Notes on ∞-categories

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August 9, 2024

These are an informal set of notes for the ∞-categories seminar I ran in Lancaster in the first half of 2024. They're pretty skeletal in places and likely contain lots of errors - while I didn't give most of the talks, I did write these notes, so any errors are mine. Our main reference was the Kerodon, and in particular, the numbering in this document is roughly supposed to reflect the numbering in the Kerodon, so that e.g. our I.2 corresponds to Kerodon 1.2.

Convention: $0 \in \mathbb{N}$ (monoids are better than semigroups).

Thanks to all the participants of the seminar, but especially the speakers: Callum Galvin, Bjorn Eurenius, and Alex Mallon.

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Part I

The language of ∞-categories

1 Introduction and motivation

In mathematics things often come organised into categories. When dealing with homotopy-theoretic objects, or more generally objects with 'higher structure', one wants a theory of 'higher categories' to deal with them effectively. Here's a simple motivating example.

Example 1.1. If X is a topological space, one would like to form a category whose objects are the points of X and whose morphisms are the paths. Composition of paths isn't associative, so the usual fix is to consider paths only up to homotopy. We then obtain a category, the **fundamental groupoid** of X. The endomorphism group of an object x of this category is then the fundamental group $\pi_1(X, x)$ of X based at x.

However, usually it is better to remember information rather than quotient out by it: if G is a group acting on a set S it is much more profitable to remember the set with action (S, ρ) rather than the quotient set S/G. So we'd like to think of homotopies as 'higher morphisms' that we can then quotient by to obtain the fundamental groupoid.

Heuristically, a higher category is a gadget with

- 0-morphisms = objects
- 1-morphisms between 0-morphisms
- 2-morphisms between 1-morphisms
- Et cetera. Maybe we stop at some point or maybe we go off to infinity.

Here's a naive definition. A **strict 1-category** is a category. A **strict n-category** is a category enriched in strict $(n-1)$-categories[1]; i.e. any two objects have a strict $(n-1)$-category of morphisms between them, and composition is a functor of such categories.

This definition is not fit for purpose if $n > 2$. To see why, consider the following example.

Example 1.2. If X is a topological space, we should be able to cook up a higher category with

- Objects = points of X
- 1-morphisms = paths between points
- 2-morphisms = paths between paths = homotopies
- 3-morphisms = homotopies between homotopies
- etc.

This should deserve to be called the **fundamental ∞-groupoid** of X.

There's a serious problem here that we've already encountered: **composition of paths is still not associative**. So this example can't fit into any kind of enriched world. We need a structure where composition of 1-morphisms is not associative, but is only associative "up to 2-morphisms". These 2-morphisms are themselves only associative up to 3-morphisms, et cetera.

Some terminology: an (n, r)-category is a category with m-morphisms for all $0 \leq m \leq n$, and such that every m-morphism is invertible whenever $m > r$. It is useful to allow $n = \infty$ and even $r = \infty$ in these definitions.

Example 1.3. a $(1, 0)$-category is a groupoid. A $(1, 1)$-category is the usual notion of a category.

Remark 1.4. The 'correct' axiomatisation of a $(2, 2)$-category is the concept of a **2-category**. However every 2-category is equivalent to a strict 2-category - this is a phenomenon unique to $n = 2$ and does not hold in generality.

[1] This leads one to the conclusion that a 0-category should be a set.

The goal of this seminar is to axiomatise $(\infty, 1)$**-categories**: i.e. objects with n-morphisms for all n, but such that all k-morphisms are invertible for $k \geq 2$. Whenever I say "∞**-category**" I mean this as shorthand for the former. Similarly we will call $(\infty, 0)$-categories ∞**-groupoids**.

As an example: whatever an ∞-groupoid is, a topological space should define an ∞-groupoid via some sort of path category construction. Grothendieck's **homotopy hypothesis** (which appears in 1983's *Pursuing Stacks*) states that topological spaces should be **equivalent** to ∞-groupoids (for some notion of "equivalence"[2]).

We will take the homotopy hypothesis as a desideratum that our theory of ∞-categories should satisfy. The upshot is that ∞-categories should be objects that simultaneously generalise both categories and topological spaces. In this course, we're going to use **simplicial sets** as our common generalisation. These are combinatorial objects that provide discrete models for topological spaces. A **quasicategory** is a certain kind of simplicial set; these will be our preferred model for ∞-categories. Quasicategories were originally introduced by Joyal and developed extensively by Lurie. Somewhat abusively I will often treat 'quasicategory' and '∞-category' as synonyms.

Loosely, a simplicial set X is an object which consists of a set X_n of n**-simplices** for all n, linked together by **face** and **degeneracy** maps. A topological space Y has a **singular simplicial set** which at level n is the set of n-simplices in Y. This is closely related to the singular chain complex of Y which computes homology. A category C has a **nerve** which at level n is the set of strings of n composable morphisms in C. A **quasicategory** is then a certain kind of simplicial set which generalises both of these constructions.

Example 1.5. Any ∞-category C has a homotopy category hC, defined roughly by flattening all of the higher morphisms to force composition to be strictly associative. For example, if A is a ring then there's an infinity-category $D(A)$ whose homotopy category $hD(A)$ is the usual derived category. If C is any infinity-category with enough (co)limits, then one can define suspension and loop functors, and one says that C is stable if they are inverse autoequivalences. For example, $D(A)$ is stable, with suspension given by $[1]$ and looping by $[-1]$. If C is a stable infinity-category, then hC is canonically a triangulated category, and the usual problems that one faces with triangulated categories disappear: stable infinity-categories have functorial cones, functor categories are stable, et cetera. So $D(A)$ is much nicer to work with than $hD(A)$, and indeed one is implicitly using the higher-categorical structure when one computes things like

[2]Precisely what 'equivalence' should mean here is thorny: ∞-groupoids should naturally fit into a ∞-category, and the ∞-categories of ∞-groupoids and topological spaces should be equivalent. But now we need to decide what it means for two ∞-categories to be equivalent, so we've just moved the problem. The solution taken in practice is to prove all of these statements with respect to a specific fixed model of higher categories.

$\mathbb{R}$Hom (as opposed to just Ext). The canonical non-algebraic example of a stable infinity-category is Sp, the category of spectra (in fact this is actually the derived category of modules over the sphere spectrum, which is a ring in an appropriate sense).

1.1 Other models

The most naive definition of an ∞-category is a category enriched in topological spaces. This can be made to work but has some difficulties - topological spaces are difficult analytic objects whereas we'd prefer something more discrete. A better definition, worked on by Bergner, is as a category enriched in simplicial sets.

Rezk introduced (complete) Segal spaces as models for ∞-categories. These are certain kinds of bisimplicial set, which at level n is supposed to behave like the space of strings of n composable morphisms. These are particularly amenable for generalisation to a definition of to (∞, k)-categories.

DG categories are models for linear ∞-categories.

Model categories are presentations for certain kinds of ∞-categories. More generally, a category with weak equivalences can be viewed as a presentation of an ∞-category: given a category C with weak equivalences W there is a **simplicial localisation** (or **hammock localisation** $C[W^{-1}]$ which is an ∞-category which 'enhances' the usual 1-categorical localisation.

Riehl and Verity are developing a synthetic theory of ∞-categories.

