Work over $\mathbb{C}$ (can probably get away with k algebraically closed, characteristic zero – any issues will be with the birational geometry).

1 The contraction algebra

- Let's say we have a projective birational morphism $f : X \to X_{\text{con}}$ with X and X_{con} noetherian, normal, integral threefolds. Let's also say that they have Gorenstein terminal singularities; these are natural types of singularities to consider when running the MMP. are Say that f is a flopping contraction if it's small (doesn't contract divisors), K_X is trivial over the base, and it's of relative Picard number 1 (the relative Picard number is the difference between the Picard numbers; the Picard number is the rank of the Picard group). The upshot is that f is an isomorphism away from a finite set of points on X_{con}, above which are trees of rational curves. In this situation, the flop X^+ of X exists: informally, one cuts out the exceptional curves and sews them back in with the opposite orientation. X^+ comes with a flopping contraction to X_{con}, and is birational to X over the base. So one should think of flops as some kind of codimension two birational surgery. MMP types care: flops link all minimal models of a given variety (in all dimensions this is a theorem of Kawamata; I believe for threefolds the first unconditional proof is due to Kollár).

- A theorem of Van den Bergh says that if the base $X_{\text{con}} = \operatorname{Spec} R$ is affine, then there exists a tilting bundle $\mathcal{V} = \mathcal{O}_X \oplus \mathcal{M}$ on X. Put $A := \operatorname{End}_X \mathcal{V}$ – it's a noncommutative ring. 'Tilting' basically means that $\mathcal{V}$ comes with an equivalence of categories $\mathbb{R}\operatorname{Hom}_X(\mathcal{V}, -) : D^b(\operatorname{Coh}(X)) \to D^b(\operatorname{mod} - A)$. In the spirit of noncommutative derived geometry, think of A as a noncommutative model for X.

- In fact, one can compute A on the base as $\operatorname{End}_R(R \oplus M)$ for some (maximal Cohen-Macaulay) module $M = f_*\mathcal{M}$ (snappy definition: a module is MCM if the natural map $\mathbb{R}\operatorname{Hom}_R(M, R) \to \operatorname{Hom}_R(M, R)$ is a quasi-isomorphism). A comes with an idempotent $e = \operatorname{id}_R$, and we have $A/AeA \cong \underline{\operatorname{End}}_R(M)$ the stable endomorphism algebra, essentially by definition.

- If we vary our affine neighbourhoods, then M may change, and so A/AeA may change. However, its Morita equivalence class will remain the same. In fact if R is small enough (by which I mean complete local) then one can fix it so that A/AeA is a finite dimensional algebra with $\#\{\text{curves above } p\}$ one-dimensional simple modules (recall that if B is a finite-dimensional $\mathbb{C}$-algebra, then $B/\operatorname{rad}(B)$ is a copy of $\mathbb{C}^n$. As a B-module, this splits as a direct sum of n pairwise non-isomorphic one-dimensional simple modules, and these are all of its 1dim. simples.). So for example if f contracts just one irreducible curve, then this algebra will have exactly one such simple module, which is equivalent to being Artinian local. More generally, the terminology is Artinian n-pointed (so 1-pointed means local). Donovan-Wemyss define the **contraction algebra** A_{con} to be this fin. dim. alg. A/AeA.

- Why do I care so much about these simples? Suppose there are n irreducible rational curves $C_1, \ldots, C_n$ above p. They're each copies of $\mathbb{P}^1$, so one can consider the twists $\mathcal{O}_{C_i}(-1)$ as sheaves on X. In fact, these sheaves are all simple. Moreover, across the derived equivalence, they map to the n distinct one-dimensional simple A-modules $S_1, \ldots, S_n$.

2 Good properties for threefolds

- A_{con} recovers all known invariants of the flop (e.g. normal bundles, width (Reid), Gopakumar-Vafa invariants, anything else you can name). Donovan and Wemyss conjecture that it classifies X_{con} complete locally around p. In general it's a noncommutative algebra!

- Derived categories are closely linked to birational geometry. For example, flops induce derived equivalences: given a variety X and its flop X^+, one obtains an induced derived equivalence $D^b(X) \to D^b(X^+)$. When X is smooth, this is a theorem of Bridgeland, and when X is allowed to be singular, this is a theorem of Chen. The equivalence is a Fourier-Mukai transform: loosely, one takes a complex on X, pulls it back to the product, twists, and pushes down to X^+. Now, flops are symmetric: if X^+ is the flop of X, then X is the flop of X^+. This gives us a flop-flop-autoequivalence of $D^b(X)$.

