

Differential Graded Algebras and Applications

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Chapter 1

Introduction

In this chapter we will describe the basics of various dg-objects (e.g. dga's, cdga's, dgla's) and important categorical structures connected to these objects, for example the derived category of a dga. We will conclude by describing the model structure on the category of negatively graded cdga's, the corresponding $(\infty, 1)$ -category, as well as using the model structure to provide a new perspective on the derived dg-category itself.

1.1 Differential Graded ‘Objects’

*‘Let k be a commutative ring, for example a field or the rings of integers.’
- B. Keller*

Graded k -modules

To describe dg (= differential graded) objects, we will first specify the grading. By a **graded k -module** V we will mean a \mathbb{Z} -graded k -module

$$V = \bigoplus_{n \in \mathbb{Z}} V^n,$$

i.e. V is a direct sum of k -modules. Elements $v \in V^n$ will be called **homogeneous of degree n** , we write $\bar{v} = n$, or $|v| = n$ or simply $\deg(v) = n$ for the degree of a homogeneous element v .

Remark. Graded modules are ubiquitous in mathematics, one can think of the homogeneous grading of polynomials in n -variables, or the singular homology of some space X , $H(X, \mathbb{Z}) = \bigoplus_n H_n(X, \mathbb{Z})$ a graded \mathbb{Z} -module.¹

A **morphism** between two graded k -modules V and W is a k -linear map $f : V \rightarrow W$. The space of morphisms is naturally graded

$$\text{Hom}_{\text{gr } k\text{-mod}}(V, W)^n = \{f \in \text{Hom}_k(V, W) : f(V^p) \subset W^{p+n} \forall p \in \mathbb{Z}\}.$$

We have the forgetful functor from the category of graded k -modules to k -modules. The forgetful functor has infinitely many left and right adjoints

$$\begin{aligned} [n] : k\text{-mod} &\rightarrow \text{gr } k\text{-mod} \\ V &\mapsto V[n], \\ \text{where } V[n]^m &= \begin{cases} V & \text{if } n = -m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Of course adjoints are unique up to unique (natural) isomorphism. The isomorphisms between the adjoints are given by the so called shift functors defined below.

¹Note that both these examples only have non-zero modules in non-negative degree.

For two graded k -modules, the **tensor product** $\otimes = \otimes_k$ **of two graded modules** inherits a natural grading

$$(V \otimes W)^n = \bigoplus_{i+j=n} V^i \otimes W^j.$$

The **tensor product** $f \otimes g$ **of two morphisms** $f : V \rightarrow V'$, $g : W \rightarrow W'$ of graded k -modules is defined using the *Koszul sign rule*

$$f \otimes g(v \otimes w) := (-1)^{\bar{v}\bar{g}} f(v) \otimes g(w).$$

Remark. Observe that in the above we have assume v and g are homogeneous elements, so that they have a well defined degree. From now on when we write \bar{v} , \bar{g} , etc, it will be implied the elements are homogeneous. By k -linear extension this also defines what to do for non-homogeneous elements.

We can define the **shift-functors**, also denoted $[n]$, by

$$\begin{aligned} [n] : \text{gr } k\text{-mod} &\rightarrow \text{gr } k\text{-mod}, \\ V &\mapsto V[n] := V \otimes k[n] \end{aligned}$$

Concretely, $V[n]^m = V^{n+m}$ for $V \in \text{gr } k\text{-mod}$.²

A **graded algebra over** k is a graded k -module A endowed with a degree 0 morphism $m : A \otimes A \rightarrow A$ that satisfies associativity, and admits a unit $1_A \in A_0$. A **morphism of graded algebras** is an algebra morphism, that is degree 0 as a k -linear map.

Differential Graded Algebras

A **differential graded k -module** (or **dg k -module**) is a graded k -module V endowed with a degree one morphism $d_V : V \rightarrow V$, called the **differential**, such that $d_V^2 = 0$. For example, in the case where $k = \mathbb{Z}$, the dg \mathbb{Z} -modules are exactly cochain complexes of abelian groups.

A **morphism of dg k -modules** is a morphism of the underlying graded k -modules i.e.

$$\text{Hom}_{\text{dg } k\text{-mod}}(V, W) := \text{Hom}_{\text{gr } k\text{-mod}}(V, W).$$

Note that we do not ask morphisms to commute with the differentials on V and W . The grading of $\text{Hom}_{\text{gr } k\text{-mod}}$ induces a grading on $\text{Hom}_{\text{dg } k\text{-mod}}$, moreover, we have the following natural differential ∂ defined by:

$$\partial f = d_W \circ f - (-1)^{\bar{f}} f \circ d_V.$$

We conclude that the **category of dg k -modules**, denoted $\mathcal{C}_{dg}(k)$, is naturally enriched over dg k -modules (this exactly means that the hom-spaces carry the structure of dg k -modules).

We can upgrade the **shift functors** to endofunctors of $\mathcal{C}_{dg}(k)$:

$$\begin{aligned} [n] : \mathcal{C}_{dg}(k) &\rightarrow \mathcal{C}_{dg}(k), \\ (V, d_V) &\mapsto (V[n], (-1)^n d_V). \end{aligned}$$

Moreover, $\mathcal{C}_{dg}(k)$ is naturally endowed with a **tensor product**:

$$(V, d_V) \otimes (W, d_W) := (V \otimes W, d_V \otimes 1 + 1 \otimes d_W).$$

An important observation is that the composition maps can be seen as degree 0 morphisms of dg k -modules

$$\mathcal{C}_{dg}(k)(V, W) \otimes \mathcal{C}_{dg}(k)(W, Z) \rightarrow \mathcal{C}_{dg}(k)(V, Z),$$

where $\mathcal{C}_{dg}(k)(V, W)$ denotes the set of morphisms from V to W in the category $\mathcal{C}_{dg}(k)$.

A **differential graded algebra over** k (or **dga/ k**) is a dg k -module (A, d_A) endowed with a degree 0 morphism $m : A \otimes A \rightarrow A$ such that $\partial m = 0$. Writing $ab := m(a \otimes b)$, we can rephrase the condition $\partial m = 0$ as

$$d_A(ab) = d_A(a)b + (-1)^{\bar{a}} ad_A(b).$$

²Notice that we have defined the two functors $[n]$ in such a way that $k[n] = k[0][n]$.

We will write $d = d_A$ if no confusion is possible. A **morphism of dga's** $f : A \rightarrow A'$ is a degree 0 morphism of algebras that commutes with the differentials $d_{A'} \circ f = f \circ d_A$. Let A be a dga/ k , then a **left dg A -module** is a $M \in \mathcal{C}_{dg}(k)$ endowed with a degree 0 morphism

$$\begin{aligned} A \otimes M &\rightarrow M, \\ a \otimes m &\mapsto a \cdot m \end{aligned}$$

of dg k -modules, that defines an action of the k -algebra A on the k -module M . Equivalently, an action is given by collection of k -linear maps $A^n \otimes M^m \rightarrow M^{n+m}$ such that $1_a \cdot m = m$, and $a \cdot (b \cdot m) = (ab) \cdot m$.

Remark. We can consider any k -algebra A as a dga situated in degree 0, with 0 differential: $(A[0], 0)$.

Remark. Note that we do not ask the action map to commute with the differentials. However, in the case of a k -algebra A a dg k -modules is a complex of A -modules i.e. a dg k -module (M, d) such that the M^n are A -modules, and $d(a \cdot m) = a \cdot d(m)$.

For A a dga, a **morphism of dg A -modules** is of course a morphism of dg k -modules, that respects the A -action.

1.2 Singular Cohomology and CDGAs

In this section we shall cover an extended example of a dgas coming from topology; that of singular cohomology. We'll also generalise a property of singular cohomology to define a commutative differential graded algebra (cdga). We shall begin with chains, cochains and singular homology.

In topology singular homology is a functor from topological spaces to graded abelian k -modules which contains information as to the number and dimension of holes in a topological space. Despite singular homology capturing exactly the type of topological information wanted topologist often pass to singular cohomology, the dual of singular homology, and use results linking homology and cohomology (for example Poincaré duality). The reason for this unusual behaviour is that singular cohomology is 'nicer' than singular homology as it is dga. Another powerful cohomology theory in (differential) topology is the De Rham cohomology of smooth manifolds which is also a dga.³ Thus a dga can be thought of as a distillation of what makes the nicest cohomology theories.

Singular Homology

Definition 1.2.1. A **chain complex** C_* is a graded k -module with grading

$$C_* = \bigoplus_{n \in \mathbb{Z}} C_n$$

equipped with a degree (-1) -morphism $d_* : C \rightarrow C$ satisfying $d_*^2 = 0$ called the **differential**. An element $\sigma \in C_n$ is called an **n -chain**.

A chain complex differs from a differential graded k -module only in the degree of its differential; chain complex differentials reduce dimension whereas the differential of a differential graded k -module increases dimension. Another name for a differential graded k -module is a **cochain complex of k -modules**. We shall denote cochain complexes by $C^* = \bigoplus_{n \in \mathbb{Z}} C^n$ to distinguish it from a chain complex. Elements $\alpha \in C_n$ are called **cochains**.

In topology the purpose of a chain complex $C(X)$ it to separate all the subspaces of a topological space X by dimension and show how these subspaces fit together. We shall show how this works by constructing the **singular chain complex of a topological space**. The first problem is how to decide on the dimension of a topological subspace. To solve this we shall instead of considering subspaces consider maps into X from something clearly n -dimensional.

³We shall not cover De Rham cohomology here, see [4]

Definition 1.2.2. The **standard n -simplex** is

$$\Delta_n = \left\{ (x_0, \dots, x_n) \mid \sum x_i = 1, 0 \leq x_i \leq 1 \right\}.$$

A **singular n -chain**⁴ of X is a map

$$\sigma : \Delta_n \rightarrow X$$

The differential maps of the chain complex should show how a subspace maps onto its boundary thus showing how the 'subspaces' fit together. The condition $d \circ d = 0$ says the boundary of a boundary is trivial.

Definition 1.2.3. The i^{th} face map

$$F_i^n : \Delta_{n-1} \rightarrow \Delta_n; (x_1, \dots, x_{n-1}) \mapsto (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1})$$

is the inclusion of Δ_{n-1} into Δ_n by adding a 0 in the i^{th} dimension. If $\sigma : \Delta_n \rightarrow X$ is a singular n -chain, the then i^{th} **face** of σ is $\sigma^{(i)} = \sigma \circ F_i^n$. The **boundary** of σ is the $(n-1)$ -chain

$$d_n(\sigma) = \sum_{i=0}^n (-1)^i \sigma^{(i)}.$$

So the boundary of a n -chain is the sum of all its faces consistently oriented. This boundary satisfies the required property that:

$$d_{n-1} \circ d_n = 0.$$

The problem with singular chain complexes is they are massive as they are a lot of possible n -chains for even simple spaces so we make things more manageable by quotienting. In particular as we wish to detect holes we only want chains that surround a hole; if there is a higher dimensional simplex between two simplices then they should be equivalent.⁵

Definition 1.2.4. Let C_* be a chain complex. The **homology** of C_* is a graded k -module

$$H_* = \bigoplus_{n \in \mathbb{Z}} H_n; H_n = \frac{\ker(d_n)}{\text{im}(d_{n-1})}$$

with differential

$$\delta_n : H_n \rightarrow H_{n+1}; [\sigma] \mapsto [d_n \circ \sigma].$$

As $\delta^2 = 0$ this is also a differential graded k -module with the indices reversed as in chain complexes. If the singular chain complex of a topological space X is used we call the homology the **singular homology** of X .

Singular Cohomology and the Cup product

We shall now construct the **singular cochain complex** which is a dual of the singular chain complex.

Definition 1.2.5. Let X be a topological space and C_* be the singular chain complex of X . The **singular cochain complex** of X is the differential graded k -module with n -cochains

$$\alpha \in \text{Hom}(\sigma, k)$$

for all $\sigma \in C_n$ and differentials

$$d^n : C^n \rightarrow C^{n+1}; \alpha \mapsto \alpha \circ d_n.$$

⁴There is no requirement that the map is injective and can be highly singular hence the name 'singular homology'. Thus considered as a subspace it is at most n -dimensional rather than strictly n -dimensional.

⁵This condition is satisfied in the construction of homotopy groups which also measure holes in a space however these are difficult to calculate.

Definition 1.2.6. Let C^* be a cochain complex. The **cohomology** of C^* is a graded k -module

$$H^* = \bigoplus_{n \in \mathbb{Z}} H^n; \quad H^n = \frac{\ker(d^n)}{\text{im}(d^{n-1})}$$

with differential

$$\delta^n : H^n \rightarrow H^{n+1}; \quad [\alpha] \mapsto [d_n \circ \alpha].$$

As $\delta^2 = 0$ this is also a **differential graded k -module**. If the singular cochain complex of a topological space X is used we call the cohomology the **singular cohomology** of X .

The singular cohomology, unlike general cohomology, cochain complexes and singular cochain complexes, is not merely a differential graded k -module but a differential graded **algebra**. This, as has already been discussed, is an important part of its power and utility. The product is called the **cup product**, \cup .

The cup product $\alpha \cup \beta$ of two cochains α and β is a function (as it must be a cochain) which restricts a simplex to two sub-simplices and feeds each sub-simplex into the cochains (α and β) before multiplying the result.

Definition 1.2.7. We need to restrict simplices to sub-simplices. Let Δ_n be a standard n -simplex then $[e_{r_0}, \dots, e_{r_p}]$ is the subspace with all but the specified basis having 0 coefficients:

$$[e_{r_0}, \dots, e_{r_p}] = \{(x_0, \dots, x_n) \mid x_{r_i} = 0, 0 \leq i \leq r\}.$$

Now, for cochains $\alpha \in C^p(X)$ and $\beta \in C^q(X)$, the **cup product** $\alpha \cup \beta \in C^{p+q}(X)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{p+q} \rightarrow X$ is given by the formula

$$(\alpha \cup \beta)(\sigma) = \alpha(\sigma|_{[e_0, \dots, e_p]})\beta(\sigma|_{[e_p, \dots, e_{p+q}]})$$

where the right hand side is the product in k .

This product has been defined at the cochain level rather than on cohomology so we need to show it descends to a product

$$H^p(X) \times H^q(X) \rightarrow H^{p+q}(X).$$

This follows from it satisfying the **graded Leibniz rule** (which in turn follows from bashing out the definitions):

$$\delta(\alpha \cup \beta) = \delta(\alpha) \cup \beta + (-1)^p \alpha \cup \delta(\beta).$$

We can follow the analogy of dgas and cohomology further by defining analogues of Poincaré duality ($H_* \cong H^{n-*}$, where n is the dimension of the space) and we shall pursue this in a later chapter.

Commutative DGAs

Note that the singular cochain complex of a topological space satisfies the condition that $x \cup y = (-1)^{\bar{x} \cdot \bar{y}} y \cup x$ for homogeneous elements. Generalising this gives the following

Definition 1.2.8. We define a **commutative differential graded algebra (cdga)** to be a dga satisfying the graded-commutativity property $xy = (-1)^{\bar{x} \cdot \bar{y}} yx$ for homogeneous elements.⁶ We say a cdga is **strictly commutative** if $x^2 = 0$ whenever x is in odd degree.

Example 1.2.9. The singular cochain complex of any topological space is a cdga.

Example 1.2.10. Work over the ring $\mathbb{C}[t]$. Let A be the cdga that is just a copy of $\mathbb{C} \cong \mathbb{C}[t]/(t)$ concentrated in degree zero. Let B be the cdga

$$\cdots \rightarrow 0 \rightarrow \mathbb{C}[t] \xrightarrow{t} \mathbb{C}[t] \rightarrow 0 \rightarrow \cdots$$

concentrated in degrees -1 and 0. Then A and B have the same homology, and are in fact quasi-isomorphic. Since all of the modules at each level of B are free over our base ring $\mathbb{C}[t]$, we say that B is a **quasi-free resolution** of A .

⁶So a commutative dga is not a dga where multiplication is commutative. Unfortunately this terminology is standard, especially in algebraic topology.

1.3 Differential Graded Lie Algebras

In this section let k be a field.