A subtle question is: in what sense are these "models" of a "theory" of ∞-categories? In all of the above cases, the models themselves form a quasicategory (often - even better - a model category), and it has been proved that these quasicategories are all equivalent. Bergner has a nice survey article for those interested in the comparisons between different models. Toën also has some good papers.

2 Simplicial sets

We develop the basic theory of simplicial sets, which will form the backbone of our approach to ∞-categories.

2.1 First definitions

Let Δ denote the category of nonempty finite ordinals. Concretely, the objects of Δ are the posets $[n] = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$ and the morphisms are the monotone maps. We call Δ the **simplex category**.

A **simplicial object** in a category C is a functor $\Delta^{\mathrm{op}} \to C$. The collection of simplicial objects in C forms a category $\mathbf{s}C \coloneqq \mathrm{Fun}(\Delta^{\mathrm{op}}, C)$.

A **cosimplicial object** in a category C is a functor $\Delta \to C$. The collection of simplicial objects in C forms a category $\mathbf{c}C \coloneqq \mathrm{Fun}(\Delta, C)$.

We will mostly be interested in simplicial sets. These are discrete/combinatorial models for topological spaces and are the basic objects of the theory of quasi-categories. Briefly, simplicial sets are to ∞-category theory what sets are to 1-category theory.

A simplicial set X is thus a collection X_n of sets, one for each n, with a collection of maps between them (which we will describe soon). We refer to X_n as the set of **n-simplices** of X. A **vertex** is a 0-simplex.

We tend to use subscripts to denote simplicial things and superscripts to denote cosimplicial things.

Example 2.1. The Yoneda embedding $\Delta \to \mathbf{sSet}$ yields for each n a simplicial set $\mathrm{Hom}(-, [n])$. We refer to this simplicial set as the **standard n-simplex** Δ^n. The Yoneda lemma yields an isomorphism $\mathrm{Hom}(\Delta^n, X) \simeq X_n$.

sSet is complete and cocomplete; this follows from it being a presheaf category.

2.2 Topological simplices

For all n, let $\underline{\Delta}^n \subseteq \mathbb{R}^{n+1}$ be the subspace cut out by the equations $x_0 + \cdots x_n = 1$ and $x_i \geq 0$. So $\underline{\Delta}^0$ is a point, $\underline{\Delta}^1$ is a line segment in $\mathbb{R}^2$, $\underline{\Delta}^2$ is a triangle in $\mathbb{R}^3$, and $\underline{\Delta}^3$ is homeomorphic to a solid tetrahedron in $\mathbb{R}^3$ (which is of course itself homeomorphic to a solid 3-ball).

We picture the standard Δ^n as the topological space $\underline{\Delta}^n$. Note that the simplices of a simplicial set come with an order, so this picture does not capture all the data. Typically when drawing simplicial sets, we assign directions to the 1-simplices but not the higher ones - this captures the invertibility of r-morphisms for $r \geq 2$.

2.3 Structure of Δ

For each $n \geq 0$ and each $0 \leq i \leq n+1$ there is an injective **coface map** $\sigma : [n] \to [n+1]$ that misses i. For each $n > 0$ and each $0 \leq i \leq n$ there is a surjective **codegeneracy map** $\delta : [n+1] \to [n]$ which sends both i and $i+1$ to i. These maps generate Δ. One can work out the identities they satsify; these are known as the **cosimplicial identities**. A cosimplicial object in C is thus the same thing as a collection $X_n \in C$ together with a collection of coface and codegeneracy maps satisfying the cosimplicial identities.

Dually a simplicial object in C is thus the same thing as a collection $X_n \in C$ together with a collection of **face** and **degeneracy** maps satisfying the **simplicial identities**.

Call a simplex of X **degenerate** if it is in the image of a degeneracy map. A degenerate n-simplex of X is precisely an $n+1$-simplex of X that has one of its edges collapsed to a point - this forces some adjacent 2-simplices, 3-simplices, etc. to also collapse. When drawing simplicial sets one typically omits the degenerate simplices.

Example 2.2. Fix $n \in \mathbb{N}$. For $m > n$, all m-simplices of Δ^n are degenerate. There is a unique nondegenerate n-simplex, corresponding to the interior.

Exercise: count the nondegenerate m-simplices of Δ^n.

2.4 Categories of simplices

If X is a simplicial set, its **category of simplices** $\mathrm{el}(X)$ is the category of elements of the functor X: concretely, the objects are the morphisms $\Delta^n \to X$ and a morphism from $\Delta^n \to X$ to $\Delta^m \to X$ is a morphism $\Delta^n \to \Delta^m$ making the obvious diagram commute.

There is a forgetful functor $J : \mathrm{el}(X) \to \mathbf{sSet}$ which sends a morphism to its domain.

Proposition 2.3. *Every simplicial set is the colimit of its simplices: there is a natural isomorphism* $X \cong \operatorname{colim} J$*.*

This proposition should be intuitively clear: a simplicial set is an object consisting of a bunch of simplices, and if one takes all of the simplices of X and glues them together in the way prescribed by X, then one obtains X again.

We will often somewhat abusively write $X \cong \operatorname{colim}_{\mathrm{el}(X)} \Delta^n$.

2.5 New simplicial sets from old

One can view a set X as a simplicial set by taking $X_0 = X$ and adding in degenerate higher simplices - one of each dimension for each element of X. One calls such simplicial sets **discrete**.

If X is a simplicial set, the n**-skeleton** of X is obtained from X by throwing away all nondegenerate m-simplices of X for $m > n$. We denote this $\mathrm{sk_n} X$.

We have $\mathrm{sk}_0 X = X_0$.

The **boundary** $\partial\Delta^n$of Δ^n is defined by taking Δ^n and removing the unique n-dimensional nondegenerate simplex. Geometrically, this looks like an $(n-1)$-sphere. There's a more combinatorial description of this.

The **spine** of Δ^n is the subset of $\mathrm{sk}_1\Delta^n$ containing all the vertices and the 1-simplices of the form $[i, i+1]$. Note that this looks like the poset $[n]$.

Note that we have an ascending filtration

$$\mathrm{sk}_0 X \hookrightarrow \mathrm{sk}_1 X \hookrightarrow \mathrm{sk}_2 X \hookrightarrow \cdots$$

whose colimit is X. This is a cell decomposition of X which exhibits X as a filtered colimit of its simplices.

2.6 Horns and fillers

Let $n \geq 1$. The (n, i)-horn Λ^n_i is obtained from $\partial\Delta^n$ by removing the top-dimensional face (which is an $(n-1)$-simplex) opposite the vertex i. Again, there is a more combinatorial description of this. Note that a 2-horn Λ^2 looks like the horns of a stag, whereas a 3-horn Λ^3 looks like a drinking horn.

Exercise: show that the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ induces an isomorphism on $(n-2)$-skeleta.

A horn is **outer** if $i = 0$ or $i = n$ and **inner** otherwise. The 1-skeleton of an inner 2-horn looks like a pair of composable morphisms, whereas the 1-skeleton of an outer 2-horn has simplices which go the 'wrong way'.

Let $\Lambda^n_i \hookrightarrow X$ be a horn in a simplicial set X. We say that this horn **admits a filler** if this map extends to a map $\Delta^n \to X$.

