Derived categories

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1 The derived category of a ring

Definition 1.1. A complex (of vector spaces) is a $\mathbb{Z}$-graded sequence of vector spaces V^i together with differentials $d^i : V^i \to V^{i+1}$ satisfying $d^2 = 0$. We can play the same game with abelian groups, or with modules over a ring, ...

I'm using cohomological indexing here. Complexes are useful in geometry and topology (e.g. the de Rham complex of a smooth manifold or the singular chain complex of a topological space). They're designed to be objects that we can take cohomology of:

Definition 1.2. If V is a complex, then its i^{th} cohomology group is the space $H^i(V) \coloneqq \frac{\ker d^i}{\operatorname{im} d^{i-1}}$ (so it's a subquotient of V^i).

Examples: singular homology, de Rham cohomology. Motivation: 'complexes good, cohomology bad' [Tho01].

Example 1.3. Suppose we have some topological space whose homology is computed by the complex $V \coloneqq \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ with the right-hand $\mathbb{Z}$ in degree zero. You can check that $H^*(V)$ is concentrated in degree zero, where it's $\mathbb{Z}/2$. To compute the singular cohomology of V, you take the cohomology of the dual complex $V^\vee \coloneqq \operatorname{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{2} \mathbb{Z}$ where this time the right-hand $\mathbb{Z}$ lives in degree one. We get $H^1(V^\vee) \cong \mathbb{Z}/2$. If we'd just taken the dual of $H(V)$, we'd have killed all the torsion and wouldn't get the right answer.

Since we mainly care about complexes as vehicles to compute cohomology, the following is a natural definition to make:

Definition 1.4. A morphism of two complexes $f : V \to W$ is a set of maps $f^i : V^i \to W^i$ commuting with the differentials. We get induced morphisms on cohomology. Say that f is a quasi-isomorphism if, for all i, the induced map $H^i f : H^i V \to H^i W$ is an isomorphism.

Example 1.5. The complex $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ from earlier is quasi-isomorphic to the complex $\mathbb{Z}/2$ (i.e. the group $\mathbb{Z}/2$ placed in degree zero, with no differential) via the obvious map $\mathbb{Z} \to \mathbb{Z}/2$. Idea: $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ is a complex of nice things (free abelian groups) which 'resolves' a nastier thing (a torsion group).

Remark 1.6. Beware! Two complexes being quasi-isomorphic is a stronger condition than them having the same cohomology (just like two spaces with isomorphic cohomology groups don't have to be homotopy equivalent).

Remark 1.7. Over a field k, every complex is quasi-isomorphic to its cohomology as, by choosing sections, one can build a chain map $V \to H(V)$ which is the identity on cohomology. This is not true in general! If R is a k-algebra then a complex of R-modules always admits a k-linear map to its cohomology, but this may fail to be R-linear.

Definition 1.8. Let R be a ring. Its bounded derived category is the category $D^b(R)$ with objects complexes of finitely generated R-modules with bounded cohomology. We identify two complexes when they are quasi-isomorphic. The morphisms are given by equivalence classes of chain maps; we'll see a concrete description of them soon.

Remark 1.9. More formally, $D^b(R)$ is the localisation of the category of (cohomologically bounded, finitely generated in each degree) chain complexes at the quasi-isomorphisms.

Remark 1.10. The derived category contains a shift functor: let $X[i]$ denote 'X shifted to the left by i places' (i.e. $X[i]^n = X^{i+n}$). Clearly we have $X[i][j] \cong X[i+j]$.

The derived category has a bunch of extra structure. Firstly, it's triangulated, meaning essentially that one can take mapping cones in a reasonable manner. We won't explore this futher. Moreover, it has a natural enhancement to a dg category, where one has morphism complexes between objects. We won't think about this in detail; we'll content ourselves with thinking about the homotopy category, where one has graded spaces of morphisms (but no differentials) given by taking cohomology of the morphism complexes.