A **differential graded Lie algebra**, or **dgla** for short, is a dg k -module with a graded Lie bracket. More precisely, a dgla L over a field k is the data of a \mathbb{Z} -graded k -vector space $L = \bigoplus_{i \in \mathbb{Z}} L^i$, a differential $d : L \rightarrow L$ and a bilinear bracket $[-, -] : L \times L \rightarrow L$ satisfying the following:

- The differential d makes L into a dg-module over k , i.e. L is a cochain complex of k -vector spaces.
- $[-, -]$ respects the grading: $[L^i, L^j] \subset L^{i+j}$
- $[-, -]$ is graded-anticommutative: $[x, y] = -(-1)^{\bar{x} \cdot \bar{y}} [y, x]$
- $[-, -]$ satisfies the graded Jacobi identity: $[[x, y], z] = [x, [y, z]] - (-1)^{\bar{x} \cdot \bar{y}} [y, [x, z]]$
- the graded Leibniz rule: $d[x, y] = [dx, y] + (-1)^{\bar{x}} [x, dy]$

Remark. Since we require our dgas to be associative, a dgla is not necessarily a dga.

Given any dgla as above then L^0 and $L^e := \bigoplus_{i \in \mathbb{Z}} L^{2i}$ are Lie algebras. Conversely we can view any Lie algebra L as a dgla concentrated in degree 0. Note that if x is an element of odd degree, then $[x, x]$ may be nonzero but we do have the identity $[[x, x], x] = 0$.

A linear map $\varphi : L \rightarrow L$ satisfying the graded Leibniz rule is called a **derivation of degree n** if φ takes L^i to L^{i+n} . The differential d is a derivation of degree 1. Fixing a homogeneous element x , the map $\text{ad}(x) : L \rightarrow L$ defined by $\text{ad}(x)(y) = [x, y]$ is a derivation of degree \bar{x} , and in fact this statement is equivalent to the graded Jacobi identity. If $D : L \rightarrow L$ is a derivation then $\ker(D)$ is a graded Lie subalgebra of L .

Example 1.3.1. Let $\text{Der}^n(L)$ be the algebra of derivations of degree n on a dgla L . Then the algebra $\text{Der}^*(L) := \bigoplus_{n \in \mathbb{Z}} \text{Der}^n(L)$ admits the structure of a dgla with bracket $[f, g] = fg - (-1)^{\bar{f} \cdot \bar{g}} gf$ and differential $\partial(f) = [d, f]$.

Example 1.3.2. If A is a commutative k -algebra and $V \subseteq \text{Der}_k(A, A)$ is an A -submodule closed under $[-, -]$ then we can define a dgla L concentrated in nonnegative degrees by setting $L^0 = A$ and $L^i = \bigwedge^i V$. The differential is zero and the bracket is characterised by the following properties:

- $[-, -]$ is the usual bracket on $L^1 = V$
- If $x \in L^1$ and $a \in L^0$ then $[x, a] = x(a)$
- If x, y, z have degrees l, m, n respectively then $[x, y \wedge z] = [x, y] \wedge z + (-1)^{(l-1)m} y \wedge [x, z]$ and $[x \wedge y, z] = x \wedge [y, z] + (-1)^{(n-1)m} [x, z] \wedge y$.

Example 1.3.3. The tensor product of a dgla (L, d_L) and a DGA (A, d_A) is defined to be the dgla $L \otimes A$ with $(L \otimes A)^n = \bigoplus_i (L^i \otimes A^{n-i})$, differential $d(l \otimes a) = d_L(x) \otimes a + (-1)^{|x|} x \otimes d_A(a)$ and bracket $[x \otimes a, y \otimes b] = (-1)^{|a| \cdot |y|} [x, y] \otimes ab$.

DGLAs will be one of the main objects of study in Chapter Four.

1.4 The Derived Category of a DGA

For a dga A , we denote the **category of dg A -modules** by $\mathcal{C}_{dg}(A)$ and note that it coincides with $\mathcal{C}_{dg}(k)$ when k is viewed as a dga concentrated in degree 0. Note that the morphism spaces in this category will be dg k -modules with the differential and grading induced from $\text{Hom}_{\text{dg } k\text{-mod}}(M, N)$. However, when we take

M (respectively N) to be A itself, $\text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)$ will be a left (respectively right) dg A -module with left action

$$\begin{aligned} A \otimes \text{Hom}_{\mathcal{C}_{dg}(A)}(A, N) &\rightarrow \text{Hom}_{\mathcal{C}_{dg}(A)}(A, N) \\ a \otimes f &\rightarrow a.f : A \rightarrow N \end{aligned}$$

where $(a.f)(b) = af(b)$ and similarly for the right action.

Example 1.4.1. We consider $\mathcal{C}_{dg}(A)$ when A is a k -algebra viewed as a dga concentrated in degree 0. The objects of $\mathcal{C}_{dg}(A)$ are just cochain complexes of A -modules and the morphism spaces are dg k -modules, $\text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)$, with grading

$$\text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)^n = \{f : M \rightarrow N \mid f(M^p) \subseteq N^{n+p} \forall p \in \mathbb{Z}\}$$

and differential

$$df = d_W \circ f - (-1)^{\bar{f}} f \circ d_V.$$

Then $\ker(d^0 : \text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)^0 \rightarrow \text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)^1)$ consists of precisely the cochain maps between the complexes M and N of A -modules. Further, $\text{Im}(d^{-1} : \text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)^1 \rightarrow \text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)^0)$ consists of precisely the null-homotopic cochain maps. Define the category $Z^0(\mathcal{C}_{dg}(A))$ (respectively $H^0(\mathcal{C}_{dg}(A))(M, N)$) to have the same objects as $\mathcal{C}_{dg}(A)$ but morphism spaces

$$Z^0(\mathcal{C}_{dg}(A))(M, N) = \ker(d^0) \quad (\text{respectively } H^0(\mathcal{C}_{dg}(A))(M, N) = \frac{\ker(d^0)}{\text{Im}(d^{-1})}).$$

Then it is clear that

$$\begin{aligned} Z^0(\mathcal{C}_{dg}(A))(M, N) &= \mathcal{C}(A) \\ H^0(\mathcal{C}_{dg}(A))(M, N) &= \mathcal{H}(A) \end{aligned}$$

where $\mathcal{C}(A)$ is the category of cochain complexes and cochain maps over A and $\mathcal{H}(A)$ is the associated homotopy category.

To get the derived category of a k -algebra A , one defines a class of morphisms in $\mathcal{H}A$ known as quasi-isomorphisms and then localises with respect to them. We would like to generalise this to dga's and so motivated by the above example, for a dga A we define

$$\begin{aligned} \mathcal{C}(A) &:= Z^0(\mathcal{C}_{dg}(A))(M, N) \\ \mathcal{H}(A) &:= H^0(\mathcal{C}_{dg}(A))(M, N) \end{aligned}$$

where Z^0 and H^0 are defined as in the example (the example shows there is no ambiguity when A is a dga concentrated in degree 0).

As a dg A -module is, in particular, a dg k -module, we can view it as a complex of k -modules and hence take the cohomology of this complex. Then any morphism of dg A -modules, f , induces a map on the cohomology groups and we say f is a quasi-isomorphism if the induced maps are all isomorphisms. Then the **derived category of a dga** A , denoted $\mathcal{D}(A)$, is defined to be the localisation of $\mathcal{H}(A)$ with respect to the class of quasi isomorphisms. This means that the objects of $\mathcal{D}(A)$ are the same as those in $\mathcal{H}(A)$ (and so consists of all dg A -modules) but the morphism classes are obtained by inverting all quasi-isomorphisms and applying some equivalence relation. In particular, a general morphism $X \rightarrow Y$ in $\mathcal{D}(A)$ is a diagram of the form

$$\begin{array}{ccc} & M & \\ f \nearrow & & \nwarrow s \\ X & & Y \end{array}$$

where s is a quasi-isomorphism and M is another dg A -module. Two such diagrams, say (f, s) above and (g, t) , are defined to be equivalent if there exists the following commutative diagram

$$\begin{array}{ccccc}
 & & M & & \\
 & f \nearrow & \downarrow & \nwarrow s & \\
 X & \xrightarrow{h} & N & \xleftarrow{u} & Y \\
 & g \searrow & \uparrow & \swarrow t & \\
 & & M & &
 \end{array}$$

where u is a quasi-isomorphism. One can check that given two morphisms $(f, s) : X \rightarrow Y$ and $(g, t) : Y \rightarrow Z$ we can construct the diagram

$$\begin{array}{ccccc}
 & & L & & \\
 & h \nearrow & & \nwarrow u & \\
 & M & & N & \\
 f \nearrow & & & & \\
 X & & Y & & Z \\
 & \nwarrow s & \nearrow g & \nwarrow t &
 \end{array}$$

where u is also a quasi-isomorphism. Thus, we can define the composition of (f, s) and (g, t) to be $(t \circ u, h \circ f)$. This can be shown to be well defined and associative with the identity morphism on a dg A module X given below.

$$\begin{array}{ccc}
 & X & \\
 \text{id} \nearrow & & \nwarrow \text{id} \\
 X & & X
 \end{array}$$

Remark. When A is a k -algebra viewed as a dga concentrated in degree zero the definition of quasi isomorphism agrees with the usual definition and thus the derived category of A considered as a dga is the same as the usual derived category.

This construction is very formal and has several disadvantages. Describing morphisms in this way can be difficult to work with, as many do not actually exist as morphisms of dg A -modules, and although the composition of two such morphisms always exists via the above diagram, actually being able to determine what it is can be a problem. This construction also has the set theoretic issue in that the morphism class between two dg A -modules in $\mathcal{D}(A)$ may not be a set. Therefore, it can be useful to define a model structure on $\mathcal{C}(A)$ and obtain the derived category using model categories. This method both ensures that the morphism classes are sets and somehow throws out the difficult morphisms.

1.5 Model structures on DG-objects

We give a very short primer on model category theory - for a more detailed exposition, see e.g. [8] or [9]. A **model category** is a category with three distinguished classes of morphisms - weak equivalences, fibrations, and cofibrations - satisfying certain axioms, among them the existence of finite limits and colimits. The distinguished classes of morphisms should be considered in analogy with weak homotopy equivalences, fibrations and cofibrations in topology - see e.g. [5] for detail. The framework of a model category allows us to localise at the weak equivalences whilst retaining some control over what happens in the localised category.

We can also build an $(\infty, 1)$ -category from a model category by defining **mapping spaces** between objects, not just sets of morphisms.

In a model category, we can define an abstract notion of homotopy between morphisms. Call an object x **cofibrant** if the unique map $0 \rightarrow x$ from the initial object is a cofibration. Call x **fibrant** if the unique map $x \rightarrow 1$ to the terminal object is a fibration. If both x and y are fibrant and cofibrant then a morphism $f : x \rightarrow y$ is a weak equivalence if and only if it has a homotopy inverse, i.e. there exists a morphism $g : y \rightarrow x$ such that gf and fg are both homotopic to the identity map.

A **fibrant-cofibrant replacement** for x is a weakly equivalent object x' that is both fibrant and cofibrant. We will use a prime to denote fibrant-cofibrant replacement.⁷ The **homotopy category** of a model category \mathcal{C} is the category $\text{Ho}(\mathcal{C})$ with the same objects, and with $\text{Hom}_{\text{Ho}(\mathcal{C})}(x, y) := [x', y']$ the set of homotopy classes of maps from x' to y' . Then the category $\text{Ho}(\mathcal{C})$ is a localisation of \mathcal{C} at the weak equivalences. Note that the hom-sets really are sets here. This also solves the problem of composition of morphisms.

It is possible to build an $(\infty, 1)$ -category from a model category by constructing hom-spaces that are simplicial sets. Call these simplicial sets $\text{Map}(x, y)$. Then the set of connected components of $\text{Map}(x, y)$ is precisely the set $\text{Hom}_{\text{Ho}(\mathcal{C})}(x, y)$. We say that a model category is a **strictification** of its associated $(\infty, 1)$ -category.

Example 1.5.1. If \mathcal{A} is any abelian category then the category $\text{Ch}(\mathcal{A})$ of chain complexes in \mathcal{A} has a model structure where the weak equivalences are the quasi-isomorphisms. The homotopy category is the usual derived category.

Example 1.5.2. The category of topological spaces has a model category structure where the weak equivalences are the weak homotopy equivalences (maps inducing isomorphisms on all homotopy groups). The homotopy category is equivalent to the homotopy category of simplicial sets. So from a homotopy-theoretic perspective, topological spaces are the same as simplicial sets.

In what follows we suppose that k has characteristic zero. Let $\text{dgmod}_k^{\leq 0}$ be the category of nonpositively graded dg k -modules. The Dold-Kan Correspondence ([7], §8.4) gives us an equivalence of categories $\text{dgmod}_k^{\leq 0} \cong \text{sk-mod}$ where sk-mod is the category of simplicial k -modules. Note that $\text{dgmod}_k^{\leq 0}$ is just the category of nonnegatively graded chain complexes of k -modules.

Theorem 1.5.3. *The category $\text{dgmod}_k^{\leq 0}$ has a model structure (the **projective model structure**) where the weak equivalences are the quasi-isomorphisms and the fibrations are levelwise surjections in strictly negative degree. The cofibrations are the levelwise injective maps with projective cokernel. The homotopy category is the usual derived category.*

Remark. A cofibrant replacement of an object in $\text{dgmod}_k^{\leq 0}$ is just a projective resolution in the usual sense of homological algebra.

Let $\text{dga}_k^{\leq 0}$ be the category of nonpositively graded dgas. We have a forgetful functor $U : \text{dga}_k^{\leq 0} \rightarrow \text{dgmod}_k^{\leq 0}$ forgetting the algebra structure. A monoidal version of the Dold-Kan Correspondence gives us an equivalence of categories between $\text{Ho}(\text{dga}_k^{\leq 0})$ and the category $\text{Ho}(\text{sAlg}_k)$, where sAlg_k is the category of simplicial k -algebras (with appropriate model structure).

Let $\text{cdga}_k^{\leq 0}$ be the category of nonpositively graded cdgas.

Theorem 1.5.4. *The category $\text{cdga}_k^{\leq 0}$ admits a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are levelwise surjections in strictly negative degree. Moreover, if Sym is the symmetric product functor then the pair of maps*

$$U : \text{cdga}_k^{\leq 0} \longleftarrow \text{dgmod}_k^{\leq 0} : \text{Sym}$$

⁷Fibrant-cofibrant replacement is not necessarily functorial; however in most cases of interest we can choose replacements functorially and this is sometimes taken as an axiom.

induce an adjunction on the homotopy categories.

Remark. The cofibrant objects in $\text{cdga}_k^{\leq 0}$ are precisely the quasi-free objects.

Nonnegatively graded cdgas provide local charts for **derived schemes** in **derived algebraic geometry**, just as usual k -algebras provide local charts for schemes in algebraic geometry. The category $k\text{-dAff} := (\text{cdga}_k^{\leq 0})^{\text{op}}$ is called the **category of derived affine k -schemes**.

Chapter 2

Rickard's Theorem

2.1 Motivation

A classical question in mathematics is to ask when two objects, in our case k -algebras where k is a field, are “the same”. Of course, this depends completely on how you define “the same”. The most common notion of this for two k -algebras is to be isomorphic. However, this is often too strong and so over the years mathematicians have defined weaker notions of equivalence.

As every ring has an associated left module category it is reasonable to suggest that we consider two rings to be “the same”, or **left Morita equivalent**, if their corresponding left module categories are equivalent as categories. Right Morita equivalence is defined similarly and it turns out that two rings are left Morita equivalent if and only if they are right Morita equivalent allowing us to consider just Morita equivalence [12]. In 1958, Morita proved the following theorem, characterising completely when two k -algebras would be Morita equivalent.

Theorem 2.1.1. [3] *The following statements are equivalent for two k -algebras A and B :*

1. *There exists a k -linear equivalence $F : \text{Mod}A \rightarrow \text{Mod}B$;*
2. *There exists an A - B bimodule X such that $- \otimes_A X$ is an equivalence from $\text{Mod}A$ to $\text{Mod}B$;*
3. *There exists a B -module P such that:*
 - (a) *P is a finitely generated projective module;*
 - (b) *P generates $\text{Mod}B$;*
 - (c) *$A \cong \text{End}_B(P)$.*

In this case, we can consider the object P , called a **progenerator**, as a left A -module with A acting by endomorphisms. Hence, P is a A - B bimodule and a very basic outline of the proof is:

- (3) \implies (2) Take $X = P$.
- (2) \implies (1) Obvious.
- (1) \implies (3) Take $P = FA$.