Exercise: show that Δ^n is what you get if you start from its spine and iteratively fill inner horns.

2.7 Singular simplicial sets

The collection $\underline{\Delta}^\bullet$ of standard topological simplices fits together into a cosimplicial space, the **standard cosimplicial space**. By the machinery of the nerve-realisation adjunction detailed in a later section, we obtain a functor $\mathrm{Sing}_\bullet : \mathbf{Top} \to \mathbf{sSet}$ which sends Y to the simplicial set whose n-simplices are $\mathrm{Hom}(\underline{\Delta}^n, Y)$.

Again by the general nerve-realisation machinery, we obtain an adjoint to $\mathrm{Sing}_\bullet$, the **geometric realisation** functor. Informally, it works as follows. To construct $|X|$, first take the disjoint union of a bunch of standard topological simplices - one for every simplex of X. Then one glues these together along the face and degeneracy maps: face maps identify one simplex as a face of a larger one and degeneracy maps collapse edges of simplices.

Example 2.4. if X is a discrete simplicial set, then $|X|$ is homeomorphic to the discrete space X.

Exercise: let Y be a simplicial complex. Define a simplicial set X such that $|X|$ is homeomorphic to Y.

Theorem 2.5. *If Y is a topological space then the adjunction unit $Y \to |\mathrm{Sing}_\bullet Y|$ is a weak homotopy equivalence.*

The proof of the above theorem is nontrivial and requires some simplicial homotopy theory. The singular simplicial set-geometric realisation adjunction is a Quillen equivalence between topological spaces and simplicial sets. (Really, when I say topological spaces I mean a 'convenient category' of topological spaces; something like CGWH spaces).

3 Nerves of categories

Quasicategories - our models for ∞-categories - will be certain kinds of simplicial sets. In particular we'd better be able to view a category as a simplicial set. In this section we describe how to do this.

3.1 Nerves

Definition 3.1. Let C be a (small) category. The **nerve** of C is the simplicial set whose n-simplices are given by the strings of n composable morphisms in C. We think of such a string as the spine of the n-simplex. The face maps insert identity morphisms. The inner degeneracy maps compose adjacent morphisms and the outer degeneracy maps remove the outermost morphisms. One can check using the simplicial identities that this makes NC into a simplicial set. Or one can use the nerve-realisation machinery described in A.10 with the standard cosimplicial category.

So $N(C)_0$ is the set of objects of C, and $N(C)_1$ is the set of morphisms.

Exercise: if x is an object of c, then we have $s_0 x = x \xrightarrow{\mathrm{id}} x$. If $e = (x \to y)$ is a morphism of C, then we have $d_0 e = y$ and $d_1 e = x$.

Definition 3.2. A simplicial set X is a **Kan complex** if all horns in X have fillers. A simplicial set is a **weak Kan complex** if all inner horns in X have fillers.

Remark 3.3. We will later see that "∞-groupoid" is a synonym for Kan complex and "∞-category" is a synonym for weak Kan complex.

Theorem 3.4 (the Nerve Theorem[3])**.** *A simplicial set is the nerve of a category if and only if all inner horns admit unique fillers.*

Proof. Let C be a category and $\sigma : \Lambda^n_i \to N(C)$ be an inner n-horn. Note that since σ is inner we must have $n \geq 2$. The 1-skeleton of an inner n-horn in NC is a string of n composable morphisms, and this corresponds to an n-simplex filling the horn. The filler is unique as composition is unique.

Conversely, if X is a simplicial set for which all inner horns admit unique fillers, we're going to write down a category C whose nerve is X. The objects of C are the vertices of X. The morphisms of C are the 1-simplices; source and target are given by the degeneracy maps. Filling inner 2-horns uniquely gives us composition and filling inner 3-horns uniquely tells us that composition is associative. □

3.2 Groupoids and Kan complexes

Our goal in this section will be to prove the following theorem:

Proposition 3.5. *Let C be a category. Then $N(C)$ is a Kan complex if and only if C is a groupoid.*

The following Lemma will be useful.

Lemma 3.6. *Let $n \geq 2$ and let $f : \mathrm{sk}_2\Delta^n \to N(C)$ be any map. Then f has a unique extension to a map $\tilde{f} : \Delta^n \to N(C)$.*

Proof. The map f determines objects $x_0, \ldots, x_n$ in C and maps $f_{ij} : x_i \to x_j$ for $i < j$ such that $f_{jk} \circ f_{ij} = f_{ik}$. This defines a functor $[n] \to C$ given by sending $[i]$ to x_i and a map $i \to j$ to f_{ij}, i.e. a map $\Delta^n \to N(C)$. □

Before we begin the proof we fix some terminology: a **left horn** is a horn of the form Λ^n_0 and a **right horn** is a horn of the form Λ^n_n.

Proof of 3.5. Suppose first that $N(C)$ is a Kan complex. A left 2-horn in NC is the same as a span $y \leftarrow x \to z$ in C. A filling of this left 2-horn is the same as a morphism $y \to z$ making the triangle commute. In particular if $f : x \to y$ is any morphism then filling the span $y \xleftarrow{f} x \xrightarrow{\mathrm{id}} x$ shows that f is a split mono. Similarly, filling right 2-horns shows that every morphism is a split epi. In particular every morphism in C is an isomorphism - that is, C is a groupoid.

[3]This does not appear to be a standard term, but it's a very fundamental theorem and needs a name.

Now we need to show the converse. Suppose that C is a groupoid. We know by the Nerve Theorem that $N(C)$ has fillers for inner horns, so we just need to check that outer horns admit fillers. Outer 1-horns in NC are just objects in C, so we can fill these with identity morphisms. Similarly to the above, we can fill outer 2-horns since C is a groupoid.

If $n \geq 4$ then by 3.6 we see that any n-horn admits a filler. So we just need to check that we can fill outer 3-horns.

A left 3-horn in NC is the same as a collection of objects $x_0, \ldots, x_3$ and morphisms $[ij] : x_i \to x_j$ for $i > j$ such that the triangles [012], [013] and [023] all commute. To fill this to a 3-simplex, by 3.6 it is enough to show that the triangle [123] commutes. To do this, consider the composition

$$[32][21][10] = [32][20] = [30] = [31][10]$$

Since [10] is a split epi by assumption, this implies that $[32][21] = [31]$ as desired. Similarly, one can fill right horns using that every morphism in C is a split mono. □

Corollary 3.7. *A simplicial set X is the nerve of a groupoid if and only if all horns in X admit unique fillers.*

Proof. If the condition on fillers holds, then the Nerve Theorem implies that X is the nerve of a category C and 3.5 implies that C is a groupoid. Conversely if C is a groupoid then the fact that inverses are unique implies that the horns in $N(C)$ must have unique fillers. □

3.3 Homotopy categories

Since **Cat** is cocomplete, the nerve-realisation technology of A.10 gives us an adjoint $h : \mathbf{sSet} \to \mathbf{Cat}$ to the nerve functor. Let's examine this in more detail.

Definition 3.8. Let X be a simplicial set. A map $u : X \to N(C)$ exhibits C as the **homotopy category of** X if for all categories D the composition

$$\mathbf{Cat}(C, D) \xrightarrow{\cong} \mathbf{sSet}(N(C), N(D)) \xrightarrow{u^*} \mathbf{sSet}(X, N(D))$$

is a bijection.

