergh gives an explicit construction of V, but we would like A_{con} to be independent of this. We're in luck:

Theorem 4.4. *The algebra* A_{con} *is intrinsic to the contraction; i.e. depends only on the data of the map* $f : X \to X_{\text{con}}$*.*

Theorem 4.5. *If* S *is the image of* $\mathcal{O}_C(-1)$ *across the derived equivalence, then there is an isomorphism* $\text{ncDef}_X^C \cong \text{ncDef}_A^S$ *of deformation functors. Moreover, the algebra* A_{con} *prorepresents the above functor.*

Returning to 4.1, one proves that A_{con} is supported exactly on the locus $L \cap \text{Spec}(R)$. So the set $L \cap \text{Spec}(R)$ is zero-dimensional if and only if A_{con} is finite-dimensional over $\mathbb{C}$. Hence we get exactly the statement required.

Example 4.6. The Atiyah flop has contraction algebra $\mathbb{C}$. The pagoda flop has contraction algebra $\mathbb{C}[x]/x^n$.

5 Multiple curves

The situation is similar when we have multiple curves $C_1, \dots, C_n$ above the point p. As before, give them all the reduced scheme structure.

Question. Pick $I \subset \{1, \dots, n\}$. When does $C_I := \bigcup_{i \in I} C_i$ contract to a point without contracting a divisor?

[6] Zariski locally, A_{con} may fail to be a local ring, because N may be decomposable. See [DW16], §2.4. However, if we're willing to define A_{con} only up to Morita equivalence, then we can work Zariski locally.

The answer is similar to the $|I| = 1$ case. Van den Bergh's tilting bundle V now acquires more summands: it becomes $V = \mathcal{O}_U \oplus_{i=1}^n \mathcal{N}_i$. To get A_{con} from the ring $A = \text{End}_R(R \oplus_{i=1}^n M_i)$, one quotients out by the ideal $[R \oplus_{i \notin I} M_i]$.

Theorem 5.1. *The collection of curves C_I contracts to a point without contracting a divisor if and only if* $\dim_{\mathbb{C}} A_{\text{con}} < \infty$.

What deformation functor does A_{con} prorepresent? It turns out in the multiple curves case to prorepresent the functor of **simultaneous noncommutative deformations** of the curves $(C_i)_{i \in I}$. The test objects for this functor are the $|I|$**-pointed** noncommutative Artinian $\mathbb{C}$-algebras; such an algebra Γ is m-pointed if and only if it has precisely m simple modules, each one-dimensional over $\mathbb{C}$. Note that the sheaves $\mathcal{O}_{C_i}(-1)$ correspond across the derived equivalence to (one-dimensional) simple modules S_i over A, so informally we are deforming the $\mathcal{O}_{C_i}(-1)$ while keeping track of the Ext groups between them.

Example 5.2 ([DW15], 6.3). Let G be the alternating group A_4. The group G acts on $\mathbb{C}^4$ by permutations; there is an invariant subspace $\{(x, y, z, w) : x + y + z + w = 0\}$ which is an irreducible three-dimensional representation of G. Pick a basis $\{X, Y, Z\}$. Let R be $\mathbb{C}[\![X, Y, Z]\!]^G$ and take the crepant resolution $f : G - \text{Hilb} \to \text{Spec}(R)$. Then the fibres of f are at most one-dimensional, and the fibre above the origin is three curves that meet in an A_3 configuration. Contracting the middle curve C_2, we get an infinite-dimensional algebra A_{con} with the property that the abelianisation $A^{\text{ab}}_{\text{con}}$ is finite-dimensional. The commutative deformation functor of the sheaf $\mathcal{O}_{C_2}(-1)$ is prorepresented by $A^{\text{ab}}_{\text{con}}$; hence the representability of the commutative deformation functor does not detect divisorial contractions.

Example 5.3. Sticking with the previous example, if we choose to contract the outer curves C_1 and C_3, we still get get an infinite-dimensional algebra A_{con}. In this case, the unpointed noncommutative deformation functor of $C_1 \cup C_3$ is representable; we see that for more than one curve, representability of the unpointed noncommutative deformation functor does not detect divisorial contractions. Of course, when we're just dealing with one curve the two functors are the same (note that a 1-pointed Artinian $\mathbb{C}$-algebra is just a local Artinian $\mathbb{C}$-algebra with residue field $\mathbb{C}$).

Remark 5.4. One can extract a presentation for A out of the geometry of the curves C_i, in a similar manner to the McKay correspondence. In fact, A is the path algebra of a quiver Q which has exactly $n + 1$ vertices: n corresponding to the sheaves $\mathcal{O}_{C_i}(-1)$ and an extra corresponding to R. One links up vertices if the corresponding curves intersect, and there are some rules dictating where to place loops on vertices (corresponding to Ext groups) and how the vertex R connects to the others. To get the quotient A_{con}, one simply removes all vertices not in I.

References

[Bra] Lucas Braune. geometric motivation for negative self-intersection. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/787715 (version: 2016-06-03).

[CCJ+05] Charles Cadman, Izzet Coskun, Kelly Jabbusch, Michael Joyce, Sándor J. Kovács, Max Lieblich, Fumitoshi Sato, Matt Szczesny, and Jing Zhang. A first glimpse at the minimal model program. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 17–42. Amer. Math. Soc., Providence, RI, 2005.

[DW15] Will Donovan and Michael Wemyss. Contractions and deformations. *arXiv preprint arXiv:1511.00406*, 2015.

[DW16] Will Donovan and Michael Wemyss. Noncommutative deformations and flops. *Duke Math. J.*, 165(8):1397–1474, 2016.

[Ele] Georges Elencwajg. geometric motivation for negative self-intersection. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/51196 (version: 2011-07-13).

[Har77] Robin Hartshorne. *Algebraic Geometry.* Number 52 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1977.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[HM10] Christopher D. Hacon and James McKernan. Flips and flops. In *Proceedings of the International Congress of Mathematicians. Volume II*, pages 513–539. Hindustan Book Agency, New Delhi, 2010.

[Kaw08] Yujiro Kawamata. Flops connect minimal models. *Publ. Res. Inst. Math. Sci.*, 44(2):419–423, 2008.

[Rei83] Miles Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 131–180. North-Holland, Amsterdam, 1983.

[VdB04] Michel Van den Bergh. Three-dimensional flops and noncommutative rings. *Duke Math. J.*, 122(3):423–455, 2004.

[Wem18] Michael Wemyss. Flops and clusters in the homological minimal model programme. *Invent. Math.*, 211(2):435–521, 2018.