However, even the notion of Morita equivalence can be too strong. For example, while all non-commutative crepant resolutions over a Cohen-Macaulay ring of dimension 2 are Morita equivalent, the same is not true in dimension 3, but it is true that they will be **derived equivalent**[15] i.e. the derived categories of their module categories will be equivalent.

Thus, our question now becomes “when are two k -algebras derived equivalent?”. Answering this question led to the development of tilting theory, which generalises the object P in Morita's Theorem. We generalise

each concept in Morita's Theorem one by one. First note that, as derived categories are triangulated categories, we now ask for the equivalence to be a triangle equivalence. Also, instead of a module P , we are now looking for an object T of $\mathcal{D}(B)$ i.e. a complex of B -modules.

Recall that $-\otimes_A X$ has a right adjoint, namely $\mathrm{Hom}_B(X, -)$. We wish to generalise these to be maps of complexes. Given two complexes of A -modules, M and N , we define $M \otimes_A N$ to be the complex with the n^{th} term

$$(M \otimes_A N)^n = \bigoplus_{p+q=n} M^p \otimes_A N^q$$

and differential

$$d(m \otimes n) = dm \otimes n + (-1)^p m \otimes dn$$

where $m \in M^p$. For two complexes of B -modules, K and L we define $\mathcal{H}om_B(K, L)$ to be the complex with terms

$$\mathcal{H}om_B(K, L)^n = \prod_{-p+q=n} \mathrm{Hom}_B(K^p, L^q)$$

and differential

$$df(x) = d(f(x)) - (-1)^n f(dx).$$

where $f \in \mathcal{H}om_B(K, L)^n$. If X is a complex of A - B bimodules, it can be shown that $-\otimes_A X$ and $\mathcal{H}om_B(X, -)$ are a pair of adjoint functors between $\mathcal{H}(A)$ and $\mathcal{H}(B)$; the categories of chain complexes up to homotopy of A and B . Moreover, the derived functors $-\otimes_A^L X$ and $\mathbf{R}\mathcal{H}om_B(X, -)$ are adjoint functors between the derived categories. Thus, $-\otimes_A^L X$ will take the place of $-\otimes_A X$ in Morita's Theorem.

Notice that condition (c) on P in Morita's Theorem is asking that the right adjoint of the equivalence in (b), namely $\mathrm{Hom}_B(P, -)$, applied to P is A . Thus, to coincide with replacing $-\otimes_A X$ with $-\otimes_A^L X$, we want to generalise this to asking that $\mathbf{R}\mathcal{H}om_B(T, T) \cong A$, where we consider A as a complex concentrated in degree 0. This turns out to be equivalent to asking that

$$\mathrm{Hom}_{\mathcal{D}(B)}(T, T[n]) \cong \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

Condition (a) on P asked that it was finitely generated and projective and so quite naturally, we generalise this to asking that T is a bounded complex of finitely generated projective B -modules. In fact, we don't even need to ask this much as we are working in the derived category and so it will be enough to ask for T to be quasi-isomorphic to such a complex. We define the full subcategory of $\mathcal{D}(B)$ consisting of such complexes to be the **category of perfect complexes**, denoted $\mathrm{per} B$.

Finally, we ask that T generates $\mathcal{D}(B)$ as a category. As we have already required that T is perfect, this is equivalent to asking that the smallest full triangulated subcategory of $\mathcal{D}(B)$ containing T and closed under taking infinite direct sums is $\mathcal{D}(B)$.

Thus, by generalising Theorem 2.1.1 we obtain the following theorem which was first proved by Rickard in 1989.

Theorem 2.1.2 (Rickard's Theorem). *[10] Let A and B be two k -algebras. Then the following are equivalent:*

1. *There is a triangle equivalence $F : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$;*
2. *There is a complex of A - B bimodules X such that the functor $-\otimes_A^L X : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is a triangle equivalence;*
3. *There is an object $T \in \mathcal{D}(B)$ such that:*

$$(a) \mathrm{Hom}_{\mathcal{D}(B)}(T, T[n]) \cong \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases};$$

- (b) $T \in \text{per} B$;
- (c) *The smallest full triangulated subcategory of $\mathcal{D}(B)$ containing T and closed under taking infinite direct sums is $\mathcal{D}(B)$.*

We call the object T a tilting complex. Although this seems a straight forward generalisation of Morita's Theorem 2.1.1, actually proving it is much more difficult. The problem comes with proving (3) \implies (2). Following the proof of Morita, we would like to take $X = T$. However, T has no natural structure as a complex of A -modules and so this makes no sense. Getting around this problem made Rickards original proof complicated as there was no natural functor associated to T to work with so he instead had to explicitly construct one. However, Keller later noticed that the problem actually disappeared if you instead consider dgas. This observation resulted in a significantly simpler proof of Rickard's Theorem which we present here, but for this, we require some more technology.

2.2 More Technology for DG Algebras

Recall from Chapter 1 that for a dga A , we define $\mathcal{C}_{dg}(A)$ to be the category with dg A -modules as objects and with morphism classes $\mathcal{H}om_A(M, N) := \text{Hom}_{\mathcal{C}_{dg}(A)}(M, N)$ which are differential graded k -modules.

Then $\mathcal{H}(A)$ was defined to be $H^0(\mathcal{C}_{dg}(A))$ which, in the case A was a dga concentrated in degree 0, agreed with the definition of $\mathcal{H}(A)$ being the category of cochain complexes over A up to homotopy. Notice that $H^n(\mathcal{H}om_A(M, N)) = \text{Hom}_{\mathcal{H}(A)}(M, N[n]) = \text{Hom}_{\mathcal{H}(A)}(M[n], N)$.

We define a dg A -module M , to be **homotopically projective** (respectively **homotopically injective**) if

$$\text{Hom}_{\mathcal{H}(A)}(M, N) = 0 \quad (\text{respectively } \text{Hom}_{\mathcal{H}(A)}(N, M) = 0)$$

for all acyclic dg A -modules N . If A is just a k -algebra, then $\text{Hom}_{\mathcal{H}(A)}(A, M) \cong H^0(M)$ and so A is a homotopically projective A -module. More generally, a homotopically projective dg A -module for a k -algebra is simply a (possibly unbounded) complex of projective A -modules [1].

We define the full triangulated subcategory of $\mathcal{H}(A)$ consisting of the homotopically projective (respectively homotopically injective) dg A -modules to be $\mathcal{H}_p(A)$ (respectively $\mathcal{H}_i(A)$). Then we get the following theorem:

Theorem 2.2.1. [1] *There exists triangle functors $\mathbf{p} : \mathcal{H}(A) \rightarrow \mathcal{H}(A)$ and $\mathbf{i} : \mathcal{H}(A) \rightarrow \mathcal{H}(A)$ such that:*

1. $\mathbf{p}M$ (respectively $\mathbf{i}M$) is a homotopically projective (respectively homotopically injective) dg A -module quasi-isomorphic to M ;
2. they both commute with infinite direct sums;
3. they vanish on acyclic dg A -modules. Hence, they induce functors $\mathbf{p} : \mathcal{D}(A) \rightarrow \mathcal{H}(A)$ and $\mathbf{i} : \mathcal{D}(A) \rightarrow \mathcal{H}(A)$ which are left and right adjoints respectively, to the canonical functor $\mathcal{H}(A) \rightarrow \mathcal{D}(A)$.

Note that part (3) tells us that we have isomorphisms

$$\text{Hom}_{\mathcal{H}(A)}(M, \mathbf{i}N) \cong \text{Hom}_{\mathcal{D}(A)}(M, N) \cong \text{Hom}_{\mathcal{H}(A)}(\mathbf{p}M, N)$$

and so if M is homotopically projective we have $\text{Hom}_{\mathcal{D}(A)}(M, N) \cong \text{Hom}_{\mathcal{H}(A)}(M, N)$.

These functors allow us to define the derived functors in Rickard's Theorem 2.1.2 and show they are still an adjoint pair. For any category \mathcal{C} , dga A and functor $F : \mathcal{H}(A) \rightarrow \mathcal{C}$, we define the **total left derived functor** $\mathbf{L}F$ as the composition $F \circ \mathbf{p} : \mathcal{D}(A) \rightarrow \mathcal{C}$. Similarly, the **total right derived functor** $\mathbf{R}F$ is $F \circ \mathbf{i}$.

Returning to the case where A and B are dga's concentrated in degree zero, recall that a dg A - B -bimodule X is simply a complex of A - B -modules and we defined the adjoint functors $F := - \otimes_A X$ and $G := \mathcal{H}om_B(X, -)$. Then by the above we have

$$\text{Hom}_{\mathcal{D}(A)}(\mathbf{L}F(M), N) \cong \text{Hom}_{\mathcal{H}(A)}(F(\mathbf{p}M), \mathbf{i}N) \cong \text{Hom}_{\mathcal{H}(A)}(\mathbf{p}M, G(\mathbf{i}N)) \cong \text{Hom}_{\mathcal{D}(A)}(M, \mathbf{R}G(N))$$

and hence the associated derived functors also form an adjunction. For two dga's, A and B , given a dg A - B bimodule X , the functors F and G can be defined in exactly the same way as for complexes and the result above still holds.

The final bit of technology we need is to define a perfect dg A -module, but to motivate this, we take a closer look at the condition that T generates $\mathcal{D}(A)$. A set of objects $\{G_i\}$ of a category \mathcal{C} is a **set of generators** for \mathcal{C} if the functors $\text{Hom}_{\mathcal{C}}(G_i, -)$ are jointly faithful. If \mathcal{C} is an additive category, as all the categories we consider here are, this is equivalent to asking

$$\text{Hom}_{\mathcal{C}}(G_i, X) = 0 \forall i \iff X = 0.$$

However, we can find different characterisations for the particular cases we are interested in. For example, it is possible to show a module M is a generator for $\text{Mod}R$ if and only if every other module is a quotient of some direct sum of copies of M .

In the case of triangulated categories, as we like to consider shifts of objects as being closely related to the object, we say that an object G is a **generator** of \mathcal{C} if the set of objects $\{G[n] \mid n \in \mathbb{Z}\}$ forms a set of generators. For triangulated categories with infinite direct sums, in particular derived categories, to find a nice characterisation of generators, we need to put an extra condition on the generator. An object G of a triangulated category \mathcal{C} with infinite direct sums is **compact** if $\text{Hom}_{\mathcal{C}}(G, -)$ commutes with infinite direct sums. In the case A is a k -algebra and $\mathcal{C} = \mathcal{D}(A)$ we can view compact objects in terms of something we already know.

Lemma 2.2.2. [13] *A complex $M \in \mathcal{D}(A)$ is compact if and only if $M \in \text{per}A$.*

With this extra condition of compact we get the following theorem.

Theorem 2.2.3. [14] *If \mathcal{C} is a triangulated category with infinite direct sums, an object G is a compact generator if and only if the smallest full triangulated subcategory of \mathcal{C} containing G and closed under taking infinite direct sums is \mathcal{C} .*

For a k -algebra A , we show that A viewed as a complex concentrated in degree zero is a compact generator of $\mathcal{D}(A)$. It is clear that $A \in \text{per}A$ and so by Lemma 2.2.2, it is enough to show $\text{Hom}_{\mathcal{D}(A)}(A[n], X) = 0$ for all $n \in \mathbb{Z}$ implies $X = 0$. But, as A is homotopically projective

$$\text{Hom}_{\mathcal{D}(A)}(A[n], X) = \text{Hom}_{\mathcal{H}(A)}(A[n], X) = H^n(X)$$

for all $n \in \mathbb{Z}$ (which also provides an alternative proof that A is compact as homology commutes with infinite direct sums). Therefore, if this is zero for all n , the complex X is exact and hence zero in $\mathcal{D}(A)$. Using this, Theorem 2.2.3 tells us the smallest triangulated subcategory of $\mathcal{D}(A)$ containing A and closed under infinite direct sums is $\mathcal{D}(A)$. Note that this can be shown to be equivalent to the smallest triangulated subcategory of $\mathcal{D}(A)$ containing A and closed under direct summands being $\text{per}A$. This is more generally true for all compact generators.

Motivated by the above, we define **per A of a dga A** , to be the smallest triangulated subcategory of $\mathcal{D}(A)$ containing A and closed under direct summands. Then it can be shown that we have the analogue to Lemma 2.2.2

Lemma 2.2.4. [10] *A dg A -module M is compact if and only if $M \in \text{per}A$.*

Now, with the technology we have built up, we are ready to state and prove the following proposition.

Proposition 2.2.5. [10] *Let A and B be dga's, X a dg A - B bimodule such that $X \in \text{per}(B)$ and $F = -\otimes_A X$. Then the following are equivalent.*

1. *The functor $\mathbf{L}F : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is a triangle equivalence;*
2. *The functor $\mathbf{L}F$ induces an equivalence $\text{per}A \rightarrow \text{per}B$;*
3. *The object $T = \mathbf{L}F(A) \in \mathcal{D}(B)$ satisfies:*

$$(a) \operatorname{Hom}_{\mathcal{D}(B)}(T, T[n]) \cong \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases};$$

(b) $T \in \operatorname{per} B$;

(c) *The smallest full triangulated subcategory of $\mathcal{D}(B)$ containing T and closed under taking infinite direct sums is $\mathcal{D}(B)$.*

Proof. (1) \implies (2): This follows from Lemma 2.2.4 as compact is an intrinsic definition.

(2) \implies (3): As $\mathbf{L}F$ is a equivalence, it is clearly fully faithful. Using that and the fact it is a triangle functor, we get

$$\operatorname{Hom}_{\mathcal{D}(B)}(T, T[n]) \cong \operatorname{Hom}_{\mathcal{D}(B)}(\mathbf{L}F(A), \mathbf{L}F(A)[n]) \cong \operatorname{Hom}_{\mathcal{D}(A)}(A, A[n])$$

which gives condition (a). Condition (b) is obvious from (2) and condition (c) follows as an equivalence must send a generator to a generator.

(3) \implies (1): We need to show that $\mathbf{L}F$ is fully faithful and essentially surjective on objects. Since $\mathbf{L}F$ is a left adjoint, to prove it is fully faithful, it is enough to show the unit of the adjunction is an isomorphism i.e.

$$\varphi_M : M \rightarrow \mathbf{R}G\mathbf{L}F M$$

is an isomorphism for each $M \in \mathcal{D}(A)$. Let \mathcal{U} be the full subcategory of $\mathcal{D}(A)$ consisting of objects for which the above holds. We wish to show $\mathcal{U} = \mathcal{D}(A)$. By Theorem 2.2.3 and the comments afterwards, it is enough to show \mathcal{U} is a triangulated subcategory containing A and closed under infinite direct sums. First we show $A \in \mathcal{U}$. We have

$$\mathbf{R}G\mathbf{L}F(A) = \mathbf{R}G(T) = \mathbf{R}\mathcal{H}om_B(X, T)$$

but as B -modules $T = A \otimes_A^{\mathbf{L}} X \cong X$, this becomes

$$\mathbf{R}G\mathbf{L}F(A) = \mathbf{R}\mathcal{H}om_B(T, T).$$

But using the definition of the right derived functor and Theorem 2.2.1 we have

$$H^n(\mathbf{R}\mathcal{H}om_B(T, Y)) = H^n(\mathcal{H}om_B(T, \mathbf{i}Y)) = \operatorname{Hom}_{\mathcal{H}(B)}(T, \mathbf{i}Y[n]) \cong \operatorname{Hom}_{\mathcal{D}(B)}(T, Y[n])$$

for all $Y \in \mathcal{D}(B)$. Therefore, $H^n(\mathbf{R}\mathcal{H}om_B(T, -)) \cong \operatorname{Hom}_{\mathcal{D}(B)}(T, -[n])$ and so condition (a) tells us that $H^0(\mathbf{R}\mathcal{H}om_B(T, T)) \cong A$ and $H^n(\mathbf{R}\mathcal{H}om_B(T, T)) = 0$ for $n \neq 0$. Therefore, in $\mathcal{D}(A)$ we have

$$\mathbf{R}\mathcal{H}om_B(T, T) \cong A$$

where A is concentrated in degree 0. To show that \mathcal{U} is closed under infinite direct sums it is enough to show that $\mathbf{R}G\mathbf{L}F$ commutes with infinite direct sums. But $\mathbf{L}F$ commutes with infinite direct sums as it is a left adjoint and by the observation above we know $\mathbf{R}G \cong \mathbf{R}\mathcal{H}om_B(T, -)$. To show this commutes with infinite direct sums it is enough to show $H^n(\mathbf{R}\mathcal{H}om_B(T, -))$ commutes with infinite direct sums (as homology also does) but this holds as $T \in \operatorname{per} B$ and so is compact by Lemma 2.2.4. Therefore, $\mathbf{L}F$ is fully faithful and so all that remains to show is that it is essentially surjective on objects. But the image of $\mathbf{L}F$ is a triangulated subcategory (as $\mathbf{L}F$ is a triangle functor) which contains T and is closed under infinite direct sums (as $\mathbf{L}F$ commutes with them). Thus, by condition (c), $\mathbf{L}F$ is essentially surjective on objects and hence is an equivalence. \square

2.3 Proof of Rickard's Theorem

Using Proposition 2.2.5, we are now able to present Keller's proof of Rickard's Theorem, which we restate below for convenience.