It follows directly from the definition that such a homotopy category C exists: indeed it is the value of the functor h on X. Moreover, the map u is the unit of the nerve-realisation adjunction.

Let's see an explicit construction of the homotopy category. Take the objects of hX to be X_0. For every edge e of X, add a morphism $e : d_1 e \to d_0 e$. This gives us a directed graph. Complete this under composition to obtain a category. We then quotient this category by relations coming from the 2-simplices of X: if x is a vertex then we impose $[s_0 x] = \mathrm{id}_x$ and if σ is a 2-simplex then we impose $[d_1 \sigma] = [d_0 \sigma][d_2 \sigma]$.

Exercise: the counit $hN(C) \to C$ is an isomorphism.

Remark 3.9. For every simplicial set X the natural map $\mathrm{sk}_2 X \to X$ induces an isomorphism $h(\mathrm{sk}_2 X) \to h(X)$.

Example 3.10. Let X be a topological space. Then $\pi_{\leq 1} X$ is the homotopy category of $\mathrm{Sing}_\bullet X$.

Example 3.11. Let X be a 1-skeletal simplicial set, i.e. a directed graph. Then hX is the path category of X; i.e. the free category on the directed graph X.

Example 3.12. If X is a quasicategory (i.e. weak Kan complex) then hX has a reasonable description: the morphisms are certain equivalence classes of the edges of X. We'll see more next week.

4 Quasicategories

4.1 Basic definitions

Definition 4.1. An ∞**-category** is a simplicial set C for which every inner horn admits a filler.

Definition 4.2. An ∞**-groupoid** is a simplicial set C for which every horn admits a filler.

Remark 4.3. Note that an ∞-category is the same thing as a weak Kan complex, and an ∞-groupoid is the same thing as a Kan complex.

Example 4.4. If C is a category then NC is an ∞-category. If C is a groupoid then NC is an ∞-groupoid.

Example 4.5. If X is a topological space then $\mathrm{Sing}X$ is an ∞-groupoid.

An **object** of an ∞-category C is a vertex of C. A **morphism** of C is a 1-simplex of C. The **source** of a morphism f is $d_1 f$ whereas the **target** is $d_0 f$. The **identity morphism** of an object x is the degenerate 1-simplex $s_0 x$. Exercise: check that these definitions agree with the usual ones when C is the nerve of a 1-category. Exercise: work out what these mean explicitly for $C = \mathrm{Sing}X$.

Let $f : x \to y$ and $g : y \to z$ be two morphisms in an ∞-category C. A **composition** of f and g is a morphism $h : x \to z$ such that the induced map $\partial\Delta^2 \to C$ admits a lift to a map $\Delta^2 \to C$. We picture a map $\partial\Delta^2 \to C$ as a possibly noncommutative triangle in C and a map $\Delta^2 \to C$ as a commutative triangle.

Lemma 4.6. *Compositions exist.*

Proof. Let f, g be as above. They define a map $\Lambda^2_1 \to C$ which, since C is an ∞-category, admits a lift to a 2-simplex $\sigma : \Delta^2 \to C$. Then $d_1\sigma$ is a composition of f and g. □

Remark 4.7. Warning: compositions need not be unique! In particular, there need not be any sort of function $\circ$ that sends a pair f, g to a composition. We will see later that compositions are unique up to a notion of homotopy, and moreover that compositions are unique precisely when C is a 1-category.

Example 4.8. Let X be a topological space. Let f, g be two composable morphisms of X, i.e. a path $x \to y$ and a path $y \to z$. Then a path $x \to z$ is a composition of f and g if and only if it is homotopic to gf.

4.2 Opposites

Recall that we can regard the simplex category Δ as a cosimplicial object in categories, i.e. a functor $\Delta \to \mathbf{Cat}$. Consider the composition of this functor with the endofunctor op : $\mathbf{Cat} \to \mathbf{Cat}$. Observe that $[n]^{\text{op}}$ is canonically isomorphic to $[n]$ (just send $i \mapsto n - i$), so the image of this functor lands in Δ. Hence we obtain an endofunctor Op :$\Delta \to \Delta$.

Let S be a simplicial object in a category C. The **opposite** of S is the simplicial object S^{op} given by $\Delta^{\text{op}} \xrightarrow{\text{Op}} \Delta^{\text{op}} \xrightarrow{S} C$. Concretely, if S is a simplicial set then S^{op} is the simplicial set with n-simplices those of S, and whose face and degeneracy maps are given by $d_i^n = d_{n-1}^n$ and $s_i^n = s_{n-1}^n$.

If C is an ∞-category then the **opposite** of C is the simplicial set C^{op}. Note that this is also an ∞-category, since the opposite of an inner horn inclusion is an inner horn inclusion. Moreover C^{op} is an ∞-groupoid if and only if C was - the opposite of a left horn inclusion is a right horn inclusion, and vice versa.

Example 4.9. If C is a category, an n-simplex of NC determines an n-simplex of $N(C^{\text{op}})$. This gives an isomorphism $N(C^{\text{op}}) \cong N(C)^{\text{op}}$.

Example 4.10. Let X be a topological space. Then there is a canonical isomorphism $\text{Sing}(X) \cong \text{Sing}(X)^{\text{op}}$, induced by the homeomorphism $\underline{\underline{\Delta}}^n \to \underline{\underline{\Delta}}^n$ which sends $(t_0, \ldots, t_n)$ to $(t_n, \ldots, t_0)$.

4.3 Homotopy

Example 4.8 indicates that there ought to be some notion of homotopy between morphisms of an ∞-category such that composition is well-defined up to homotopy. We mimic the topological definitions in the simplicial world.

Definition 4.11. Let $f, g : x \to y$ be two parallel morphisms in an ∞-category C. Adding the identity morphism on y defines a noncommutative triangle $\sigma : \partial\Delta^2 \to C$. A **homotopy** from f to g is a 2-simplex $\sigma' : \Delta^2 \to C$ lifting σ. Say that f and g are **homotopic** if there exists a homotopy between them.

Exercise: if C is a 1-category, then two parallel morphisms in NC are homotopic if and only if they are equal.

Exercise: if X is a topological space, then two parallel paths in X are homotopic as morphisms if and only if they are homotopic relative to their common endpoints. (Hint: $\underline{\Delta}^2$ is homeomorphic to $[0,1] \times [0,1]$.)

Proposition 4.12. *Homotopy is an equivalence relation on morphisms.*

Proof. Reflexivity is clear: the degenerate 2-simplex $s_1 f$ is a homotopy from f to itself. For symmetry, let σ be a 2-simplex witnessing a homotopy $f \to g$ of maps $x \to y$. Glue σ to the two degenerate 2-simplices $s_1 f$ and $s_0 \operatorname{id}_y$ to obtain a 3-horn Λ^3_1 in C with x at 0 and y at every other vertex. Fill in this 3-horn with a 3-simplex σ'. The 2-simplex $d_1\sigma'$ is then a homotopy $g \to f$. For transitivity, let f, g, h be three parallel morphisms $x \to y$ and assume that f is homotopic to g and g is homotopic to h. Glue the two 2-simplices witnessing these homotopies together along g and glue on the constant 2-simplex on y to obtain a 3-horn Λ^3_2. Fill this horn with a 3-simplex σ'; the desired homotopy $f \to h$ is then $d_2\sigma'$. □

Exercise: two parallel morphisms of C are homotopic if and only if they are homotopic as morphisms of C^{op}. (Hint: this is a similar diagram chase. The hardest part is unwinding what it means for f, g to be homotopic in C^{op}).