A good motivational reference for derived categories is [Tho01]. For the keen, [Wei94] is a comprehensive textbook on homological algebra.

2 Ext functors and mapping spaces

How do we actually work out the graded morphism spaces in the derived category? Let R be a ring. Recall that a projective module is a summand of a free module (e.g. $\mathbb{Z}/2$ is a projective but not free $\mathbb{Z}/6$-module). If M is a finitely generated R-module, then a projective resolution for M is a complex P of finitely generated projectives with cohomology concentrated in degree 0, where it is $H^0P \cong M$. Necessarily P is then quasi-isomorphic to M. Example: $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ resolves $\mathbb{Z}/2$.

Definition 2.1. If V and W are two complexes, a degree j map takes $V^i \to W^{i+j}$. If $\operatorname{Hom}^j(V,W)$ denotes the set of degree j maps, then the set of maps between the two underlying modules is $\operatorname{Hom}(V,W) = \prod_j \operatorname{Hom}^j(V,W)$ (if both of your complexes are bounded you can use the sum instead). This becomes a complex upon setting $\partial f = f d_V \pm d_W f$ where the sign depends on the degree of f.

Definition 2.2. Let M and N be two modules. Let $P \to M$ be a projective resolution. The i^{th} Ext group is $\operatorname{Ext}^i(M,N) := H^i(\operatorname{Hom}(P,N))$.

One can check that Ext extends to a well-defined bifunctor on the derived category, although I've only told you how to compute it for modules. Fact: if M and N are modules then $\operatorname{Ext}^0(M,N) \cong \operatorname{Hom}(M,N)$, but this isn't true for complexes in general. If $M = N$ then the complex $\operatorname{Hom}(P,P)$ becomes a differential graded algebra, and this makes $\operatorname{Ext}^*(M,M)$ into an algebra. The following is a nontrivial theorem:

Theorem 2.3. *Let M and N be two modules. Then there is an isomorphism $\operatorname{Hom}^*_{D(R)}(M,N) \cong \operatorname{Ext}^*(M,N)$. Moreover, if $M = N$ then this is an isomorphism of graded algebras.*

This is valid for all complexes, not just modules.

Remark 2.4. If you forget that the derived category has graded hom-spaces, you get $\operatorname{Hom}_{D(R)}(M,N) \cong \operatorname{Ext}^0(M,N)$. It is not hard to see that using this convention, there are isomorphisms $\operatorname{Hom}_{D(R)}(M,N[i]) \cong \operatorname{Ext}^i(M,N)$. You sometimes see this formulation as a definition of Ext.

Morally, what's going on is that there's some kind of 'total derived hom' functor $\mathbb{R}\operatorname{Hom} : D(R) \times D(R) \to D(\mathbb{Z})$ that has complexes up to quasi-isomorphism as both input and output, and Ext is its cohomology. This is closely related to the dg-category structure on $D(R)$: the mapping complexes are given by (particular models for) $\mathbb{R}\operatorname{Hom}$. To go from the dg derived category to $D(R)$, one takes H^* of the morphism complexes (this is a special case of the general construction of the homotopy category of a dg category). If you are keen to learn more about dg categories you can try [Kel06], or [Toë11] if you know a bit of homotopy theory.

Example 2.5. Ext functors show up in real life! You can check that $\operatorname{Ext}^1(M,N)$ is in bijection with the set of (equivalence classes of) 1-extensions of M by N; a.k.a. short exact sequences

$$0 \to N \to X \to M \to 0.$$

In general, $\operatorname{Ext}^n(M,N)$ is in bijection with (equivalence classes of) n-extensions

$$0 \ \to N \to X_1 \to \cdots \to X_n \to M.$$

The compositions $\operatorname{Ext}^n \times \operatorname{Ext}^m \to \operatorname{Ext}^{n+m}$ correspond to the Yoneda product: essentially, one splices together n-extensions and m-extensions to get $(n+m)$-extensions.