Theorem 2.3.1 (Rickard's Theorem). *[10] Let A and B be two k -algebras. Then the following are equivalent:*

1. There is a complex of A - B bimodules X such that the functor $-\otimes_A^L X : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is a triangle equivalence;
2. There is a triangle equivalence $F : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$;
3. There is a triangle equivalence $\text{per} A \rightarrow \text{per} B$;
4. There is an object $T \in \mathcal{D}(B)$ such that:

$$(a) \text{Hom}_{\mathcal{D}(B)}(T, T[n]) \cong \begin{cases} A & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases};$$

(b) $T \in \text{per} B$;

(c) The smallest full triangulated subcategory of $\mathcal{D}(B)$ containing T and closed under taking infinite direct sums is $\mathcal{D}(A)$.

Proof. The only implication which is not trivial or not already covered in Proposition 2.2.5 is (4) \implies (1). Taking $X = T$ is not going to work as T is not a complex of A -modules. However, Proposition 2.2.5 tells us that if we can find a complex X of A - B -bimodules such that $A \otimes_A^L X = T$, then the functor $-\otimes_A^L X$ will give the desired equivalence.

Keller's key observation was that although T is not a complex of modules over $A = \text{End}_{\mathcal{D}(B)}(T, T)$, if we instead consider the endomorphism ring of T viewed as a dg B -module, T can be considered as a dg module over this. Thus, we take $C = \mathcal{H}om_B(T, T) = \text{Hom}_{C_{\text{dg}}(B)}(T, T)$ which is a dga and then T as a dg C - B -bimodule. Thus,

$$\begin{aligned} H^0(C) &= \text{Hom}_{\mathcal{H}(C)}(C, C) \\ &\cong \text{Hom}_{\mathcal{H}(C)}(C, \mathcal{H}om_B(T, T)) \\ &\cong \text{Hom}_{\mathcal{H}(B)}(C \otimes_C T, T) \\ &\cong \text{Hom}_{\mathcal{H}(B)}(T, T) \\ &\cong A \end{aligned}$$

as $T \in \text{per} B$ and so is homotopically projective. More generally $H^n(C) \cong \text{Hom}_{\mathcal{H}(B)}(T, T[n])$ and so $H^n(C)$ viewed as a complex with zero differential is isomorphic to A . Therefore, if we consider the subalgebra C_- with

$$(C_-)^n = \begin{cases} C^n & \text{if } n < 0 \\ Z^0 C & \text{if } n = 0 \\ 0 & \text{otherwise;} \end{cases}$$

we get a map

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{-2} & \longrightarrow & C_{-1} & \longrightarrow & Z^0(C) \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow q \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A \longrightarrow 0 \longrightarrow \dots \end{array}$$

where $q : Z^0(C) \rightarrow H^0(C) = A$ is the quotient map. By the calculation above these complexes have the same homology and so the map is a quasi-isomorphism. Since we can view T as a dg C - B -bimodule, it is clear that we can view it as a dg C_- - B -bimodule. Thus, we can define the object

$$X := A \otimes_{C_-} \mathbf{p}T.$$

where $\mathbf{p} : \mathcal{H}(B \otimes C_-^{op}) \rightarrow \mathcal{H}(B \otimes C^{op})$. This means that $\mathbf{p}T$ is homotopically projective both as a complex of B -modules and as a dg C_- -module. It is clear that X is a dg A - B -bimodule and so a complex of A - B -bimodules. All that remains to prove is that A is sent to T under the functor $-\otimes_A^L X$.

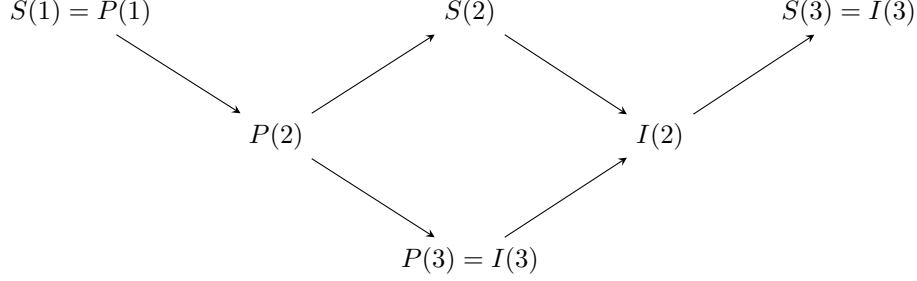


Figure 2.1: The AR quiver of the algebra A in Section 1.4.

But $A \otimes_A^L X = A \otimes_A^L (A \otimes_{C_-} \mathbf{p}T) \cong (A \otimes_{C_-} \mathbf{p}T)$ and the quasi-isomorphism $C_- \rightarrow A$ above induces a quasi-isomorphism $C_- \otimes_{C_-} \mathbf{p}T \rightarrow A \otimes_{C_-} \mathbf{p}T$. Moreover, $C_- \otimes_{C_-} \mathbf{p}T \cong \mathbf{p}T$ which is quasi-isomorphic to T by Theorem 2.2.1. So we have a chain of quasi-isomorphisms

$$T \leftarrow \mathbf{p}T \cong C_- \otimes_{C_-} \mathbf{p}T \rightarrow A \otimes_{C_-} \mathbf{p}T.$$

As all quasi-isomorphisms in $\mathcal{D}(B)$ are invertible, this implies there is an invertible map $T \rightarrow A \otimes_{C_-} \mathbf{p}T$. However, as $T \in \text{per}B$, T is homotopically projective and so $\text{Hom}_{\mathcal{D}(B)}(T, A \otimes_{C_-} \mathbf{p}T) \cong \text{Hom}_{\mathcal{H}(B)}(T, A \otimes_{C_-} \mathbf{p}T)$. Thus, the invertible map in $\mathcal{D}(A)$ must lift to a quasi-isomorphism in $\mathcal{H}(B)$. Therefore, we have $A \otimes_A^L X = T$ as required. \square

Throughout this paper we have considered the unbounded derived category $\mathcal{D}(A)$ of a k -algebra A . However, this is just a convenience as there are no issues defining the derived functors in this setting and it should be noted that Rickard's Theorem can also be stated and proved for bounded derived categories [11].

2.4 Using Rickard's Theorem

We now finish with an example of using Rickard's Theorem. This example comes from [11]. Consider the two algebras A and B both given by the quiver

$$1 \xleftarrow{\beta} 2 \xleftarrow{\alpha} 3$$

but where A has no relations and B has $\alpha\beta = 0$. Then, we have

$$A = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} k & k & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

We now show that these two algebras are derived equivalent by considering a tilting complex in A . For this it will be useful to know the AR-quiver of A which is given in Figure 2.1.

Consider the module $T = P(1) \oplus P(3) \oplus S(3)$ viewed as a complex of A -modules concentrated in degree 0. First of all, we notice that T is quasi-isomorphic to the complex

$$\dots \rightarrow 0 \rightarrow P(2) \xrightarrow{(0,0,i)} P(1) \oplus P(3) \oplus P(3) \rightarrow 0 \rightarrow \dots$$

where i is inclusion. This complex is clearly a perfect complex and so $T \in \text{per}A$. It is also clear that $\text{Hom}_{\mathcal{D}(B)}(T, T[n]) = 0$ when $n \neq 0$. To calculate $\text{Hom}_{\mathcal{D}(B)}(T, T)$ we use the AR quiver.

- $\text{Hom}_{\mathcal{D}(B)}(P(1), P(1)) = \text{Hom}_{\mathcal{D}(B)}(P(3), P(3)) = \text{Hom}_{\mathcal{D}(B)}(S(3), S(3)) = k$;

- $\text{Hom}_{\mathcal{D}(B)}(P(1), P(3)) = k, \text{Hom}_{\mathcal{D}(B)}(P(3), P(1)) = 0;$
- $\text{Hom}_{\mathcal{D}(B)}(P(1), S(3)) = 0, \text{Hom}_{\mathcal{D}(B)}(S(3), P(1)) = 0;$
- $\text{Hom}_{\mathcal{D}(B)}(S(3), P(3)) = 0, \text{Hom}_{\mathcal{D}(B)}(P(3), S(3)) = k.$

Therefore, we have

$$\text{Hom}_{\mathcal{D}(B)}(T, T) = \begin{pmatrix} k & k & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} = B.$$

Finally, we show that T generates $\mathcal{D}(A)$. Let $\langle T \rangle$ denote the smallest full triangulated subcategory of $\mathcal{D}(A)$ containing T and closed under taking direct summands. We wish to show that $\langle T \rangle = \text{per}(A)$. However, if we show that $A \in \langle T \rangle$, then, as we know the smallest full triangulated subcategory of $\mathcal{D}(A)$ containing A and closed under taking direct summands is $\text{per}(A)$, the result will follow.

Note that $A = P(1) \oplus P(2) \oplus P(3)$ and so to show that $A \in \langle T \rangle$, it is enough to show each of these terms are. Since $T = P(1) \oplus P(3) \oplus S(3)$ is clearly in $\langle T \rangle$ and $\langle T \rangle$ is closed under taking direct summands we get that $P(1), P(3), S(3) \in \langle T \rangle$. Finally, we note that there is an exact sequence

$$0 \rightarrow P(2) \xrightarrow{i} P(3) \rightarrow S(3) \rightarrow 0$$

where i is inclusion. Hence, there is a triangle, $P(2) \rightarrow P(3) \rightarrow S(3)$, in $\mathcal{D}(A)$. As $\langle T \rangle$ is triangulated subcategory of $\mathcal{D}(A)$, if two of the objects in the triangle are contained in $\langle T \rangle$, so is the third. Thus, as $P(3)$ and $S(3)$ belong to $\langle T \rangle$, so must $P(2)$. Therefore, T is a tilting complex and so by Rickard's Theorem, we get A and B are derived equivalent. The same tilting complex would also give that the bounded derived categories of A and B are also equivalent.

Chapter 3

Koszul Duality for Algebras

In this chapter we begin by developing the ideas of S. Priddy on Koszul resolutions ([21]) in the spirit of Loday and Valette's book Algebraic Operads ([16]), keeping algebras and coalgebras (mostly) on equal footing. The cobar and bar construction yield an adjunction between augmented dga's and conilpotent dgc's. The unit of this adjunction provides us with a quasi-free resolution of any algebra, thus, in theory, allowing us to compute derived functors such as Ext and Tor. However, these resolutions can be (unwieldingly) large in general. Therefore, we restrict our attention to Koszul algebras, quadratic algebras satisfying some condition, where we can do much better. For such algebras Koszul duality provides us with a minimal model as a subalgebra of the cobar-bar resolution. We conclude by discussing the derived equivalence between Koszul duals obtained in [18], and [20]. We will see that theorems improve when considering dual coalgebras instead of dual algebras.

Convention. In this chapter k will denote a characteristic 0 field. Algebras, coalgebras, vector spaces, and linear maps will abbreviate k -algebras, k -coalgebras, k -modules, and k -linear maps.

3.1 DGA's and DGC's

To define the cobar and bar constructions we will make use of the language of augmented dg algebras, and coaugmented dg coalgebras. Recall that a dg algebra is a differential graded vector space A endowed with two degree 0 morphisms, $\eta : k[0] \rightarrow A$ and $\mu : A \otimes A \rightarrow A$, such that $d_{\text{Hom}(A \otimes A, A)} \mu = 0$, satisfying the conditions (Ass) and (Un).

(Ass) The diagram

$$\begin{array}{ccc}
 A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\
 \downarrow \text{id} \otimes \mu & & \downarrow \mu \\
 A \otimes A & \xrightarrow{\mu} & A
 \end{array}$$

commutes.

(Un) The diagram

$$\begin{array}{ccccc}
 k \otimes A & \xrightarrow{\eta \otimes \text{id}} & A \otimes A & \xleftarrow{\text{id} \otimes \eta} & A \otimes k \\
 & \searrow \cong & \downarrow \mu & \swarrow \cong & \\
 & & A & &
 \end{array}$$

commutes.

We will say the dga is **augmented** when there is a given degree 0 morphism $\varepsilon : A \rightarrow k[0]$ s.t. $\varepsilon \circ \eta = id_k$, and **connected** if $A^0 \cong k$.

Example 3.1.1. (The Tensor Algebra) Let V be a (dg) vector space, we define the vector space

$$T(V) := \bigoplus_{n \geq 0} V^{\otimes n},$$

where $V^{\otimes 0} := k$. We will often denote elements $v \in T(V)_n := V^{\otimes n}$ as $v_1 \cdots v_n$, suppressing the \otimes -signs. We can equip $T(V)$ with the concatenation product

$$\begin{aligned} T(V) \otimes T(V) &\rightarrow T(V), \\ v_1 \cdots v_n \otimes w_1 \cdots w_m &\mapsto v_1 \cdots v_n w_1 \cdots w_m. \end{aligned}$$

$T(V)$ is naturally an augmented, in fact connected, graded algebra, with a grading called the **weight grading**, defined by

$$\omega(v_1 \cdots v_n) := n.$$

In the case that the vector space V also had dg structure, then $T(V)$ inherits the structure of a dga from V , with grading called the **cohomological grading** of $T(V)$, defined by

$$\begin{aligned} \deg(v_1 \cdots v_n) &:= \deg(v_1) + \dots + \deg(v_n), \\ \deg(1) &:= 0, \\ d_{T(V)_n} &:= \sum_{i=1}^n id_V \otimes \dots \otimes id_V \otimes d_V \otimes id_V \otimes \dots \otimes id_V. \end{aligned}$$

Where d_V occurs in the i th place. Note that the dg structure is compatible with the weight grading in the sense that the differential respects the weight grading i.e. $\omega(dv) = \omega(v) \forall v \in T(V)_n$, thus the complex splits as follows:

We say that $T(V)$ has the structure of a **weight graded dga**, or **wdga**.

Denote $\bar{T}(V) := \bigoplus_{n \geq 1} V^{\otimes n}$ the **restricted tensor algebra**, a non-unital subalgebra of $T(V)$.

We recall some basic properties of $T(V)$.

Proposition 3.1.2. *Let V be a (graded) k -vector space.*

1. $T(V)$ is free over V in the category of unital algebras i.e. for every (graded) k -linear map $f : V \rightarrow A$, from V to a (graded) unital algebra A there exists a unique extension $\tilde{f} : T(V) \rightarrow A$ a morphism of graded algebras.
2. $\bar{T}(V)$ is free over V in the category of non-unital algebras.
3. Every linear map $f : V \rightarrow \bar{T}(V)$ extends uniquely to a derivation $d_f : \bar{T}(V) \rightarrow \bar{T}(V)$.

□

Dually we will need the notions of (cofree) (dg) coalgebra.

Definition 3.1.3. 1. A **differential graded coalgebra** (or **dgc**) is a dg vector space C endowed with two degree 0 morphisms $\varepsilon : C \rightarrow k[0]$, $\Delta : C \rightarrow C \otimes C$ such that

$$\Delta \circ d_C = (d_C \otimes id_C + id_C \otimes d_C) \circ \Delta,$$

and the conditions (CoUn) and (CoAss) hold.