Proposition 4.13. *Let $f : x \to y$ and $g : y \to z$ be two morphisms in an ∞-category. If $h : x \to z$ is a composition of f and g, then any morphism h' parallel to h is a composition of f and g if and only if $h' \simeq h$.*

Proof. Suppose first that $h \simeq h'$. Let σ (resp. σ') be a 2-simplex in C which witnesses h (resp. h') as a composition of f and g. Glue σ and σ' along f and glue on the degenerate 2-simplex $s_0 g$ to obtain a 3-horn Λ^3_1 in C. Lifting to a 3-simplex and taking the first face gets us a homotopy $h \simeq h'$.

Conversely, suppose that h is homotopic to h' via a witnessing 2-simplex τ. This time we glue σ to τ along h and add in $s_0 g$ to get a copy of Λ^3_2 in C; lifting and taking faces gives us a 2-simplex witnessing h' as a composition of f and g. □

In other words, the homotopy classes of morphisms are precisely the 'composition classes'. This indicates that if we take C and quotient out by homotopies, we should obtain a genuine 1-category. Showing this is the goal of the next section.

4.4 The homotopy category

Homotopy respects composition:

Proposition 4.14. *Let C be an ∞-category. Let $f, f' : x \to y$ be two homotopic morphisms and let $g, g' : y \to z$ be two homotopic morphisms. If h is a composition of f and g and h' is a composition of f' and g' then h is homotopic to h'.*

Proof. If h'' is a composition of f and g', it suffices to show that h'' is homotopic to both h and h'. We will show that $h'' \simeq h$; the other case is left as an exercise to the reader. Take 2-simplices σ_3 witnessing h as a composition, σ_2 witnessing h as a composition, and σ_0 witnessing a homotopy $g \to g'$. Glue these together into an inner 3-horn $[\sigma_0, -, \sigma_1, \sigma_2]$. Extend this to a 3-simplex and take the first face to obtain a 2-simplex σ_1 which is a homotopy from h to h''. □

Definition 4.15. Let C be an ∞-category. We want to define a 1-category $\underline{h}C$ with objects the objects of C, and with morphisms the homotopy classes of morphisms in C. Composition is inherited from C and is well-defined by the previous Proposition.

Proposition 4.16. *If C is an ∞-category, then $\underline{h}C$ is a 1-category.*

Proof. We need to prove that composition is associative and admits identities. The latter is left as an exercise. To show the former, take three morphisms $f : x \to y$, $g : y \to z$, and $h : z \to w$. Let u be a composition of f and g, v a composition of g and h, and w a composition of f and v. It suffices to show that w is a composition of u and h. To do this, choose three witnessing 2-simplices for u, v, w and glue them together to get an inner 3-horn whose spine is $[f, g, h]$. Lift this horn to a 3-simplex and take the first face to obtain the desired witness. □

Example 4.17. If D is a 1-category, there is a natural equivalence $D \cong \underline{h}ND$.

The assignment $C \mapsto \underline{h}C$ is functorial: a map of simplicial sets $F : C \to C'$ determines a map F_0 on objects of $\underline{h}C$ and a map F_1 on morphisms of $\underline{h}C$, which respects composition since F is a map of simplicial sets and hence respects witnessing 2-simplices.

Theorem 4.18. *Let C be an ∞-category. Then $\underline{h}C$ is canonically equivalent to hC, the homotopy category of the simplicial set C.*

Proof. We need to check that $\underline{h}C$ satisfies the universal property of the homotopy category: we will show this by giving unit and counit morphisms and checking that the zigzag identities for an adjunction are satisfied.

If C is an ∞-category and σ is an n-simplex of C then the spine of σ defines a composable string of n morphisms in $\underline{h}C$; in other words an n-simplex of $N(\underline{h}C)$. This gives a natural map $\eta_C : C \to N(\underline{h}C)$; this will be our adjunction unit. If D is a 1-category, observe that sending a morphism $[f]$ to f defines a natural isomorphism $\epsilon_D : \underline{h}ND \to D$; this will be our adjunction counit.

We need to check that the zigzag identities hold. The natural map

$$\underline{h}C \to \underline{h}N\underline{h}C \to \underline{h}C$$

is the identity on objects and morphisms and is hence is the identity functor. For the other map, take an n-simplex σ of ND; i.e. a string of n composable morphisms in D. The natural map $ND \to N\underline{h}ND$ sends σ to the n-simplex σ' of $N\underline{h}ND$ given by the corresponding string of composable morphisms in $\underline{h}ND \cong D$. The natural map $N\underline{h}ND \to ND$ sends this n-simplex back to σ. Hence the composition

$$ND \to N\underline{h}ND \to ND$$

is the identity as required. □

4.5 Isomorphisms

Definition 4.19. A morphism f in an ∞-category C is an **isomorphism** if $[f]$ is an isomorphism in hC.

It is immediate that isomorphisms satisfy the two-out-of-three property.

Exercise: if C is a 1-category then a map is an isomorphism in NC if and only if it is an isomorphism in C.

Definition 4.20. Let f be a morphism in an ∞-category. A **left homotopy inverse** of f is a morphism g such that $[g][f] = \mathrm{id}$ in hC. A **right homotopy inverse** is a g such that $[f][g] = \mathrm{id}$ in hC. A **homotopy inverse** of f is a morphism which is both a left and a right homotopy inverse.

Proposition 4.21. *Let f be a morphism in an ∞-category. The following are equivalent:*

1. *f is an isomorphism.*
2. *f admits a homotopy inverse.*
3. *f admits both a left and a right homotopy inverse.*

Proof. The equivalence of (1) and (2) are clear. Note that (2) and (3) both only depend on the homotopy class of f. So the equivalence of (2) and (3) follows from the fact that a morphism in a 1-category is an isomorphism if and only if it has both a left and a right inverse: if $uv = \mathrm{id}$ and $vw = \mathrm{id}$ then $u = uvw = w$. □

Proposition 4.22. *If C is an ∞-groupoid, then every morphism in C is an isomorphism.*

Proof. Given a morphism $f : x \to y$, filling the left horn formed by f and s_0x yields a right homotopy inverse of f. Similarly, filling the right horn formed by f and s_0y yields a left homotopy inverse of f. So f is an isomorphism. □

If X is an ∞-groupoid then hX is hence a groupoid; we denote this groupoid by $\pi_{\leq 1}X$ and refer to it as the **fundamental groupoid** of X.

Example 4.23. If X is a topological space, we have $\pi_{\leq 1}\mathrm{Sing}X \cong \pi_{\leq 1}X$, the usual fundamental groupoid of X.

5 Functors

Definition 5.1. A functor of ∞-categories $C \to D$ is just a map of simplicial sets $F : C \to D$. For now, we denote the set of such maps by $\mathrm{fun}_\infty(C, D)$.