One can translate this into the algebraic setting and obtain an autoequivalence FF of $D^b(A)$. It's not immediately obvious that FF is nontrivial, but one can check that $FF(S_i) \cong S_i[2]$, which is clearly not isomorphic to S_i. (Equivalently in the geometric world $FF(\mathcal{O}_{C_i}(-1)) \cong \mathcal{O}_{C_i}(-1)[2]$). Donovan-Wemyss prove that FF is representable (in a derived sense; i.e. by a complex, and we have to use derived hom). In fact, they prove that the kernel $\ker(A \to A_{\text{con}})$, considered as a complex in degree zero, represents FF. So the contraction algebra controls the flop-flop autoequivalence.

3 Bad properties for sufaces

- Let's see an example. Look at the quadric cone $C = \{xy = zw\}$ inside $\mathbb{A}^4$. Blow up the singular point to obtain a birational morphism $C' \to C$ with exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$. One can contract the left-hand $\mathbb{P}^1$ to obtain a birational morphism $X \to C$, and similarly one can contract the right-hand $\mathbb{P}^1$ to obtain $Y \to C$. In fact X is birational to Y over C, and Y is the flop of X. (this is the Atiyah flop). One can compute the contraction algebra in this example to be $\mathbb{C}$.

- Now cut the whole picture along $x = y^n$, to get a partial resolution of an A_n singularity (i.e. $\operatorname{Spec} \mathbb{C}[u,v,t]/(uv - t^{n+1})$). It's a full resolution if $n = 1$. In fact, one can also think about $n = 0$, but the picture is different since the cut is already smooth. One can compute $A_{\text{con}} = \mathbb{C}$ downstairs: this can either be done directly or using that it behaves well with respect to cuts. The upshot is that it's always $\mathbb{C}$, no matter what n one chooses.

- So in this case the Donovan-Wemyss conjecture fails horribly: A_n singularities are certainly not all complete locally isomorphic near the singular point. But they all have the same contraction algebra. So it's not a very good invariant in this case. Moreover, A_{con} no longer controls FF; one can do a computation to check that FF is not representable by a module (in fact it's represented by a 2-term complex).

- Reid's general elephant principle tells us that if we start with a threefold flopping contraction, and take a generic cut, the surface we get is a partial resolution of a Kleinian singularity. The question behind my PhD: is there a different – but similarly defined – invariant that works better in this setup?

4 The derived contraction algebra

- Braun, Chuang, and Lazarev have a general homotopical construction called the **derived quotient**. Given an algebra A and an idempotent e, this produces a dga (differential graded algebra), $A/^{\mathbb{L}}AeA$. It's nonpositively cohomologically graded, and has $H^0(A/^{\mathbb{L}}AeA) = A/AeA$. I define the derived contraction algebra $A_{\text{con}}^{\text{der}}$ to be $A/^{\mathbb{L}}AeA$. This has shown up in papers before: Kalck-Yang on relative singularity categories, Hua-Zhou on noncommutative Mather-Yau, and others. It's possible to show that each H^i is finite-dimensional. $A_{\text{con}}^{\text{der}}$ is never A_{con}; it's always bigger.

- What's the point of this definition? I'll try to give you three reasons: singularity categories, a relation with FF, and a deformation-theoretic approach.

5 Singularity categories

- Let R be a commutative k-algebra. A perfect complex in $D^b(R)$ is a complex quasi-isomorphic to a bounded complex of finitely generated projective modules. A famous theorem of Serre says that if R is smooth, then every object in $D^b(R)$ is perfect. With this in mind, the **singularity category** of R is the Verdier (or Drinfeld) quotient $D_{\text{sg}}(R) := D^b(R)/\mathbf{per}(R)$. So it's a triangulated (or dg) category, such that if R is smooth then $D_{\text{sg}}(R)$ vanishes. It records information about the type of singularities of R. Now if R is Gorenstein, a theorem of Buchweitz from the 80s tells us that $D_{\text{sg}}(R)$ is the stable category $\underline{CM}(R)$ of MCM modules: the objects are the MCM modules, and we kill any morphism that factors through a free module (hence, through any projective module). In particular, projective objects are zero in the stable category. If in addition R is a hypersurface, a theorem of Eisenbud (again from the 80s) says that $\Sigma^2 = \text{id}$, where Σ is the shift functor on $D_{\text{sg}}(R)$.

- Now let R be as above (so the base of a threefold flop, or a generic cut of such a threefold to a Kleinian singularity). Let M be the MCM module defining A. Then by definition we have $A_{\text{con}} = \underline{\text{End}}(M) = \text{End}_{D_{\text{sg}}(R)}(M)$. So the contraction algebra knows a little bit about the singaularity category.