(CoUn) The diagram

$$\begin{array}{ccccc} C \otimes C & \xleftarrow{\varepsilon \otimes id} & C \otimes C & \xrightarrow{id \otimes \varepsilon} & C \otimes C \\ & \searrow \cong & \uparrow \Delta & \swarrow \cong & \\ & & C & & \end{array}$$

commutes.

(Coass) The diagram

$$\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow \Delta & & \downarrow \text{id} \otimes \Delta \\
C \otimes C & \xrightarrow{\Delta \otimes \text{id}} & C \otimes C \otimes C
\end{array}$$

commutes.

2. A **morphism of dg coalgebras** is a degree 0 linear map $f : C \rightarrow C'$ that commutes with coproducts, counits and differentials i.e. $\Delta_{C'} \circ f = (f \otimes f) \circ \Delta_C$, $\varepsilon_{C'} \circ f = f \circ \varepsilon_C$, and $d_{C'} \circ f = f \circ d_C$.
3. We will say a dgc is **coaugmented** when there is a given degree 0 morphism $\eta : k[0] \rightarrow C$ such that $\varepsilon \circ \eta = \text{id}_k$. We will write $C = k \oplus \bar{C}$, where $\bar{C} = \ker(\varepsilon)$, for coaugmented dgc's. A **morphism between coaugmented coalgebras** is a coalgebra morphism $f : C \rightarrow C'$ s.t. $f \circ \eta_C = \eta_{C'}$.
4. A coaugmented dgc is called **conilpotent** if $\bar{\Delta}^n(c) = 0 \ \forall c \in \bar{C}$ for some $n \in \mathbb{N}$. Here $\bar{\Delta}^n(c) := (\Delta \otimes \text{id} \otimes \dots \otimes \text{id}) \circ \bar{\Delta}^{n-1}$, and

$$\bar{\Delta}(c) := \Delta(c) - 1_C \otimes c - c \otimes 1_C.$$

5. A **coderivation** of a coaugmented dgc C is a linear map $d : C \rightarrow C$ such that

$$\begin{aligned}
\Delta \circ d &= (d \otimes \text{id}_C + \text{id}_C \otimes d) \circ \Delta, \\
d(1_C) &= 0.
\end{aligned}$$

6. A conilpotent coalgebra C is called **cofree** over a vector space V if it is endowed with a linear map $\pi : C \rightarrow V$ s.t. $\pi(1) = 0$. Moreover, for every linear map $\varphi : C' \rightarrow V$ with C' a conilpotent coalgebra, there exists a unique morphism of coaugmented coalgebras $\tilde{\varphi} : C' \rightarrow C$ such that $\pi \circ \tilde{\varphi} = \varphi$.
7. In the case that a differential graded coalgebra has another degree, we will refer to the auxiliary grading as **weight**, and to the differential grading as **cohomological degree**. A **weight graded dg coalgebra** or **wdgc** is a dgc where all the morphisms respect the weight (are 0 weight graded morphisms).

It is not hard to check that the map $\bar{\Delta}$, called the reduced coproduct, is also coassociative. In fact we obtain a functor $(C, \Delta, \varepsilon) \mapsto (\bar{C}, \bar{\Delta})$ sending counital, coassociative coalgebras to coassociative coalgebras; the latter not having a counit.

Remark. Even though the definitions of algebra and coalgebra are naturally dual, the duality is rather subtle. Observe that for defining cofreeness we assumed conilpotency. We need this assumption to make the cofree objects simple to describe. Another example: let C be a coalgebra. Then C^* naturally carries the structure of an algebra. However, for A an algebra, A^* only carries a coalgebra structure if A is finite dimensional. For general A one needs to take something smaller than the full linear dual to move from algebras to coalgebras. It is exactly this latter constraint why coalgebras are the right language for Koszul duality. Normally in defining Koszul duals one takes the linear dual of some coalgebra, which forces you to use finiteness assumptions. However, if one phrases Koszul duality in the language of coalgebras, one does not need any finiteness assumptions to formulate the important theorems.

Example 3.1.4. (Tensor Coalgebra) Let V be a (dg) vector space. The vector space $\oplus_{n \geq 0} V^{\otimes n}$ underlying the tensor algebra $T(V)$ naturally carries the structure of a conilpotent coalgebra, denoted $T^c(V)$. The counit ε is given by the augmentation of $T(V)$. The coproduct $\Delta : T^c(V) \rightarrow T^c(V) \otimes T^c(V)$, called the deconcatenation coproduct, and is defined by

$$\Delta(v_1 \cdots v_n) = 1 \otimes v_1 \cdots v_n + v_1 \otimes v_2 \cdots v_n + \cdots + v_1 \cdots v_n \otimes 1.$$

If V was a dg-mod then $T^c(V)$ is naturally a wdgc. We denote $\bar{T}^c(V) := \oplus_{n \geq 1} V^{\otimes n}$ the restricted tensor coalgebra, a subcoalgebra of $T^c(V)$.

Proposition 3.1.5. *Let V be a (graded) vector space.*

1. $T^c(V)$ is cofree over V .
2. Any linear map $f : T^c(V) \rightarrow V$ uniquely extends to a coderivation $d_f : T^c(V) \rightarrow T^c(V)$.

Proof. We will only prove 1, as the proof of 2 is almost identical. For $x \in T^c(V)$ denote $x_n \in V^{\otimes n}$ its weight n component. Let $\varphi : C \rightarrow V$ be a linear map s.t $\varphi(1) = 0$. If an extension Φ exists we must have the following identities for $x \in \bar{C}$:

$$\begin{array}{ll} \Phi(1) = 1 & \text{by coaugmentation,} \\ \Phi(x)_0 = 0 & \text{by counitality,} \\ \Phi(x)_1 = \varphi(x) & \text{by compatibility with } \varphi, \\ \Phi(x)_n = (\varphi \otimes \dots \otimes \varphi) \circ \bar{\Delta}^n(x) & \text{as } \Phi \text{ is a coalgebra morphism.} \end{array}$$

This last condition comes from the fact that we are asking $\Phi \otimes \Phi \circ \Delta_C = \Delta_{T^c(V)} \circ \varphi$ which implies -by functoriality of $(C, \Delta) \mapsto (\bar{C}, \bar{\Delta})$ - that $\Phi \otimes \Phi \circ \bar{\Delta}_C = \bar{\Delta}_{T^c(V)} \circ \Phi$. As $(\bar{\Delta}_{T^c(V)})^n(v_1 \cdots v_n) = v_1 \otimes \dots \otimes v_n \in T^c(V)^{\otimes n}$ the identity follows.

Thus if Φ exists it is unique. As C is conilpotent $\bar{\Delta}^n(x) = 0$ for $n > N$ for some N for every $x \in \bar{C}$. Thus we can define $\Phi(x) := \sum_n \Phi(x)_n$. It is a coalgebra morphism extending φ by construction. \square

3.2 The Cobar-Bar Resolution

Let A be an augmented dga, C be a coaugmented dgc. As the mapping space between dg k -modules we have seen that $\text{Hom}(C, A) = \text{Hom}_{k\text{-mod}}(C, A)$ naturally carries a dg k -module structure $(\text{Hom}(C, A), \partial)$. We can define the **convolution product** \star as follows

$$f \star g := \mu_A \circ f \otimes g \circ \Delta_C.$$

It is not hard to check that $(\text{Hom}(C, A), \star, \partial)$ defines a dga, called a convolution algebra. We can define a bracket $[,]$ by

$$[a, b] := a \star b - (-1)^{\bar{a}\bar{b}} b \star a,$$

giving a dgla structure on $\text{Hom}(C, A)$. For $\alpha \in \text{Hom}(C, A)$ of degree one we can define the twisted derivation

$$\partial_\alpha := \partial + [\alpha, -],$$

∂_α defines a differential exactly when α satisfies the Maurer-Carten equation¹

$$\partial\alpha + \alpha \star \alpha = 0. \tag{3.1}$$

We call such morphisms **twisting morphisms**, or say that α is twisting. We denote the set of twisting morphisms by

$$\text{Tw}(C, A) := \{\alpha \in \text{Hom}(C, A)^1 : \alpha \star \alpha + \partial\alpha = 0\}.$$

Thus the twisting morphisms are exactly the degree one morphisms that define a dga $(\text{Hom}(C, A), \star, \partial_\alpha)$, called a twisted convolution algebra. We can also twist the dg k -module $C \otimes A$ by α as follows. For $\alpha : C \rightarrow A$ define a map

$$\begin{aligned} d_\alpha^r : C \otimes A &\rightarrow C \otimes A, \\ c \otimes a &\mapsto (\text{id}_C \otimes \mu_A) \circ (\text{id}_C \otimes \alpha \otimes \text{id}_A) \circ (\Delta_C \otimes \text{id}_A). \end{aligned}$$

Then $(C \otimes A, d_{C \otimes A} + d_\alpha^r)$ is a dg k -module iff α is twisting. We denote this dg-module $C \otimes_\alpha A$, and call it the **(right) twisted tensor product**. Similarly one can define the left twisted tensor product $A \otimes_\alpha C$.

¹The equation perhaps looks more familiar if one replaces $\alpha \star \alpha$ by $\frac{1}{2}[\alpha, \alpha]$.

We will now construct an adjoint pair of functors

$$B : \{\text{augmented dga's}\} \rightleftarrows \{\text{conilpotent dgc's}\} : \Omega \quad (3.2)$$

that (co)-represents the bifunctor Tw .

Let C be a conilpotent dgc $C = k \oplus \bar{C}$. Recall the morphism

$$\bar{\Delta} : \bar{C} \rightarrow \bar{C} \otimes \bar{C}.$$

We can introduce the **Sweedler notation**:

$$\bar{\Delta}(c) = \sum c_{(1)} \otimes c_{(2)}.$$

The reduced coproduct induces a degree one morphism

$$\begin{aligned} \bar{C}[-1] &\rightarrow T(\bar{C}[-1]), \\ c &\mapsto - \sum (-1)^{\deg(c_{(1)})} c_{(1)} \otimes c_{(2)}. \end{aligned}$$

Remark. Formally, the signs are picked up from a graded flip, and the graded coproduct on $k[-1]$, but you can just take this as the definition.

This map extends to a derivation $d_2 : T(\bar{C}[-1]) \rightarrow T(\bar{C}[-1])$ by proposition 3.1.2 part 3. Recall that the cohomological grading on $\bar{T}(V)$ was defined by $\deg(\prod v_i) = \sum_i \deg(v_i)$. From the construction of d_f it is then obvious d_2 is a derivation of degree one.

Similarly d_C is a degree one morphism $C \rightarrow C$ and induces a degree one map

$$\begin{aligned} \bar{C}[-1] &\rightarrow T(\bar{C}[-1]), \\ c &\mapsto -d_C(c) \end{aligned}$$

It extends to a degree one derivation denoted $d_1 : T(\bar{C}[-1]) \rightarrow T(\bar{C}[-1])$.

Proposition 3.2.1. *Let C be a conilpotent dg coalgebra. The cobar construction*

$$(\Omega C, d_{\Omega C}) := (T(\bar{C}[-1]), d_1 + d_2)$$

defines a functor from conilpotent dgc's to augmented dga's.

Proof. There are two things that remain to be checked. Firstly, that we have defined a functor, and secondly that $d_1 + d_2$ is a differential. The functoriality is obvious as the cobar construction is just a composition of functors: $C \mapsto \bar{C}$, the shift functor, and taking the free algebra (i.e. the left adjoint to a forgetful functor). As d_1 and d_2 are both degree one derivations we only need to check that $d_1 + d_2$ squares to zero. Since $d_C^2 = 0$, it is easy to see that $d_1^2 = 0$, one only needs to take care about the signs. For d_2 it is exactly the shift $[-1]$ that kills the square, using coassociativity of $\bar{\Delta}$.

Finally using $\bar{\Delta}_C \circ d_C = (d_C \otimes \text{id} + \text{id} \otimes d_C) \circ \bar{\Delta}_C$ we find that $d_1 \circ d_2 = -d_2 \circ d_1$, again obtaining a minus sign by the shift. \square

For A an augmented dga, we can restrict the multiplication $\mu : \bar{A} \otimes \bar{A} \rightarrow \bar{A}$, it induces a degree one morphism

$$\begin{aligned} \mu : \bar{A}[1] \otimes \bar{A}[1] &\rightarrow T^c(\bar{A}[1]), \\ a \otimes a' &\mapsto -(-1)^{\bar{a}} a a'. \end{aligned}$$

By Proposition 3.1.5 part 2 we obtain a (degree one) coderivation, denoted,

$$d_2 : T^c(\bar{A}[1]) \rightarrow T^c(\bar{A}[1]).$$

Analogously, d_A induces a degree one coderivation d_1 .

Proposition 3.2.2. *let A be an augmented dga. The Bar construction*

$$(BA, d_{BA}) := (T^c(\bar{A}[1]), d_1 + d_2)$$

defines a functor from augmented dga's to conilpotent dgc's.

Proof. The tensor coalgebra is automatically conilpotent. The rest of the proof is similar to the cobar proof. \square

Theorem 3.2.3. *Let A be an augmented dga, C a conilpotent dgc. There exist natural isomorphisms*

$$Hom_{aug.dga}(\Omega C, A) \cong Tw(C, A) \cong Hom_{con.dgc}(C, BA),$$

where twisting maps are required to send $k \rightarrow 0$, and $\bar{C} \rightarrow \bar{A}$.

Proof. We will prove the first natural isomorphism. Note that since twisting morphisms $\varphi : C \rightarrow A$ are required to send $k \rightarrow 0$ and $\bar{C} \rightarrow \bar{A}$, the data of a twisting map $\varphi : C \rightarrow A$ is the same as a degree one map $\varphi : \bar{C} \rightarrow \bar{A}$ satisfying the M.C.-equation i.e. a degree zero map $\bar{\varphi} : \bar{C}[-1] \rightarrow \bar{A}$ satisfying Maurer-Cartan. We claim that the natural bijection is given by

$$\Phi : \Omega C \rightarrow A \quad \mapsto \quad \varphi = \Phi|_{\bar{C}} : \bar{C} \rightarrow A.$$

With inverse given by the extension property using freeness of $\Omega\bar{C}$. Obviously these are natural morphisms, inverse to each other, but it remains to check whether we restrict to a twisting morphism, and extend to a morphism of augmented dga's.

First observe, that as $\Phi : \Omega C \rightarrow A$ is a degree 0 morphism of augmented algebras, it restricts to a degree 1 map $\varphi : \bar{C}[-1] \rightarrow \bar{A}$. We then have the following equivalent expressions

$$\begin{aligned} d_A \circ \Phi &= \Phi \circ (d_1 + d_2), \\ &\iff \\ d_A \circ \Phi(c) &= \Phi \circ (d_1 + d_2)(c) \quad \forall c \in C[-1], \\ &\iff \\ d_A \circ \varphi(c) &= \varphi \circ (-d_C(c)) - \sum (-1)^{\deg(c_{(1)})} \varphi(c_{(1)}) \otimes \varphi(c_{(2)}) \quad \forall c \in C[-1], \\ &\iff \\ d_A \circ \varphi(c) + \varphi \circ d_C(c) &= -\varphi \star \varphi(c) \quad \forall c \in \bar{C}. \end{aligned}$$

This proves the last statement, and hence concludes the proof. \square

We now state, but will not prove, the Fundamental Theorem on Twisting Morphisms. For a proof, see for example [16, thm 2.3.2].

Theorem 3.2.4. (Fundamental Theorem on Twisting Morphisms) *Let A be a weight graded dga, with $A_0 = k$, C a weight graded dgc, s.t. $C_0 = k$, and let $\alpha : C \rightarrow A$ twisting. The following are equivalent.*

1. $C \otimes_\alpha A$ is acyclic.
2. $A \otimes_\alpha C$ is acyclic.
3. The induced dgc morphism $f_\alpha : C \rightarrow BA$ is a quasi-isomorphism.
4. The induced dga morphism $g_\alpha : \Omega C \rightarrow A$ is a quasi-isomorphism.