Example 5.2. If C and D are 1-categories, there is a natural bijection

$$\mathrm{Fun}(NC, ND) \cong \mathrm{fun}_\infty(C, D)$$

Example 5.3. If C is an ∞-category and D is a 1-category, then there is a natural bijection

$$\mathrm{Fun}(hC, D) \cong \mathrm{fun}_\infty(C, ND)$$

Example 5.4. If X is a topological space and C is an ∞-category, then there is a natural bijection $\mathrm{Fun}_\infty(C, \mathrm{Sing}X) \cong \mathbf{Top}(|C|, X)$.

Of course, functors tend to come with more structure: they are themselves organised into functor categories. The goal of this section is to prove the ∞-categorical version of this fact: we will show how to make the collection of functors between two ∞-categories into an ∞-category.

5.1 Functor categories

Definition 5.5. If S, T are simplicial sets, then we define a simplicial set $\mathrm{Fun}(S, T)$ by setting $\mathrm{Fun}_n(S, T) := \mathbf{sSet}(\Delta^n \times S, T)$. Note that this is the nerve of T with respect to the cosimplicial simplicial set $\Delta^\bullet \times S$.

Lemma 5.6. *Let U, S, T be simplicial sets. Then there exists a natural bijection*

$$\mathbf{sSet}(U, \mathrm{Fun}(S, T)) \cong \mathbf{sSet}(U \times S, T).$$

Proof. Write U as a filtered colimit of its simplices and use that filtered colimits commute with products in Set to deduce that it suffices to check on $U = \Delta^n$. But this is clear by definition. □

Remark 5.7. The construction $\mathrm{Fun}(S, T)$ is part of a Cartesian closed symmetric monoidal structure on **sSet**.

Proposition 5.8. *Let C, D be two* 1*-categories. Then there is a natural isomorphism*

$$N\,\mathrm{Fun}(C, D) \cong \mathrm{Fun}(NC, ND).$$

Proof. We have isomorphisms

$$\begin{aligned}
N_n\,\mathrm{Fun}(C, D) &\cong \mathrm{Cat}([n], \mathrm{Fun}(C, D)) && \text{by definition}\\
&\cong \mathrm{Cat}([n] \times C, D) && \text{by hom-tensor in Cat}\\
&\cong \mathbf{sSet}(N([n] \times C), ND) && \text{by adjunction}\\
&\cong \mathbf{sSet}(\Delta^n \times NC, ND) && \text{since } N \text{ preserves limits}\\
&\cong \mathrm{Fun}(NC, ND)_n && \text{by definition}
\end{aligned}$$

naturally in n. □

The main theorem of this section is:

Theorem 5.9. *Let S be a simplicial set and D be an ∞-category. Then* $\mathrm{Fun}(S, D)$ *is an ∞-category.*

Corollary 5.10. *Let C, D be ∞-categories. Then* $\mathrm{Fun}(C, D)$ *is an ∞-category.*

The main theorem is difficult to prove and will require some discussion of lifting properties.

5.2 Trivial Kan fibrations

Definition 5.11. A map $p : X \to Y$ of simplicial sets is a **trivial Kan fibration** if every commutative diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

admits a lift. In other words, if a boundary of an n-simplex in X admits a filler in Y, then it admits a filler in X.

Proposition 5.12. *A map is a trivial Kan fibration if and only if it lifts against all monos.*

Proof. The idea is that any mono of simplicial sets can be written as a filtered colimit of boundary inclusions. □

Lemma 5.13. *Let f be a trivial Kan fibration and S a simplicial set. Then* $\mathrm{Fun}(S, f)$ *is a trivial Kan fibration.*

Proof. By the hom-tensor adjunction, an arbitrary map g lifts against $\mathrm{Fun}(S, f)$ if and only if $g \times S$ lifts against f. So let g be a mono; hence $g \times S$ is also a mono. In particular, $g \times S$ lifts against f and so g lifts against $\mathrm{Fun}(S, f)$ as desired. □

5.3 Troughs

Definition 5.14. The m**-trough** is the simplicial set

$$\mathrm{Tr}^m := (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2)$$

There is a natural inclusion $\mathrm{Tr}^m \hookrightarrow \Delta^m \times \Delta^2$.

Example 5.15. Tr^0 is simply Λ^2_1. The 1-trough Tr^2 consists of $\Delta^1 \times \Lambda^2_1$ glued at each end to a copy of Δ^2.

Lemma 5.16 (Joyal)**.** *Let $f : X \to Y$ be a map of simplicial sets. The following are equivalent:*

1. *f satisfies the right lifting property with respect to all inner horn inclusions.*

2. *f satisfies the right lifting property with respect to all trough inclusions.*

The proof is rather technical and involved, so we omit it.

5.4 Proof of the main theorem

Theorem 5.17 (Joyal)**.** *Let S be a simplicial set. The following are equivalent:*

1. *S is an ∞-category.*

2. *The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial Kan fibration*

$$\mathrm{Fun}(\Delta^2, S) \to \mathrm{Fun}(\Lambda^2_1, S).$$

Proof. S is an ∞-category if and only if $S \to \Delta^0$ lifts against all inner horn inclusions. By 5.16, S is hence an ∞-category if and only if $S \to \Delta^0$ lifts against all trough inclusions. But by adjointness, a lift in the diagram

$$\begin{array}{ccc} \mathrm{Tr}^m & \longrightarrow & S \\ \downarrow & & \downarrow \\ \Delta^m \times \Delta^2 & \longrightarrow & \Delta^0 \end{array}$$

is equivalent to a lift in the diagram

$$\begin{array}{ccc} \partial\Delta^m & \longrightarrow & \mathrm{Fun}(\Delta^2, S) \\ \downarrow & & \downarrow \\ \Delta^m & \longrightarrow & \mathrm{Fun}(\Lambda^2_1, S). \end{array}$$

□

Exercise: visualise this for $m = 1$.

Proof of 5.9. Let S be any simplicial set and D an ∞-category. We wish to show that $\mathrm{Fun}(S, D)$ is an ∞-category. Joyal's theorem tells us that we need only check that the natural map

$$\mathrm{Fun}(\Delta^2, \mathrm{Fun}(S, D)) \to \mathrm{Fun}(\Lambda^2_1, \mathrm{Fun}(S, D))$$

is a trivial Kan fibration. But using the closed monoidal structure, this map is equivalent to the map

$$\mathrm{Fun}(S, \mathrm{Fun}(\Delta^2, D)) \to \mathrm{Fun}(S, \mathrm{Fun}(\Lambda^2_1, D))$$

Since $\mathrm{Fun}(S, -)$ preserves trivial Kan fibrations by 5.13, we are done. □

5.5 Uniqueness of composition

Recall that composition of morphisms in an ∞-category is commutative up to homotopy. As an application of the previous technology, we'll refine this statement: there's actually a topological space (more accurately, a simplicial set) of compositions, and it's contractible.

Definition 5.18. A Kan complex X is called **contractible** if $X \to \Delta^0$ is a trivial Kan fibration.

Example 5.19. The empty simplicial set is a contractible Kan complex.