Example 2.6. If you know about Hochschild cohomology: if R is a k-algebra, then $HH^n(R,M) \cong \operatorname{Ext}^n_{R\otimes R^{\text{op}}}(R,M)$, because the bar resolution of R resolves R as an R-bimodule.

3 Computations in the algebraic world

To do computations, we need to be able to build resolutions. In this section, we'll do an extended example. Let $R := k[x_1, \dots, x_n]$ be the coordinate ring of affine n-space $\mathbb{A}^n_k$. Put $K^p := \bigwedge^p_R R^{\oplus n}$, the p^{th} exterior product of $R^{\oplus n}$ with itself. Define a differential

$$d(e_{i_1} \wedge \cdots \wedge e_{i_p}) := \sum_k (-1)^{k+1} x_{i_k} e_{i_1} \wedge \cdots \hat{e_{i_k}} \wedge \cdots \wedge e_{i_p} \wedge e_{i_p}$$

where $\hat{e_{i_k}}$ means we omit e_{i_k}.

Theorem 3.1. *The sequence $K^n \to K^{n-1} \to \cdots \to K^0$ is a complex. Moreover $K \to k$ is a resolution.*

Call K the Koszul complex.

Example 3.2. For $n = 2$ the Koszul complex is

$$R \xrightarrow{\binom{x}{-y}} R^2 \xrightarrow{(y,x)} R.$$

Because the Koszul complex K resolves k, we can say that if N is a module over R then we have an isomorphism $\mathrm{Ext}^i_R(k, N) := H^i \mathrm{Hom}_R(K, N)$. In particular, if $N = k$ then we have $\mathrm{Hom}_R(R, k) \cong k$, and so the hom-complex has in its i^{th} degree $\mathrm{Hom}^i_R(K, k) = \mathrm{Hom}_R(K^i, k) \cong \bigwedge^i(k^n)$. The differential vanishes because each x_i has image $0 \in k$. In particular, $H^i(\mathrm{Hom}_R(K, k)) \cong \mathrm{Hom}^i_R(K, k) \cong \bigwedge^i(k^n)$. So $\mathrm{Ext}^*_R(k, k) \cong \bigwedge^*(k^n)$. In fact, this is an isomorphism of graded algebras.

Remark 3.3. One can show that $\mathrm{Ext}^*_{\bigwedge^*(k^n)}(k, k)$ is just R again (this is a manifestation of Koszul duality).

See [Wei94, 4.5] for the specifics of Koszul complexes.

4 Coherent sheaves

Now we'd like to do some geometry. Assume from now on that k is algebraically closed. Let X be an affine or projective algebraic variety over k (e.g. $\mathbb{A}^n_k$ or $\mathbb{P}^n_k$). A sheaf $\mathcal{F}$ on X is like a vector bundle on X: over each open subset U, there are a bunch of sections $\mathcal{F}(U)$, and they glue together in the appropriate manner. You can think of sheaves as vector bundles on submanifolds. For example, let x be a (closed) point of X. The skyscraper sheaf $\mathcal{O}_x$ has sections given by

$$\mathcal{O}_x(U) := \begin{cases} k & x \in U \\ 0 & \text{else} \end{cases}$$

Associated to any closed subvariety $Z \hookrightarrow X$ there is a corresponding sheaf $\mathcal{O}_Z$ of polynomial functions on Z. In particular, X itself has a structure sheaf $\mathcal{O}_X$ and its global sections are the coordinate ring of X. When X is affine n-space, the global sections of $\mathcal{O}_X$ are exactly $k[x_1, \dots, x_n]$. Projective space has no non-constant global functions (this is an analogue of Liouville's theorem).