Corollary 3.2.5. *Let A be an augmented algebra. The counit of the adjunction $\Omega BA \rightarrow A$ is a quasi-isomorphism.*

Proof. We view A as a wdga concentrated in weight degree 0 and cohomological degree 0, with trivial differential. Recall that the counit is obtained through the adjunction from $\text{id}_{BA} \in \text{Hom}_{\text{con.dgc}}(BA, BA)$. Denote the associated twisting morphism $\pi : BA \rightarrow A$. The map π is just obtained by projecting id_{BA} onto the $A[-1]$ i.e.

$$\pi : BA = T(\bar{A}[-1]) \rightarrow \bar{A}[-1] \rightarrow A.$$

By the Fundamental Theorem on Twisting Morphisms it suffices to prove that $BA \otimes_{\pi} A$ is acyclic. Since A has a trivial dg structure, we have that the complex is non-positively graded $(BA \otimes_{\pi} A)^{-n} = \bar{A}^{\otimes n} \otimes A$, with differential $d = d_{BA} \otimes \text{id}_A + d_{\pi}^r$. We will denote homogeneous elements as $[a_1 | \dots | a_n] a_{n+1} \in (BA \otimes_{\pi})^{-n}$. As $d_A = 0$, we have that $d_1 = 0$. Then the differential on BA acts as follows

$$d_{BA}[a_1 | \dots | a_n] = - \sum_i (-1)^i [a_1 | \dots | a_{i-1} | a_i a_{i+1} | a_{i+2} | \dots | a_n].$$

Combining this with the action of d_{π}^r we find that the differential acts as

$$\begin{aligned} d[a_1 | \dots | a_n] a_{n+1} &= - \sum_i (-1)^i [a_1 | \dots | a_{i-1} | a_i a_{i+1} | a_{i+2} | \dots | a_n] a_{n+1} \\ &\quad + (-1)^{n-1} [a_1 | \dots | a_{n-1}] a_n a_{n+1} \end{aligned}$$

We wish to prove $BA \otimes_{\pi} A$ is acyclic, recall that by this one actually means that the augmentation map $\varepsilon : BA \otimes_{\pi} A \rightarrow k[0]$ is a quasi-isomorphism. This is equivalent to proving $K = \ker(\varepsilon)$ is (truly) acyclic. The map

$$\begin{aligned} h : K &\rightarrow K \\ [a_1 | \dots | a_n] a_{n+1} &\mapsto (-1)^n [a_1 | \dots | a_n] a_{n+1} - \varepsilon(a_{n+1}) \end{aligned}$$

is a homotopy between id_K and 0. □

Remark. The complex $BA \otimes_{\pi} A$ we obtain is in fact the non-unital Hochschild cochain complex of \bar{A} with coefficients in A .

We have found a way to produce quasi-free resolutions of any augmented algebra A . Unfortunately, these resolutions are incredibly large. As we saw in the proof above for every step in the resolution we are multiplying by $\dim(\bar{A})$. We want to find smaller resolutions to be able to make actual computations. To be more precise we want to find minimal models. Let A be dga, a **minimal model** of A is a quasi-free dga $(T(V), d)$ endowed with a surjective quasi-isomorphism $p : T(V) \twoheadrightarrow A$ such that

1. $d : V \rightarrow T(V)_{\geq 2}$,
2. $V = \bigoplus_{n \geq 1} V^{-n}$ such that $d(V^{-n-1}) \subset T(\bigoplus_{i=1}^n V^{-i})$.

It is a non-trivial fact that if minimal models exist, they are unique up to (non-unique) isomorphism. Priddy's Koszul resolutions will provide us with minimal models for so-called Koszul algebras.

3.3 The Koszul Resolution

We will define the Koszul resolution for quadratic Koszul algebras, this is no restriction as one can show that any Koszul algebra admits a quadratic presentation.

Let V be a graded vector space over k . Let $R \subset V \otimes V$ be a graded graded subspace. The **quadratic algebra associated to** (V, R) is defined as

$$A(V, R) := T(V)/(R).$$

We can view $A(V, R)$ as a wdga. The weight is inherited from the weight grading in $T(V)$, the cohomological degree is induced by V , and a trivial differential ($d = 0$). It has the following universal property. Every surjective algebra morphism $T(V) \twoheadrightarrow A$ such that $R \rightarrow T(V) \rightarrow A$ composes to 0, factors uniquely through $A(V, R)$.

Dually we define the **quadratic coalgebra associated to** (V, R) to be the subcoalgebra of $T^c(V)$ with the following universal property. Every coalgebra C that injects into the tensor coalgebra $C \hookrightarrow T^c(V)$ such that $C \rightarrow T^c(V) \rightarrow V \otimes V/R$ composes to 0, in fact injects into $C(V, R)$. Concretely,

$$C(V, R) := k \oplus V \oplus R \oplus (R \otimes V \cap V \otimes R) \oplus \dots$$

gives the decomposition of $C(V, R)$ into its components of different weight.

For $A(V, R)$ a quadratic algebra, we define the **Koszul dual coalgebra** of $A(V, R)$ to be

$$A^! := C(V[1], R[1]^2),$$

where $R[1]^2$ is R with grading inherited from $T(V[1])$. We define the **Koszul dual algebra** to be

$$A^! := \bigoplus_n ((A^!)^*)^n[-n].$$

Proposition 3.3.1. *Let R^\perp denote the image of $(V \otimes V/R)^*$ inside $V^* \otimes V^*$. Then we have the following algebra isomorphism*

$$A(V, R)^! \cong A(V^*, R^\perp).$$

Corollary 3.3.2. *For $A = A(V, R)$ a finitely generated algebra, i.e. $\dim(V) < \infty$, then $(A^!)^! \cong A$.*

Example 3.3.3. 1. For $A(V, R) = T(V)$ i.e. $R = 0$, we have $R^\perp = V^* \otimes V^*$. Then $T(V)^! \cong k \oplus V^*$, the dual numbers on $\dim(V)$ generators.

2. For $A(V, R) = S(V)$ i.e. $R = \langle v_i \otimes v_j - v_j \otimes v_i \rangle$. Then $R^\perp = \langle \check{v}_i \otimes \check{v}_j + \check{v}_j \otimes \check{v}_i \rangle$ such that $S(V)^! \cong \bigwedge(V^*)$.

We will now introduce yet another degree on BA that will replace the cohomological grading on BA .

From now on $A = A(V, R)$ denotes a quadratic algebra i.e. V is not graded!

Then $A(V, R)$ is a wdga, concentrated in cohomological degree 0, with trivial differential. The weight grading ω on $A = A(V, R)$ induces a weight grading on BA as follows.

Notation. We will use the tensor product only in the context of BA , not in $A(V, R)$.

v_i	element of V
$v_1 \dots v_n$	element of $T(V)$
$[v_1 v_2 v_3], [v_1 v_2], v_1,$	elements in the quotient $A = A(V, R)$
$v_1 \otimes [v_2 v_3 v_4] \otimes [v_5 v_6]$	element of BA

The **weight grading** on BA is defined by

$$\omega([v]_1 \otimes \dots \otimes [v]_n) = \omega([v]_1) + \dots + \omega([v]_n),$$

the **syzygy grading** is defined by

$$\omega([v]_1 \otimes \dots \otimes [v]_k) = \omega([v]_1) + \dots + \omega([v]_k) - k.$$

Since $d_A = 0$ we have that $d_{BA} = d_2$. Recall d_2 is induced by a graded multiplication, therefore, it respects the weight grading, but increases the syzygy degree by one. The figure below depicts the split cochain complex. For clarity we use the notation

$$A(V, R) = k \oplus V \oplus R/V \oplus V^2/(RV + VR) \oplus \dots,$$

again keeping tensor products for tensors in BA .

$$(3) \quad V^{\otimes 3} \longrightarrow (V^2/R \otimes V) \oplus (V \otimes V^2/R) \longrightarrow V^3/(VR + RV) \longrightarrow 0$$

$$(2) \quad V^{\otimes 2} \longrightarrow V^2/R \longrightarrow 0$$

$$(1) \quad V \longrightarrow 0$$

$$(0) \quad k$$

$$0 \qquad \qquad \qquad 1 \qquad \qquad \qquad 2 \qquad \qquad \qquad 3$$

Proposition 3.3.4. *Let A be a quadratic algebra. We have the natural coalgebra inclusion $A^i \rightarrow BA$ which induces an isomorphism of graded coalgebras*

$$A^i \cong H^0(BA).$$

Proof. From figure it is obvious that the syzygy 0 component of BA is exactly the tensor coalgebra $T^c(V[1])$; the Koszul dual coalgebra A^i is by definition a subcoalgebra of $T^c(V[1])$. Therefore, it suffices to check that for every weight $n \geq 0$ the kernel of the boundary map $d_n^0 : V^{\otimes n} \rightarrow (BA)_n^1$ is exactly $(A^i)^n$. This is obvious for $n = 0, 1, 2$. For higher n the boundary map sends

$$v_1 \otimes \dots \otimes v_n \mapsto \sum_i v_1 \otimes \dots \otimes v_{i-1} \otimes [v_i v_{i+1}] \otimes v_{i+2} \otimes \dots \otimes v_n,$$

where all summands land in different direct summands of $(BA)_n^1$. Hence an element lies in the kernel iff $[v_i v_{i+1}] = 0 \forall i$ i.e. $v_i v_{i+1} \in R \forall i$. Those are exactly the elements in $A^i = C(V[1], R[1]^2)$. \square

We can define a degree one map κ as the following composition

$$A^i \rightarrow V[1] \rightarrow V \rightarrow A(V, R).$$

Lemma 3.3.5. *The morphism κ is twisting.*

Proof. Since A^i and A both have trivial differentials, we wish to show $\kappa \star \kappa = 0$. Since κ is only non-zero the weight one component $A_1^i = V[1]$, it suffices to check that $\kappa \star \kappa$ is 0 on the weight two component. However there $\kappa \star \kappa$ is just the 0 map $A_2^i = R[1]^2 \rightarrow A_2 = V^2/R$. \square

Theorem 3.3.6. (Koszul Criterion) *The following are equivalent.*

1. *The (right) Koszul complex $A^i \otimes_{\kappa} A$ is acyclic.*
2. *The (left) Koszul complex $A \otimes_{\kappa} A^i$ is acyclic.*
3. *The inclusion $A^i \rightarrow BA$ is a quasi-isomorphism.*
4. *$H^d(BA) = 0$ for $d \geq 1$.*
5. *The induced map $\Omega A^i \rightarrow A$ is a quasi-isomorphism.*

*When these assertions hold ΩA^i is a minimal model of A , called the **Koszul resolution**.*

Proof. The inclusion $A^i \rightarrow BA$ is a morphism of conilpotent coalgebras. Obviously composing the inclusion $A^i \rightarrow BA$ with the projection $BA \rightarrow \bar{A}$ yields κ . Hence the equivalence of the statements follows immediately from the Fundamental Theorem on Twisting Morphisms and Proposition 3.3.4.

One can now directly check that ΩA^i satisfies the properties of being a minimal model. \square

Proposition 3.3.7. *The symmetric algebra $S(V)$ is Koszul.*

Proof. The Koszul dual coalgebra of $S(V)$ is given by the alternating coalgebra $\bigwedge^c(V[-1])$. We denote homogeneous elements of $\bigwedge^c(V[1])$ by $x_1 \wedge \dots \wedge x_n$, and homogeneous elements of $S(V)$ by $s_1 \cdots s_m$. As both $S(V)$ and $\bigwedge^c(V[1])$ have trivial differentials the differential on the right Koszul complex is given by

$$d = d_\kappa : x_1 \wedge \dots \wedge x_n \otimes s_1 \cdots s_m \mapsto \sum_{i=1}^n (-1)^{n-i} x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_n \otimes x_j s_1 \cdots s_m.$$

We define a morphism h as follows

$$h : x_1 \wedge \dots \wedge x_n \otimes s_1 \cdots s_m \mapsto \sum_{i=1}^m x_1 \wedge \dots \wedge x_n \wedge s_i \otimes s_1 \cdots \hat{s}_i \cdots s_m.$$

One can check that $hd + dh = (p + q)\text{id}$, so that h is not quite a homotopy, but the complex is acyclic. \square

3.4 Koszul Duality as Derived Equivalence

We will call a quadratic algebra A **Koszul**, or a **Koszul algebra**, if the conditions from Theorem 3.3.6 hold.

Remark. More generally, a connected algebra $A = \bigoplus_{n \geq 0} A_n$, is called Koszul if $H^d(BA) = 0$ for $d > 0$. However, such algebras admit a quadratic representation as follows

$$\begin{aligned} A &\cong A(V, R), \text{ where} \\ V &= A_1, \\ R &= \ker(d_1^0 : A_1 \otimes A_1 \rightarrow A_2). \end{aligned}$$

Thus there is no restriction treating only quadratic Koszul algebras.

Proposition 3.4.1. *Let A be a finitely generated quadratic algebra. A is Koszul iff $A^!$ is Koszul.*

Proof. Let $\kappa : A^i \rightarrow (A^i)^!$ denote the Koszul twisting morphism. The left Koszul complex $A^i \otimes_{\kappa'} A$ is linearly dual to the right Koszul complex $A^i \otimes_{\kappa} A$ up to suspension. Hence if the one is acyclic, so is the other. \square

Koszul duality has deep implications for the representation theory of both algebras.

Proposition 3.4.2. *Let A be a finitely generated Koszul algebra. We have*

$$A^! \cong \text{Ext}_A(k, k)$$

canonically.

Proof. Recall that $\text{Ext}_A(k, k) := \text{RHom}_A(k, k)$. We use our canonical resolution $BA \otimes_{\pi} A \rightarrow k$ to obtain

$$\begin{aligned} \text{Ext}_A(k, k) &= H_*(\text{Hom}_A(BA \otimes_{\pi} A, k)) \\ &= H_*(\text{Hom}(BA, k)) \\ &\cong A^!. \end{aligned}$$

Where we used that $H^d(BA) = 0$ for $d > 1$ and, since A and hence $A^!$ a f.g., we find that $A^! \cong H^0(BA)^* \cong (H_0(BA^*))^*$. \square

Using the identification in Proposition 3.4.2 we can state the derived equivalence described in [18] as follows

Theorem 3.4.3. [18, thm 2.12.6] *Let A be a finitely generated Koszul algebra, let $A^! = \text{Ext}_A(k, k)^{\text{opp}} = \text{RHom}_A(k, k)^{\text{opp}}$ denote its Koszul dual algebra. The functor $K := \text{RHom}_A(k, -)$ defines an equivalence of triangulated categories*

$$K : \mathcal{D}^b(A\text{-mof}) \rightarrow \mathcal{D}^b(A^!\text{-mof})$$

the Koszul duality functor. Where $A\text{-mof}$ denotes the category of finitely generated graded left A -modules.

Remark. The derived equivalence $\mathcal{D}^b(S(V)\text{-mof})$ and $\mathcal{D}^b(\bigwedge(V^*)\text{-mof})$ was first described in [19].

One can wonder whether the derived equivalence can be upgraded to a full derived morita equivalence in the sense of 2.3.1. The following example shows that this cannot be done in general.

Proposition 3.4.4. *Let $K : \mathcal{D}^b(S(V)\text{-mof}) \cong \mathcal{D}^b(\bigwedge(V^*)\text{-mof})$ denote the derived equivalence obtained from 3.4.3. There exists no derived equivalence $\tilde{K} : \mathcal{D}(S(V)) \rightarrow \mathcal{D}(\bigwedge(V^*))$ extending K .*

Proof. An extension of the Koszul duality functor will have to send $S(V)$ to k . $S(V)$ is finitely generated and sits in cohomological degree 0, thus it is perfect by definition. However, the minimal model of k is the Koszul resolution, which is not a bounded complex. Therefore, k cannot be perfect. We conclude K cannot be extended to a functor that satisfies condition 3 in 2.3.1, and thus we cannot obtain a full derived equivalence. \square

There does exist a full derived equivalence, but it is different from the one above. We will need the language of coalgebras to describe it.

Definition 3.4.5. 1. Let C be a dgc. A **right dg C -comodule** M is a dg k -module M endowed with a cochain map $\rho : M \rightarrow M \otimes C$ such that $(\text{id} \otimes \Delta) \circ \rho = (\rho \otimes \text{id}) \circ \rho$, $(\text{id} \otimes \varepsilon) \circ \rho = \text{id}$.