Example 5.20. Let K be a connected Kan complex. If K is contractible then $|K|$ is a weakly contractible topological space, in the sense that all homotopy groups of $|K|$ vanish. This is because the geometric realisation of the boundary inclusion $\partial\Delta^n \to \Delta^n$ is the inclusion $S^{n-1} \hookrightarrow D^n$. In fact, something stronger is true: let X be a path connected topological space. Then $\mathrm{Sing}_\bullet X$ is contractible if and only if X is weakly contractible. To prove this requires a little simplicial homotopy theory: a Kan complex K has homotopy groups $\pi_* K$, defined purely combinatorially, and they agree with $\pi_*|K|$. Moreover they detect contractibility.

Proposition 5.21. *Let $p : X \to Y$ be a trivial Kan fibration. Then for all $y \in Y$, the fibre $X_y := X \times_Y y$ is a contractible Kan complex.*

Proof. Trivial Kan fibrations are closed under pullbacks, since they're defined by a lifting property. But the pullback of p along y is $X_y \to \Delta^0$. □

Let f, g be a pair of composable morphisms in an ∞-category, so that they define an inner 2-horn $\sigma : \Lambda^2_1 \to C$. Then the simplicial set

$$K := \mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Lambda^2_1, \mathrm{C})} \{\sigma\}$$

is the space of compositions of f and g: the zero-simplices are given precisely by the 2-simplices $\tau : \Delta^2 \to C$ which restrict to σ along the canonical embedding $\Lambda^2_1 \to \Delta^2$. This simplicial set is contractible:

Proposition 5.22. *Let f, g be a pair of composable morphisms in an ∞-category. Then $\mathrm{Fun}(\Delta^2, C) \times_{\mathrm{Fun}(\Lambda^2_1, \mathrm{C})} \{(g, -, f)\}$ is a contractible Kan complex.*

Proof. The simplicial set in question is the fibre of the map $p : \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$ above the vertex defined by f and g. By 5.17, the map p is a trivial Kan fibration and hence its fibres are contractible Kan complexes. □

6 Commutative diagrams

In this section we'll study commutative diagrams in ∞-categories, especially commutative diagrams indexed by 1-skeletal simplicial sets, since we will be able to compare these with 1-categorical diagrams.

An important piece of intuition is that commutativity of a given diagram should be viewed as extra **structure** rather an a **property** of that diagram. For example, let x, y, z be three objects of an ∞-category C, with morphisms $f : x \to y$, $g : y \to z$, and $h : x \to z$. This data defines a morphism $\sigma : \partial\Delta^2 \to C$. We say that this triangle **commutes** if there is a map $\sigma' : \Delta^2 \to C$ extending σ, i.e. if there is a 2-simplex $gf \implies h$ witnessing h as a composition of f and g. Since there may be many such 2-simplices σ', one may need to make the choice of a particular one. This is structure rather than property!

6.1 Diagrams in ∞-categories

Definition 6.1. Let K be a simplicial set and C an ∞-category. A **K-indexed diagram in** C is a map $K \to C$ of simplicial sets.

Example 6.2. Let $K \subseteq \Delta^1 \times \Delta^1$ be the simplicial set $(\partial\Delta^1 \times \Delta^1) \cup (\Delta^1 \times \partial\Delta^1)$. Then a K-indexed diagram in C is a possibly noncommutative square in C. We view 'commutativity' of such a diagram as extra data; i.e. a lift of this diagram to a $\Delta^1 \times \Delta^1$-indexed diagram. Concretely, a square

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle r} \\ z & \xrightarrow{s} & w \end{array}$$

in C is commutative if we are given a morphism $q : x \to w$ together with two homotopies $rf \simeq q$ and $q \simeq sg$. Since there are many such choices of morphisms, commutativity is a structure of a diagram rather than a property.

In this section we will mainly focus on diagrams indexed by one-dimensional simplicial sets, since these can be interpreted in terms of 1-categorical information.

6.2 Quivers and path categories

Recall that a **quiver** is a directed multigraph, i.e. a set of **vertices** V and **edges** E with **source** and **target** maps $s, t : E \to V$. In particular we allow multiple edges between vertices and loops at a single vertex.

There is an equivalence between the category $\mathbf{sSet}^{\leq 1}$ of one-dimensional simplicial sets and the category of quivers: given a quiver Q the corresponding simplicial set $Q_\bullet$ has vertices V and nondegenerate 1-simplices given by E.

There's a forgetful functor from the category of small categories to the category of quivers: a category is in particular a quiver where one can compose arrows. This functor has a left adjoint, the **path category** functor. Concretely, if Q is a quiver then its path category $\mathbf{Path}[Q]$ is the category with objects V and morphisms given by (finite length) paths in Q. Exercise: prove the adjointness!

Example 6.3. Let Q be the quiver $0 \to 1 \to 2$. Then the (nerve of the) path category of Q is the 2-simplex Δ^2. More generally, if Q is the quiver $1 \to \cdots \to n$ then the path category of Q is Δ^n, with Q embedded as the spine.

If Q is a quiver, there is a natural map of simplicial sets $Q_\bullet \to N_\bullet\mathbf{Path}[Q]$ which is the identity on 0-simplices and an injection on 1-simplices. Exercise: show that this exhibits $\mathbf{Path}[Q]$ as the homotopy category $hQ_\bullet$.

6.3 Commutative diagrams in 1-categories

Before going to ∞-categories, we need to set our conventions for precisely what a 'commutative diagram' in a 1-category is.

Definition 6.4. Let Q be a quiver and C a category. A **(free) diagram** in C of shape Q is a functor $\mathbf{Path}[Q] \to C$. Equivalently, by adjointness, a free diagram is a map $Q_\bullet \to N_\bullet C$ of simplicial sets. A free diagram **commutes** if any two parallel paths in Q have the same image in C.

Example 6.5. Let Q be the quiver $1 \rightrightarrows 2$. A commutative diagram of shape Q in a category C is nothing more than an arrow in C; the two edges of Q must map to the same arrow of C.

Example 6.6. Let Q be the quiver with one vertex and one loop. A commutative diagram of shape Q in a category C is nothing more than an object in C, since the loop is parallel to the identity morphism.

Proposition 6.7. *Let Q be a quiver and $F : \mathbf{Path}[Q] \to C$ a free diagram. Then F commutes if and only if it factors through a poset P.*

When Q does not look like the previous two Examples, then one can write down a universal such P concretely:

Definition 6.8. A quiver Q is **thin** if

1. Q has at most one edge between each pair of vertices.
2. Q has no (directed) cycles.

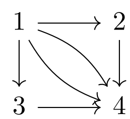
If Q is a thin quiver, then its vertex set V becomes a poset where we say that $u \leq v$ if there is a path from u to v. If $Q_<$ denotes the associated category, then there is a natural quotient map $\mathbf{Path}[Q] \to Q_< \cong$ which is the identity on vertices and sends a path $u \to v$ to the unique arrow $u \to v$ in $Q_<$.