- Theorem. (B.): Suppose that X is smooth. Then $A^{\text{der}}_{\text{con}}$ and the dg-category $D_{\text{sg}}(R)$ determine each other. A similar statement is true if X is not smooth: in general $A^{\text{der}}_{\text{con}}$ only recovers the part of the singularity category that M sees (i.e. $\mathbf{thick}_{D_{\text{sg}}(R)}(M)$).

6 The flop-flop autoequivalence

- 2-periodicity in the singularity category (recall Eisenbud's theorem) gives a central cocycle $\eta \in A^{\text{der}}_{\text{con}}$ of degree -2, which you can think of as some sort of periodicity element. Now, one can look at the quotient $A^{\text{der}}_{\text{con}}/\eta$ (more formally, take the cone of $\eta : A^{\text{der}}_{\text{con}} \to A^{\text{der}}_{\text{con}}$), which turns out to be a two-term dga. The idea is that this quotient $A^{\text{der}}_{\text{con}}/\eta$ records the unstable derived endomorphisms of $M \in D_{\text{sg}}(R)$. Put $J := \text{cone}(A \to A^{\text{der}}_{\text{con}}/\eta)$; again it's a two-term complex.

- If we're in the threefold setup, and X is a minimal model of R, then $A^{\text{der}}_{\text{con}}/\eta$ is actually just A_{con} (in other words, the degree -1 part vanishes). Hence, J is just the kernel of $A \to A_{\text{con}}$, and so J represents FF, by Donovan and Wemyss' theorem. Moreover, in the specific A_n singularity setup described earlier, I can prove that J also represents FF (the computation is long and pretty difficult).

- Of course, a natural guess to make is that $A^{\text{der}}_{\text{con}}/\eta$ always controls FF (in the same sense that A_{con} controls FF for threefolds).

7 Deformation theory

- There ought to be another interpretation of $A^{\text{der}}_{\text{con}}$ in terms of deformation theory. When doing deformation theory one cares about Artinian local algebras with residue field $\mathbb{C}$: geometrically, if Γ is such an algebra, then $\text{Spec}\,\Gamma$ is a fat point (i.e. a point together with some infinitesimal nonreduced fuzz). These are the right types of thing to give you infinitesimal information: e.g. the dual numbers $\mathbb{C}[\epsilon] := \mathbb{C}[x]/x^2$ are such an algebra, and if X is a $\mathbb{C}$-variety then a map $\text{Spec}\,\mathbb{C}[\epsilon] \to X$ is nothing more than a point together with a tangent vector at that point (this is an early exercise in Hartshorne). An infinitesimal deformation of a $\mathbb{C}$-variety X over Γ is defined to be a Γ-variety $\mathcal{X}$ together with an identification of the fibre over the closed point with X. In other words, we thicken X up infinitesimally along Γ – one can always take the trivial deformation $\mathcal{X} = X \times_{\mathbb{C}} \Gamma$.

- Deformations pull back along maps of Artinian local algebras, so one can fix an X and consider the assignment $\Gamma \mapsto \{\text{deformations of } X \text{ over } \Gamma\}$ as a functor. In general, it's not representable by an Artinian local algebra, but in many situations of geometric interest it's prorepresentable, which one can think of as meaning representable by a complete local Noetherian algebra (it equivalently means that it's a filtered colimit of representables; to get the complete local algebra take the inverse limit of the associated system of Artinian algebras).

- One can also do noncommutative or derived or pointed (or etc...) deformation theory by modifying the definition of Artinian local algebra accordingly (the formal setup is the same). The point is this: a theorem of Donovan-Wemyss says that A_{con} prorepresents the noncommutative deformation functor of the flopping curves. But of course, A_{con} is already Artinian – so the functor is actually representable – so why did I bother to introduce prorepresentability?

- Because I expect $A^{\text{der}}_{\text{con}}$ to prorepresent the derived noncommutative deformation functor, and $A^{\text{der}}_{\text{con}}$ is not an Artinian dga. This also gives a new way to do computations – in fact really the only way I know to effectively do computations – and provides local-to-global results (it would imply that $A^{\text{der}}_{\text{con}}$ is determined by $\mathbb{R}\,\text{End}_X(\oplus_i \mathcal{O}_{C_i}(-1))$. For experts: the usual Koszul duality args don't work, and the proof will be quite subtle. When I submitted my abstract I thought I had a proof of this, which turned out not to work. (I am pretty confident that it is true!)

- Assuming that the above holds, then for the A_n singularities the deformation theory computations show that $A_{\text{con}}^{\text{der}}$ is a finer invariant than A_{con}: as an A_∞-algebra it's $\mathbb{C}[\eta]\langle\zeta\rangle$ with a single Massey product $\langle\zeta^{\otimes(n+1)}\rangle = \eta^n$, and all others involving ζ are zero (in particular ζ is square-zero unless $n = 1$). This description also works if $n = 0$.