2. A **morphism of dg comodules** $f : M \rightarrow N$ is a morphism of dg k -modules that respects the C -action.

3. Let $\tau_0 : C \rightarrow \Omega C$ denote the **canonical twisting cochain** i.e. the twisting morphism corresponding to the counit of the adjunction $B\Omega C \rightarrow C$.

Lemma 3.4.6. *Let C be a dgc, A a dga, M a right dg C -comodule, and $\tau : C \rightarrow A$ twisting. We define $M \otimes_\tau A$ to be the dg k -module with differential*

$$d = d_M \otimes \text{id}_A + \text{id}_M \otimes d_A + (\text{id}_M \otimes \mu_A) \circ (\text{id}_M \otimes \tau \otimes \text{id}_A) \circ (\rho \otimes \text{id}_A).$$

Then $M \otimes_\tau A$ is naturally a right dg A -module.

Definition 3.4.7. 1. Let $f : M \rightarrow N$ be a morphism of dg C -comodules. We call f a **weak equivalence** if $f \otimes \text{id}_{\Omega C} : N \otimes_{\tau_0} \Omega C \rightarrow M \otimes_{\tau_0} \Omega C$ is a quasi-isomorphism.

2. Let C be a finite dimensional dgc. The **coderived category** $\mathcal{D}(C)$ of C is the localization of the category of dg C -comodules at the class of weak equivalences.

In his thesis [20] Lefèvre obtained the following result.

Theorem 3.4.8. *Let A be a finitely generated Koszul algebra, with Koszul morphism $\kappa : A' \rightarrow A$. The functor $-\otimes_\kappa A$ defines a derived equivalence*

$$\mathcal{D}(A) \cong \mathcal{D}(A').$$

Moreover, using the following lemma we can return this to a statement about derived categories of algebras.

Lemma 3.4.9. *Let C be a finite dimensional dgc. We have an isomorphism of categories*

$$\{\text{right dg } C\text{-comodules}\} \cong \{\text{left dg } C^*\text{-modules}\}. \quad (3.3)$$

Proof. Note that for finite dimensional coalgebras C^* naturally carries a canonical algebra structure. Let N be a dg C -comodule. We define the action map to be

$$C^* \otimes N \rightarrow \otimes C^* \otimes N \otimes C \rightarrow C^* \otimes C \otimes N \rightarrow N.$$

\square

Corollary 3.4.10. *Let V be finite dimensional. The isomorphism 3.3 induces an isomorphism*

$$\mathcal{D}(S(V)) \cong \mathcal{D}(S(V)^i) \cong \{\text{left dg } \bigwedge(V^*)\text{-modules}\}[W^{-1}]$$

where W are the morphisms corresponding to the weak equivalences.

Remark. In conclusion we have now come back to a much nicer derived equivalence between dual algebras. We saw that there were two things ‘wrong’ in Theorem 3.4.3, in the sense that they kept us from obtaining a full derived equivalence. Firstly we should have been comparing right modules of the algebra to left modules of its Koszul dual. Furthermore, we were localizing with respect to the wrong morphisms (in fact, too many morphisms were being inverted, see [17]). It seems impossible that one could have discovered this without using the language of coalgebras.

Chapter 4

DGLAs in Deformation Theory

4.1 Motivation

Deformation theory studies infinitesimal deformations of geometric objects. For example, we can look at the deformations of a scheme at a point to get local information about the scheme. Viewing a scheme as its functor of points, we can generalise this to a deformation theory of functors.

For example, suppose we have a (contravariant) functor $F : \mathbf{Sch} \rightarrow \mathbf{Set}$, viewed as encoding some kind of moduli problem. For instance F could be the functor that takes a scheme S to the set of closed subschemes of \mathbb{P}_S^n that are flat over S .¹ If F is representable then we get an associated moduli space. But if this functor is not representable, we can still get some information on what the moduli space should look like. For example we can try to ‘thicken’ points of this space to get local information, even if we have no hope of getting global information.

This is one example of a deformation problem. Every deformation problem should be governed by a suitable deformation functor, which is a functor from \mathbf{Art}_k to \mathbf{Set} satisfying certain geometrically-inspired conditions. It’s not usually the case that a given deformation functor is representable, so we introduce some weaker notions of representability and mention some criteria for these notions to hold.

Given a dglA, we can define an associated deformation functor. If two dglAs are quasi-isomorphic then they define isomorphic deformation functors. In this way, dglAs encode deformation problems. Finally, we’ll mention deformation functors in the context of derived algebraic geometry, where things work very nicely - a dglA actually defines a derived deformation functor giving us information about derived deformations. Moreover, in the derived world, every deformation functor can be obtained from some dglA.

4.2 Basic Deformation Theory

Let k be an algebraically closed field. Let \mathbf{Art}_k denote the category of Artinian local k -algebras with residue field k , and local k -algebra homomorphisms. If A is an object of this category, let \mathfrak{m}_A denote its unique maximal ideal. Note that A is a finite-dimensional k -vector space, \mathfrak{m}_A is nilpotent, and that $A/\mathfrak{m}_A \cong k$ as k -algebras.

Definition 4.2.1. A **fat point** is a scheme of the form $\mathrm{Spec}(A)$ for $A \in \mathbf{Art}_k$.

Definition 4.2.2. A **deformation** of a k -scheme X over $\mathrm{Spec}(A)$ is a scheme \tilde{X} together with a flat map $\tilde{X} \rightarrow \mathrm{Spec}(A)$ exhibiting X as the pullback of the diagram

$$\mathrm{Spec}(k) \rightarrow \mathrm{Spec}(A) \leftarrow \tilde{X}$$

¹This is the **Hilbert functor**. If we restrict to the case where S is locally noetherian, then this functor is in fact representable.

Smooth affine schemes do not deform in interesting ways:

Theorem 4.2.3 (M. Artin, [31]). *If X is a smooth affine scheme then any deformation of X over $\text{Spec}(A)$ is trivial, i.e. is isomorphic to the product $X \times \text{Spec}(A)$*

In what follows we'd like to consider particularly nice maps of Artinian algebras.

Definition 4.2.4. Let $\pi : A \twoheadrightarrow B$ be a surjection in \mathbf{Art}_k . Let I be its kernel. We say that the exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

is a **square zero extension** if $I^2 = 0$.

In a square zero extension as above, I naturally has the structure of a B -module.

Definition 4.2.5. With notation as above, we say that the extension is **semismall** if $I \cdot \mathfrak{m}_A = 0$, i.e. if I as an A -module is just a k -vector space. In addition if $\dim_k(I) = 1$ then we say that the extension is **small**.

Remark. A semismall extension is square zero.

Proposition 4.2.6 ([22]). *Every surjection in \mathbf{Art}_k factors as a finite composition of small extensions.*

4.3 Deformation Functors

We set up a general framework for studying deformations, and give some examples to see how they fit into our theory.

Definition 4.3.1. A **predeformation functor** (sometimes called a **functor of Artin rings**) is a functor $F : \mathbf{Art}_k \rightarrow \mathbf{Set}$ such that $F(k)$ is a one-element set.

Suppose we have a diagram

$$B \xrightarrow{\beta} A \xleftarrow{\gamma} C$$

in \mathbf{Art}_k and any functor $F : \mathbf{Art}_k \rightarrow \mathbf{Set}$. Taking the pullback and applying F we obtain a natural map of sets $\eta : F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$.

Definition 4.3.2. A **deformation functor** is a predeformation functor F such that

- In any diagram as above, whenever β is a surjection then η is a surjection
- If $B = k$ then η is a bijection

We say that F is homogeneous if η is a bijection whenever β is a surjection.

Remark. Since every surjection factors as a finite composition of small extensions it suffices to check these conditions on small extensions.

Example 4.3.3. Let X be a scheme over k . The functor

$$A \mapsto \{\text{deformations of } X \text{ over } A\} / \{\text{isomorphism classes of deformations}\}$$

is a deformation functor. This functor is usually called Def_X .

Example 4.3.4. The trivial functor $A \mapsto *$ is a homogeneous deformation functor.

Example 4.3.5. If N is a flat k -module then the functor $A \mapsto N \otimes_k \mathfrak{m}_A$ is a homogeneous deformation functor.

Example 4.3.6. If we have a reasonable² moduli functor \tilde{F} and an element $\eta_0 \in \tilde{F}(\text{Spec}(k))$ then the functor $F : A \mapsto \{\eta \in \tilde{F}(\text{Spec}(A)) : \eta|_{\text{Spec}(k)} = \eta_0\}$ is a deformation functor.

²See [26], §2.1

Definition 4.3.7. If F is a deformation functor then the **tangent space to F** is defined to be the set $TF := F(k[\epsilon]/(\epsilon^2))$. Its elements are called the **first-order deformations**. If k is a field then this set has the structure of a k -vector space, incuded from the vector space structure on $k[\epsilon]/(\epsilon^2)$. A natural transformation $\varphi : F \rightarrow G$ induces a linear map $d\varphi : TF \rightarrow TG$, called the **differential** of φ .

Example 4.3.8. If R is in \mathbf{Art}_k and F is the functor $h_R = \text{Hom}(R, -)$ then TF is the Zariski tangent space to $\text{Spec}(R)$.

Definition 4.3.9. Let $\varphi : F \rightarrow G$ be a natural transformation of predeformation functors. φ is said to be

- **unramified** if the differential is injective
- **smooth** if for every surjection $B \rightarrow A$ in \mathbf{Art}_k the map $F(B) \rightarrow G(B) \times_{G(A)} F(A)$ is surjective
- **étale** if it is both smooth and unramified

In addition F is said to be **smooth** if the unique map $F \rightarrow *$ is smooth.

If \sim is the equivalence relation on the category of deformation functors generated by the étale morphisms, then $h_A \sim h_B$ if and only if A and B are isomorphic as k -algebras.

Definition 4.3.10. Let F be a predeformation functor. An **obstruction theory** for F is the data of a k -vector space V (the **obstruction space**) and for every small extension

$$\xi : 0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$$

an **obstruction map** $v_\xi : F(A) \rightarrow V \otimes I$ satisfying the following properties:

- If $x \in F(A)$ can be lifted to $F(B)$ then $v_\xi(x) = 0$
(the obstruction theory is called **complete** if whenever $v_\xi(x) = 0$ then x can be lifted.)
- The maps v_ξ must commute with morphisms of small extensions: concretely, if we have a commutative diagram

$$\begin{array}{ccccccccc} \xi_1 : & 0 & \longrightarrow & I_1 & \longrightarrow & B_1 & \longrightarrow & A_1 & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow f_I & & \downarrow f_B & & \downarrow f_A \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \xi_2 : & 0 & \longrightarrow & I_2 & \longrightarrow & B_2 & \longrightarrow & A_2 & \longrightarrow & 0 \end{array}$$

then $v_{\xi_2}(f_A(x)) = (\text{id}_V \otimes f_I)(v_{\xi_1}(x))$ for all $x \in F(A_1)$.

Remark. If F is smooth then all the obstruction maps are trivial.

If a functor admits a complete obstruction theory V then it admits infinitely many: clearly it is sufficient to embed V in a larger space. So we'd like to find some minimal complete obstruction theory.

Definition 4.3.11. A morphism of obstruction theories is a linear map $\varphi : V \rightarrow W$ such that $w_\xi = (\varphi \otimes \text{id})v_\xi$ for every small extension ξ . An obstruction theory is said to be **universal** if it admits a morphism to every other obstruction theory.

Theorem 4.3.12 (Fantechi, Manetti). *Every deformation functor admits a universal obstruction theory.*

4.4 Representability

We'd like to know when deformation functors are representable; i.e. when there is some geometric object encoding a deformation problem. All of the results in this section come from [28].

Definition 4.4.1. Let \mathcal{C} be any category. A functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ is said to be **representable** if there exists an object X of \mathcal{C} such that F is isomorphic to the functor $\text{Hom}(X, -)$.

In fact the definition of a deformation functor is a weakening of the definition of a representable functor:

Proposition 4.4.2. *A representable functor $\mathbf{Art}_k \rightarrow \mathbf{Set}$ is a homogeneous deformation functor.*

Unfortunately deformation functors are not in general representable (they're not even in general homogeneous), so we have to consider some weaker notions.

Definition 4.4.3. Let $\hat{\mathbf{Art}}_k$ be the category of complete Noetherian local k -algebras with residue field k , and homomorphisms of local k -algebras. Given a functor $F : \mathbf{Art}_k \rightarrow \mathbf{Set}$ define an associated functor $\hat{F} : \hat{\mathbf{Art}}_k \rightarrow \mathbf{Set}$ by $\hat{F}(A) = \varprojlim_n (A/\mathfrak{m}_A^n)$. We say that F is **prorepresentable** if \hat{F} is representable.

Let F be a deformation functor and define \hat{F} as above. Let R be a ring in $\hat{\mathbf{Art}}_k$. Let $h_R : \hat{\mathbf{Art}}_k \rightarrow \mathbf{Set}$ be the functor $\mathrm{Hom}(R, -)$ and let h_R be the restriction of h_R to \mathbf{Art}_k . Yoneda's Lemma tells us that we have a bijection $\hat{F}(R) \cong \mathrm{Hom}(h_R, \hat{F})$. In fact, we also have a bijection $\hat{F}(R) \cong \mathrm{Hom}(h_R, F)$, defined by the following procedure:

Fix an element $\zeta \in \hat{F}(R)$. Then we can write $\zeta = \varprojlim_n \zeta_n$ for $\zeta_n \in F(R/\mathfrak{m}_R^n)$. Take any ring $A \in \mathbf{Art}_k$. Take a map $u \in h_R(A)$. Since u is a local homomorphism we can define maps $u_n : R/\mathfrak{m}_R^n \rightarrow A/\mathfrak{m}_A^n$ for all n . Then since \mathfrak{m}_A is nilpotent there exists some N for which the map u factors through u_N . Set $\eta(\zeta)_A(u)$ to be the element $F(u_N)(\zeta_N)$ of $F(A)$. Then the collection $\eta(\zeta) := \{\eta(\zeta)_A : A \in \mathbf{Art}_k\}$ is a natural transformation from h_R to F , and moreover the map η is a bijection from $\hat{F}(R)$ to $\mathrm{Hom}(h_R, F)$.

Proposition 4.4.4. *A deformation functor F is prorepresentable if and only if there exists a pair (R, ζ) with $R \in \hat{\mathbf{Art}}_k$ and $\zeta \in \hat{F}(R)$ such that the natural transformation $\eta(\zeta) : h_R \rightarrow F$ is an isomorphism.*

We can use this characterisation of prorepresentability to define a new, even weaker notion of representability.

Definition 4.4.5. Let F be a predeformation functor. A **hull** for F is a pair (R, ζ) as above such that the map $\eta(\zeta)$ is smooth, with bijective differential.

Proposition 4.4.6. *If R prorepresents F then R is unique up to unique isomorphism. If (R, ζ) and (R', ζ') are both hulls for F , then there exists a (not necessarily unique) isomorphism $\varphi : R \rightarrow R'$ such that $F(\varphi)$ takes ζ to ζ' .*

Schlessinger's conditions tell us when a deformation functor is representable:

Theorem 4.4.7 (Schlessinger). *A deformation functor F has a hull if and only if $\dim_k(TF)$ is finite. In addition, F is prorepresentable if and only if it is homogeneous.*

Remark. Schlessinger's theorem is the slightly stronger statement that a predeformation functor has a hull if and only if it is a deformation functor with a finite-dimensional tangent space.

Example 4.4.8. Let X be a scheme over k and consider Def_X , the functor of deformations of X . If X is proper over k then Def_X has a hull. X is prorepresentable if and only if for each small extension $A' \rightarrow A$ and every deformation Y of X over A' , every automorphism of Y over A extends to an automorphism of Y over A' .

4.5 Background on Lie algebras and DGLAs

We're going to need a few preliminary results on Lie theory before we can study dglas in the context of deformation theory.

Definition 4.5.1. The **lower central series** of a Lie algebra V is the sequence of Lie subalgebras $V^1 = V$, $V^i := [V, V^{i-1}]$. The lower central series of a dglA is defined in the same way.

Say that a Lie algebra (or a dglA) V is **nilpotent** if the lower central series terminates (i.e. there exists some n for which $V^n = 0$).