Formally, a coherent sheaf on $\mathbb{A}^n_k$ is the same thing as a finitely generated module over the coordinate ring $R := k[x_1, \ldots, x_n]$. The ring R itself corresponds to the structure sheaf $\mathcal{O}_X$. Ideals $I \hookrightarrow R$ define subvarieties $V(I) \hookrightarrow \mathbb{A}^n_k$ by looking at where the functions in I vanish. The coordinate ring of $V(I)$ is exactly the quotient R/I. Under the above correspondence, R/I corresponds to the coherent sheaf $\mathcal{O}_{V(I)}$.

More generally, let X be any variety. Cover it with finitely many open affine pieces U_α. Then a coherent sheaf on X is the same thing as a compatible collection of coherent sheaves on each of the U_α. A module over the coordinate ring of X is a coherent sheaf, but there may be many more.

Let's think about $\mathbb{P}^1$ in detail. It's covered by two copies of $\mathbb{A}^1$ (with coordinate rings $k[x]$ and $k[y]$, say) and they meet in a puctured copy of $\mathbb{A}^1$, which has coordinate ring $k[z, z^{-1}]$. They glue along the maps $x = 1/z$ and $y = z$. A coherent sheaf on $\mathbb{P}^1$ is given by a $k[x]$-module M and a $k[y]$-module N together with a transition function; we'll encode the data of this transition function by the restriction maps.

Let's think specifically about line bundles (i.e. locally free sheaves of rank one). In this case, $M = k[x]$ and $N = k[y]$. Write $\mathcal{O}(d)$ for the line bundle on $\mathbb{P}^1$ with transition function given by multiplication by z^d. A global section of $\mathcal{O}(d)$ is hence given by a pair of polynomials $p(x)$ and $q(y)$ such that $p(1/z)z^d = q(z)$. It follows that, if $d > 0$, then p must have degree $\leq d$, and indeed this is the only condition we need. So if $d > 0$, the space of global sections of $\mathcal{O}(d)$ has dimension $d + 1$, and if $d < 0$ then there are no global sections.

The bundle $\mathcal{O} = \mathcal{O}(0)$ is the structure sheaf of $\mathbb{P}^1$ (this is easy to see!). You can easily check that $\mathcal{O}(i) \otimes \mathcal{O}(j) \cong \mathcal{O}(i + j)$, since tensoring corresponds to multiplying transition functions. The bundle $\mathcal{O}(1)$ is called Serre's twisting sheaf. People often write $\mathcal{F}(d)$ to mean $\mathcal{F} \otimes \mathcal{O}(d)$.

There is a similar story for $\mathbb{P}^n_k$, which has a family of line bundles $\mathcal{O}(d)$, and they have no global sections if $d < 0$. If $d \geq 0$, the dimension of the space of global sections of $\mathcal{O}(d)$ on $\mathbb{P}^n_k$ is the binomial coefficient $\binom{n+d}{n}$. In fact, the $\mathcal{O}(d)$ are all of the line bundles on $\mathbb{P}^n_k$.

Now we've seen some coherent sheaves, we proceed very similarly: look at complexes of sheaves on X and invert quasi-isomorphisms to get the derived category $D(X) := D^b(\mathrm{Coh}(X))$. If X is affine space, with coordinate ring $R = k[x_1, \ldots, x_n]$, then we have an equivalence $D(X) \cong D(R)$. In the next section we'll do some computations.

The standard English reference for schemes and sheaves is [Har77], although [Vak] is more readable. See [Căl05] for an introduction to derived categories of coherent sheaves and [Huy06] for a more comprehensive textbook account.