Theorem 4.5.2 (Baker-Campbell-Hausdorff Formula). *Let G be a Lie group and \mathfrak{g} its Lie algebra. Let X, Y be elements of \mathfrak{g} . Then the element $\log(\exp(X)\exp(Y))$ is expressible as a sum of commutators in X and Y , whose first few terms are*

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\{[X, [X, Y]] + [Y, [Y, X]]\} + \dots$$

Observe that if V is any nilpotent Lie algebra then the right hand side of the Baker-Campbell-Hausdorff Formula defines a binary product on V which we write as $*$.

Proposition 4.5.3. *If V is a nilpotent Lie algebra then the product $*$ induces the structure of a group on V . We often write the group $(V, *)$ as $\exp(V)$.*

Equivalently we can define $\exp(V)$ to be the group with elements the formal symbols e^v indexed by $v \in V$ and product $e^v e^w = e^{v+w}$. We use this definition in the sequel.

Example 4.5.4. If V is the Lie algebra of $n \times n$ upper-triangular matrices then the map from $\exp(V)$ to GL_n given by

$$e^X \mapsto \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

is a homomorphism of groups.

For a dgla L we want to examine the group $\exp(L^0)$ and how it acts on the vector spaces L^i .

Definition 4.5.5. Say that a dgla is **ad₀-nilpotent** if for all i the image of the adjoint action $\text{ad} : L^0 \rightarrow \text{End}(L^i)$ is contained in a nilpotent associative subalgebra.

Example 4.5.6. A nilpotent dgla is ad₀-nilpotent. Similarly if a dgla L is ad₀-nilpotent then L^0 is nilpotent. The converses of these statements are false.

If L is an ad₀-nilpotent dgla then the exponential of the adjoint action gives us group homomorphisms $\text{Ad}^i : \exp(L^0) \rightarrow GL(L^i)$ for all $i \in \mathbb{Z}$ defined by

$$\text{Ad}^i(e^X) = \sum_{n=0}^{\infty} \frac{(\text{ad}(X))^n}{n!}$$

These homomorphisms give us an action of the group $\exp(L^0)$ on the vector spaces L^i given concretely by

$$e^X Y = 1 + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots$$

Definition 4.5.7. Fix a dgla (L, d) over k . The **Maurer-Cartan equation** of L is the equation

$$d(x) + \frac{1}{2}[x, x] = 0$$

for $x \in L^1$. Solutions are called the **Maurer-Cartan elements** of L .

Given a dgla (L, d) , we can define a new dgla $(\text{aff}(L), \partial)$ with $\text{aff}(L)^1 = L^1 \oplus kd$ (we are considering d as a formal symbol of degree one, so that kd is a one-dimensional k -vector space with basis vector d) and $\text{aff}(L)^i = L^i$ for $i > 1$. The bracket is $[x + vd, y + wd] = [x, y] + vd(y) + (-1)^{|x|}wd(x)$, where the bracket on the right is the bracket from L , and the differential is $\partial(x + vd) = d(x)$. The map aff is functorial.

The name $\text{aff}(L)$ arises because we have an **affine embedding** $\varphi : L^1 \rightarrow \text{aff}(L)^1$ given by $\varphi(x) = x + d$. Then x is a Maurer-Cartan element of L if and only if $[\varphi(x), \varphi(x)] = 0$.

If L is an ad_0 -nilpotent dgla then the map Ad^1 induces an action of $\exp(L^0)$ on the set of solutions of the Maurer-Cartan equation via the affine embedding. We call this action the **gauge action**, and it is given by the formula $e^x y = \varphi^{-1}(\text{Ad}^1(e^x)(\varphi(y)))$. Concretely, the gauge action is given by

$$e^x y = y + \sum_{n=0} \frac{\text{ad}(x)^n}{(n+1)!} ([x, y] - d(x))$$

4.6 DGLAs and Deformation Functors

We begin this section with a motivating example looking at deformations of a cochain complex.

Example 4.6.1 ([27], §2). Let k be a field and let V be a finite cochain complex

$$0 \rightarrow V^1 \xrightarrow{\partial^2} V^2 \xrightarrow{\partial^3} \dots \xrightarrow{\partial^n} V^n \rightarrow 0$$

of k -vector spaces. If $A \in \mathbf{Art}_k$ then define a deformation of V to be any cochain complex of A -modules of the form

$$0 \rightarrow V^1 \otimes A \xrightarrow{\delta^2} V^2 \otimes A \xrightarrow{\delta^3} \dots \xrightarrow{\delta^n} V^n \otimes A \rightarrow 0$$

such that modulo \mathfrak{m}_A this complex is isomorphic to V via the canonical isomorphism $V \cong V \otimes k$. Define an isomorphism between two such deformations to be an isomorphism of chain complexes that when restricted to the residue field k is the identity. We can then define the deformation functor

$$\text{Def}_V : A \mapsto \{\text{deformations of } V \text{ over } A\} / \{\text{isomorphism}\}$$

Let $\text{Hom}^n(V, V)$ be the set of linear maps from V to V of degree n . The cochain complex $H(V) = \text{Hom}(V, V)$ that at level n is $\text{Hom}^n(V, V)$ admits the structure of a dgla with bracket $[f, g] = fg - (-1)^{|f| \cdot |g|} gf$ and differential $d(f) = [\partial, f]$. I claim that the dgla $H(V)$ completely determines the deformation functor Def_V :

Since any deformation $(V \otimes A, \delta)$ modulo \mathfrak{m}_A has to be isomorphic to the original complex (V, ∂) we obtain the relation $\delta = \partial + \zeta$ for some $\zeta \in \text{Hom}^1(V, V) \otimes \mathfrak{m}_A$. The condition that $\delta^2 = 0$ translates to the condition that $d(\zeta) + \frac{1}{2}[\zeta, \zeta] = 0$, i.e. that ζ is a Maurer-Cartan element of $H(V) \otimes \mathfrak{m}_A$.

If we have an isomorphism of deformations $\psi : (V \otimes A, \delta) \rightarrow (V \otimes A, \delta')$ then write $\psi = \exp(a)$ for $a \in \text{Hom}^0(V, V) \otimes \mathfrak{m}_A$. We get an equation

$$\zeta' = \zeta + \sum_{n=0} \frac{(\text{ad}(a))^n}{(n+1)!} ([a, \zeta] - d(a))$$

i.e. ζ' and ζ are related by a gauge action. We conclude that Def_V is isomorphic to the group of Maurer-Cartan elements of $H(V) \otimes \mathfrak{m}_A$ modulo the gauge action.

The above example gives us a clue that every dgla L should define a deformation functor in a similar manner, as a quotient of the Maurer-Cartan elements by the action of the group $\exp(L^0 \otimes \mathfrak{m}_A)$.

Definition 4.6.2. Fix a dgla L over a field k . The **Maurer-Cartan functor** MC_L is the functor from \mathbf{Art}_k to \mathbf{Set} defined by the assignment

$$A \mapsto \{x \in L^1 \otimes \mathfrak{m}_A : d(x) + \frac{1}{2}[x, x] = 0\}$$

The **Gauge Group functor** G_L is the functor from \mathbf{Art}_k to \mathbf{Grp} defined by $G_L(A) = \exp(L^0 \otimes \mathfrak{m}_A)$.

Proposition 4.6.3. *The Maurer-Cartan functor is a homogeneous deformation functor. The gauge group functor is a smooth homogeneous deformation functor. If L is abelian (has trivial bracket) then MC_L is smooth.*

Using the result at the end of Section 5, we see that the group $G_L(A)$ acts on the set $MC_L(A)$. Let $\text{Def}_L : \mathbf{Art}_k \rightarrow \mathbf{Set}$ be the corresponding quotient functor given on objects by $\text{Def}_L(A) = MC_L(A)/G_L(A)$.

Proposition 4.6.4. *Def_L is a deformation functor. It is not in general homogeneous.*

Remark. If $L \rightarrow N$ is a morphism of dglas then we get associated natural transformations $MC_L \rightarrow MC_N$, $G_L \rightarrow G_N$ and $\text{Def}_L \rightarrow \text{Def}_N$.

For the rest of the section, fix a dgl L over a field k . We'd like to examine some of the geometric properties of the three functors MC_L , G_L and Def_L .

Proposition 4.6.5.

1. The functor G_L has tangent space L^0 .
2. The functor MC_L has tangent space $Z^1(L)$.

Moreover the gauge action on the tangent spaces reduces to the action by differentials. Hence the tangent space $T\text{Def}_L$ is just the space $H^1(L)$.

So $H^1(L)$ has a nice geometric interpretation: it is the tangent space to the functor Def_L . In fact, $H^2(L)$ has a similar interpretation: it encodes the smoothness of Def_L in some sense.

Proposition 4.6.6. *If $H^2(L)$ vanishes then the functors MC_L and Def_L are smooth.*

In fact we can say more: $H^2(L)$ is a complete obstruction space for Def_L .

Theorem 4.6.7. *Let*

$$\xi : 0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$$

be a small extension in \mathbf{Art}_k . Define the map $v_\xi : MC_L(B) \rightarrow H^2(L) \otimes I$ by

$$x \mapsto \left[d(\tilde{x} + \frac{1}{2}[\tilde{x}, \tilde{x}]) \right]$$

where \tilde{x} is any lift of x to $L \otimes \mathfrak{m}_A$. Then this map is well-defined and the space $H^2(L)$ together with the maps v_ξ is a complete obstruction theory for MC_L . Moreover the maps v_ξ are invariant under the gauge action and hence form a complete obstruction theory for Def_L .

Suppose we have a map $f : L \rightarrow N$ of dglas. Write $H^i(f) : H^i(L) \rightarrow H^i(N)$ for the induced maps on cohomology. Knowing $H^1(f)$ and $H^2(f)$ is enough to tell us some properties of the induced map $\text{Def}_f : \text{Def}_L \rightarrow \text{Def}_N$:

Theorem 4.6.8 (Main Theorem of Deformation Theory). *1. If $H^1(f)$ is surjective and $H^2(f)$ is injective then Def_f is smooth.*

2. *If in addition to (1) the map $H^1(f)$ is injective then Def_f is étale.*
3. *If, in addition to (1) and (2) the map $H^0(f)$ is surjective then Def_f is an isomorphism.*
4. *In particular, if f is a quasi-isomorphism then Def_f is an isomorphism.*

Remark. The assignment $L \mapsto \text{Def}_L$ determines a functor from dglas to deformation functors. The above shows that this functor descends to a functor defined on the homotopy category of dglas.

Remark. Differential graded Lie algebras are Koszul dual to conilpotent cocommutative (non-counital) coalgebras. If C is such a coalgebra then $A := k \oplus C^\vee$ is a unital Artinian k -algebra with maximal ideal $\mathfrak{m}_A = C^\vee$. The duality between dglas and deformation functors is in some sense just the above Koszul duality.

A natural question to ask is ‘Does every deformation functor come from a dgl?’ We’ll see next that this question has a positive answer in the setting of derived geometry.

4.7 Derived Deformation Theory

There are a few problems with Schlessinger's framework of deformation functors. Firstly, it's not clear that every deformation functor is obtained as Def of some dgla, as we'd like. Secondly, $H^2(L)$ is not in general a universal obstruction space for the functor Def_L . In fact, sometimes there are no obstructions but $H^2(L) \neq 0$. We'll see that in the derived world, every (derived) deformation functor is obtained from Def , and that $H^2(L)$ actually measures 'derived deformations'. Throughout, k is a field of characteristic zero.

Write \mathbf{dgArt}_k for the category of **local Artinian dg-algebras over k** : these are the dg-algebras A over k such that

- $\dim(A) < \infty$
- The ideal $\mathfrak{m}_A := \ker(A \rightarrow k)$ is nilpotent.

Definition 4.7.1. The concept of a semismall extension readily generalises to \mathbf{dgArt}_k . Let $\pi : A \twoheadrightarrow B$ be a surjection in \mathbf{dgArt}_k and let $I = \ker \pi$ be its kernel. The extension

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

is **semismall** if $I \cdot \mathfrak{m}_A = 0$. The extension is said to be **acyclic** if I is an acyclic complex.

Definition 4.7.2. Say that a functor $F : \mathbf{dgArt}_k \rightarrow \mathbf{Set}$ is a **deformation functor** if

- $F(k)$ is a one-element set,
- For every $A \twoheadrightarrow B$ semismall and every $C \rightarrow B$ in \mathbf{dgArt}_k , the natural map $F(A \times_B C) \rightarrow F(A) \times_{F(B)} F(C)$ is surjective,
- for every A and B in \mathbf{dgArt}_k , the natural map $F(A \times_k B) \rightarrow F(A) \times F(B)$ is an isomorphism,
- If $A \twoheadrightarrow B$ is an acyclic semismall extension then the map $F(A) \rightarrow F(B)$ is an isomorphism.

Remark. This definition is due to Manetti.

We define the Maurer-Cartan functor, gauge group functor, and deformation functor Def_L of a dgla L in exactly the same manner as in the underived case. These are indeed deformation functors in the sense of Manetti.

Theorem 4.7.3 (Manetti). *The functor Def_L is the universal deformation functor under MC_L , i.e. for all deformation functors F , every map $MC_L \rightarrow F$ factors through Def_L .*

Definition 4.7.4. Let $k[n]$ be the local Artinian dg-algebra that is a copy of k concentrated in degree $-n$. If F is a deformation functor then define, for all $i \in \mathbb{Z}$, the set $T^i F := F(k[i-1])$. These sets naturally admit the structure of vector spaces. Let TF be the graded vector space that in degree i is $T^i F$. We call $T^1 F$ the **tangent space** and $T^2 F$ the **obstruction space** to F .

Remark. If we compute $T^1 F$ when F is a functor of Artin rings then we recover the usual definition of tangent space.

So we may think of the obstruction space, and more generally the spaces $T^n F$ as spaces of 'higher deformations' representing geometry of the deformation problem that is perhaps not apparent when we are working in the underived setting.

Proposition 4.7.5. *If L is a dgla then $T^i \text{Def}_L = H^i(L)$.*

This tells us in particular that $H^2(L)$ is the 'space of derived obstructions': the elements lying in it that are not first-order obstructions are in fact higher-order obstructions.

Let \mathbf{Def}_k^M be the category of derived deformation functors in the sense of Manetti. Then \mathbf{Def}_k^M admits a model structure where the weak equivalences are the natural transformations $F \rightarrow G$ inducing isomorphisms $T^i F \rightarrow T^i G$ for all $i \in \mathbb{Z}$.

Moreover the category \mathbf{dgLie}_k of dglas over k admits the **projective model structure** where the weak equivalences are the quasi-isomorphisms and the fibrations are the levelwise surjections.

Remark. Note that the projective model structure on \mathbf{dgLie}_k is obtained by transport of the projective model structure on $\mathbf{dg-mod}_k$.

Theorem 4.7.6 (Manetti, Pridham). *The functor $\text{Def} : \mathbf{dgLie}_k \rightarrow \mathbf{Def}_k^M$ induces an equivalence of homotopy categories $\text{Ho}(\mathbf{dgLie}_k) \cong \text{Ho}(\mathbf{Def}_k^M)$.*

As a corollary, we get the following result:

Proposition 4.7.7. *For every deformation functor F in the sense of Manetti, there exists a dgl L such that F is isomorphic to the functor Def_L .*

Unfortunately Manetti's deformation functors are not the end of the story. They're not left exact since they don't preserve fibre products; in particular they don't sheafify and so there's no global version. If we're looking at some moduli problem, Manetti's deformation functors will not tell us whether or not we have some kind of moduli space.

The solution to this, proposed by Hinich, is to 'stackify' our deformation functors, that is to consider functors from $\mathbf{dgArt}_k^{\leq 0}$ (nonpositively graded local Artinian dg-algebras over k) to \mathbf{sSet} (simplicial sets). These functors encode the stacky geometry of our moduli problems. The category \mathbf{Def}_k^H of such functors is then an ∞ -category, and we have the following result:

Theorem 4.7.8 (Pridham, Lurie). *The functor $\text{Def} : \mathbf{dgLie}_k \rightarrow \mathbf{Def}_k^H$ is an equivalence of ∞ -categories.*

In particular Def induces an equivalence of homotopy categories, and we recover our earlier isomorphism at the homotopy level.

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