5 Computations in the geometric world

Theorem 5.1. *Let p and q be (closed) points of $\mathbb{A}^n_k$. Then there is an isomorphism*

$$\mathrm{Ext}^*_{\mathbb{A}^n_k}(\mathcal{O}_p, \mathcal{O}_q) \cong \begin{cases} 0 & p \neq q \\ \bigwedge^*(k^n) & p = q \end{cases}$$

Proof. If $p \neq q$, then there are no maps $\mathcal{O}_p \to \mathcal{O}_q$ in the derived category. Geometrically this is clear, because the sheaves have stalks supported at p and q respectively. If $p = q$, we may as well assume by a linear transformation that p is the origin. Now use the equivalence of $\mathrm{Coh}(\mathbb{A}^n_k)$ with $\mod$-$k[x_1, \ldots, x_n]$ to see that $\mathrm{Ext}^*_{\mathbb{A}^n_k}(\mathcal{O}_0, \mathcal{O}_0) \cong \mathrm{Ext}^*_{k[x_1,\ldots,x_n]}(k, k)$. But we've done this computation already. □

Remark 5.2. Why do we care about this in the context of HMS? The idea is that X is a moduli space of points on X, so if we want to 'build' X as this moduli space we want to know what the Ext-algebras of the sheaves $\mathcal{O}_p$ are for (closed) points $p \in X$. One the symplectic side of the mirror, points ought to correspond to Lagrangian tori, whose Ext-algebras are again exterior algebras, so things do indeed match up.

Let's do some computations on non-affine varieties. Things get more difficult now, because projective resolutions might not exist. So we'll have to work a bit harder. We'll need some facts:

Proposition 5.3. *Let X be a variety.*

- *If $\mathcal{L}$ is a vector bundle on X, there is a dual vector bundle $\mathcal{L}^\vee$. If $\mathcal{L}$ is in addition a line bundle then there is a canonical isomorphism $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}$.*
- *If $X = \mathbb{P}^n_k$, the dual of $\mathcal{O}(d)$ is $\mathcal{O}(-d)$.*
- *Let $\mathcal{F}, \mathcal{G}$ be any coherent sheaves on X and let $\mathcal{L}$ be a vector bundle. Then there is an isomorphism*

  $$\mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}^\vee, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F}, \mathcal{L} \otimes \mathcal{G}).$$

  In particular, if $\mathcal{L}$ is a line bundle then putting $\mathcal{F} = \mathcal{L}$ we get

  $$\mathrm{Ext}^i(\mathcal{O}, \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{L}, \mathcal{L} \otimes \mathcal{G}).$$

- *If $\mathcal{F}$ is a coherent sheaf on X, then there is a canonical isomorphism*

  $$\mathrm{Ext}^i(\mathcal{O}, \mathcal{F}) \cong H^i(X, \mathcal{F})$$

  where the right-hand side denotes sheaf cohomology.

If you don't know about sheaf cohomology, the above is basically the definition.

Remark 5.4. One can also define a sheaf $\mathscr{E}\mathrm{xt}$, which is the derived functor of sheaf $\mathscr{H}\mathrm{om}$. Because $\mathscr{H}\mathrm{om}(\mathcal{O}, -)$ is the identity functor, the higher $\mathscr{E}\mathrm{xt}(\mathcal{O}, -)$ functors vanish. There is a local-to-global Ext spectral sequence with E^2 page

$$H^p(X, \mathscr{E}\mathrm{xt}^q(\mathcal{F}, \mathcal{G})) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G})$$

comparing sheaf $\mathscr{E}\mathrm{xt}$ to usual Ext. One can think of Ext as being built from a purely homological part (the $\mathscr{E}\mathrm{xt}^q$ sheaves) and a purely geometric part (the sheaf cohomology functors).

Putting the previous facts together, we see that

$$\mathrm{Ext}^l_{\mathbb{P}^n_k}(\mathcal{O}(i), \mathcal{O}(j)) \cong \mathrm{Ext}^l_{\mathbb{P}^n_k}(\mathcal{O}(i), \mathcal{O}(i) \otimes \mathcal{O}(j-i)) \cong H^l(\mathbb{P}^n_k, \mathcal{O}(j-i)).$$

So we need to do a sheaf cohomology computation! We're going to use Čech cohomology (that this is a valid way of computing sheaf cohomology is a nontrivial theorem of Leray). We'll just do it for $n = 1$.

Recipe for Čech cohomology: cover your space X by affine opens U_α. Let $\mathcal{F}$ be a sheaf on X, and put $U_{\alpha_1 \cdots \alpha_n} := U_{\alpha_1} \cap \cdots \cap U_{\alpha_n}$. The Čech complex is the complex

$$\prod_\alpha \mathcal{F}(U_\alpha) \to \prod_{\alpha, \beta} \mathcal{F}(U_{\alpha\beta}) \to \cdots$$

where the differentials are given by the alternating sums of the restriction maps. Then the cohomology of the Čech complex is the same as the sheaf cohomology of $\mathcal{F}$. So if U_0, U_1 is the standard cover of $\mathbb{P}^1$, and $\mathcal{F}$ is any sheaf on $\mathbb{P}^1$, then the cohomology of $\mathcal{F}$ is the cohomology of the complex

$$\mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{r_0 - r_1} \mathcal{F}(U_0 \cap U_1)$$

where the $r_i : \mathcal{F}(U_i) \to \mathcal{F}(U_0 \cap U_1)$ are the restriction maps. In particular the cohomology of $\mathcal{F}$ vanishes outside of degrees 0 and 1.

Because $\mathrm{Hom}(\mathcal{O}, -)$ is the global sections functor, it follows that $H^0(X, -) \cong \mathrm{Ext}^0(\mathcal{O}, -)$ is also the global sections functor. We can also see this directly (at least for $\mathbb{P}^1$) from the Čech complex, since a global section is given by sections on U_0 and U_1 which agree on the intersection.

Since we know the global sections of $\mathcal{O}(d)$, to complete the description of the Čech cohomology of $\mathcal{O}(d)$ we just need to work out what H^1 is. Looking at the Čech complex gives us an isomorphism

$$H^1(\mathbb{P}^1, \mathcal{O}(d)) \cong \mathrm{coker}\left(k[x] \times k[y] \to k[z, z^{-1}]\right)$$

where the differential sends $(p(x), q(y)) \mapsto p(z^{-1})z^d - q(z)$. The point is now that $p(z^{-1})z^d$ can hit all Laurent polynomials of the form $\cdots + a_{d-1}z^{d-1} + a_d z^d$ and that $q(z)$ can hit all Laurent polynomials of the form $b_0 + b_1 z + \cdots$. So if $d \geq -1$ then we can obtain all Laurent polynomials with appropriate sums of this form, and the cokernel vanishes, and so there is no H^1. But if $d < -1$ then we cannot hit polynomials of the form $c_{d-1}z^{d-1} + \cdots + c_{-1}z^{-1}$. So the cokernel consists of these polynomials and hence has dimension $-1-d$. So we've proved:

Theorem 5.5. $\mathrm{Ext}^n_{\mathbb{P}^1}(\mathcal{O}(i), \mathcal{O}(j)) \cong H^n(\mathbb{P}^1, \mathcal{O}(j-i))$ *is zero unless* n *is zero or one. When* $n = 0$ *we have*

$$\mathrm{Ext}^0_{\mathbb{P}^1}(\mathcal{O}(i), \mathcal{O}(j)) \cong \begin{cases} k^{j-i+1} & i \leq j \\ 0 & \textit{else} \end{cases}$$

and when $n = 1$ *we have*

$$\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(i), \mathcal{O}(j)) \cong \begin{cases} k^{i-j-1} & i-1 > j \\ 0 & \textit{else.} \end{cases}$$

There are similar statements for $\mathbb{P}^n$; in particular $H^*(\mathbb{P}^n, \mathcal{O}(d))$ vanishes outside of degrees 0 and n.

See e.g. [Har77, Chapter III] for material on sheaf cohomology as well as the other facts we use here. A more friendly reference for sheaf cohomology (especially Čech cohomology) is [Vak, Chapter 18].

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