Proposition 6.9. *Let Q be a thin quiver and $F : \mathbf{Path}[Q] \to C$ a free diagram in C of shape Q. Then F commutes if and only if it factors through the quotient $\mathbf{Path}[Q] \to Q_<$.*

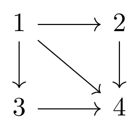
Example 6.10. Let Q be the thin quiver

$$\begin{array}{ccc} 1 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 3 & \longrightarrow & 4 \end{array}$$

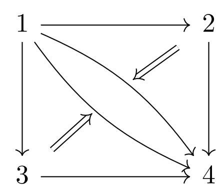
The non-identity arrows in the path category of Q are



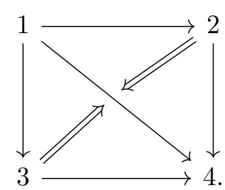
whereas the non-identity arrows in $Q_<$ are given by



In terms of simplicial sets, $N_\bullet\mathbf{Path}[Q]$ looks like



whereas $N_\bullet Q_<$ looks like



6.4 Path categories revisited

Let Q be a quiver. Recall that $\mathbf{Path}[Q]$ is the homotopy category of the simplicial set $hQ_\bullet$. In particular, if C is a 1-category then we have an isomorphism $\mathrm{Fun}(\mathbf{Path}[Q], C) \cong \mathrm{Fun}(Q_\bullet, N_\bullet C)$. We'd like an ∞-categorical version of this statement. And indeed we have:

Theorem 6.11. *Let Q be a quiver and C an ∞-category. Then the adjunction unit $u : Q_\bullet \to N_\bullet \mathrm{Path}[Q]$ induces trivial Kan fibration*

$$\mathrm{Fun}(N_\bullet \mathrm{Path}[Q], C) \to \mathrm{Fun}(Q_\bullet, C).$$

The proof relies on some technical calculations involving lifting properties.

Corollary 6.12. *Let Q be a quiver and C an ∞-category. Given a vertex $F \in \mathrm{Fun}(Q_\bullet, C)$, the fibre*

$$\mathrm{Fun}(N_\bullet \mathrm{Path}[Q], C) \times_{\mathrm{Fun}(Q_\bullet, C)} \{F\}$$

is a contractible Kan complex.

Example 6.13. Let Q be the poset $[n]$, so that $Q_\bullet \to N_\bullet \mathrm{Path}[Q]$ becomes the spine inclusion $\mathrm{sp}\Delta^n \hookrightarrow \Delta^n$. If C is an ∞-category then $\mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(\mathrm{sp}\Delta^n, C)$ is hence a trivial Kan fibration. When $n = 2$ this recovers one direction of 5.17. In particular $f : 0 \xrightarrow{f_1} 1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} n$ is a $Q_\bullet$-shaped diagram in C, it then follows that the space $\mathrm{Fun}(\Delta^n, C) \times_{\mathrm{Fun}(\mathrm{sp}\Delta^n, C)} \{f\}$ of 'n-fold compositions of f' is contractible.

Part II

Examples of ∞-categories

7 2-categories

Consider the category **Cat** of all small categories. The notion of isomorphism here is too strict: we care whether two categories are equivalent, not whether they're isomorphic. What's the natural framework to handle this?

One solution is to use a notion of **2-category**; then **Cat** should become a 2-category, with 2-morphisms the natural transformations and invertible 2-morphisms the equivalences.

One other solution is to regard **Cat** as a category with weak equivalences and formally invert them to obtain an ∞-category. Note that this discards information about non-invertible natural transformations. One can also put extra structure on top of this to get better control of the localisation, e.g. a model structure (and indeed **Cat** does admit such a model structure, the most famous being the **folk** or **canonical** model structure).

In fact, 2-categories have been well studied and are reasonably well understood. From the above discussion, we expect a 2-category to give an $(\infty, 1)$-category by discarding the noninvertible 2-morphisms and then taking some sort of nerve. In this section, we'll show that this can indeed be done; in fact we'll produce a simplicial set from any 2-category C, which will be an ∞-category precisely when C is a $(2, 1)$-category. We'll also show that this nerve construction, the **Duskin nerve**, is fully faithful.

Really, a 2-category should be a certain kind of $(\infty, 2)$-category via some sort of nerve construction, and then one can truncate to obtain an $(\infty, 1)$-category: this won't bother us here.

7.1 Strict 2-categories

The following is the most naïve definition of a 2-category.

Definition 7.1. A **strict 2-category** is a category C enriched in categories, i.e. a gadget with

- Objects x.
- For every pair of objects x, y a category $C(x, y)$. We call objects of $C(x, y)$ **1-morphisms** or just **morphisms**. We call morphisms of $C(x, y)$ **2-morphisms**.
- For every triple of objects x, y, z, composition functors

$$\circ_{xyz} : C(y, z) \times C(x, y) \to C(x, z).$$

  Note the order of the factors in the product.
- Identity 1-morphisms $\mathrm{id}_x \in C(x, x)$. Note that we can also view these objects as functors $* \to C(x, x)$.

Satisfying the following conditions:

- Composition is strictly associative.
- There are equalities $\mathrm{id}_x \circ - = \mathrm{id}_{C(y,x)}$ and $- \circ \mathrm{id}_x = \mathrm{id}_{C(x,y)}$.

Example 7.2. **Cat** is itself a 2-category (indeed the prototypical example). The objects are the categories, and the hom-categories are given by functor categories.

Example 7.3. A category is a strict 2-category, where the hom-categories are just regarded as sets (i.e. discrete categories).

Remark 7.4. There are some set-theoretic difficulties here which we will brush under the rug. In particular the hom-categories should be small, in order to be able to use standard category-theoretic arguments.

Given an object x in a strict 2-category, it has an endomorphism category $C(x, x)$ which is strict monoidal (since it's a monoid in categories). Similarly, given a strict monoidal category C, it has a delooping BC which is a one-object strict 2-category. These constructions give an equivalence (in an appropriate sense...) between strict monoidal categories and one-object 2-categories. In particular, it is often useful to think of a strict 2-category as a 'many-object monoidal category'.

7.2 2-categories

Just like strict monoidal categories are often too rigid (think of $\mathbf{Vect}_k$ with the tensor product - it is only associative up to natural isomorphism!), strict 2-categories are often too rigid for our purposes. We need to introduce a weaker concept. The following definition is due to Bénabou.

Definition 7.5. A **2-category** C consists of:

- Objects x.
- Hom-categories $C(x, y)$.
- Composition functors and identity 1-morphisms as before.
- For every object x an invertible 2-morphism $\upsilon_x : \mathrm{id}_x \circ \mathrm{id}_x \to \mathrm{id}_x$. Note that there is no sense in which these are 'natural in x'. These 2-morphisms are known as the **units**.
- For every quadruple x, y, z, w, natural isomorphisms α_{xyzw} between the functors $- \circ (- \circ -)$ and $(- \circ -) \circ -$. Exercise: work out the subscripts on the composition functors. We denote the component of α_{xyzw} on a triple of 1-morphisms h, g, f by $\alpha_{hgf} : h \circ (g \circ f) \to (h \circ g) \circ f$. These natural transformations are known as the **associators**.

Satisfying the following conditions:

(C) : Pre- and post-composition with id_x is fully faithful.

(P) : The pentagon identity for associators; this asserts that the two ways of going from the fourfold composition $h(g(fe)) \to ((hg)f)e$ using the associators are equal. Exercise: explicitly draw out this diagram to see why this is called the pentagon identity.

Example 7.6. if C is a strict 2-category, then it is a 2-category: associators and units are just the identity maps.

Example 7.7. If C is a 2-category then $C(x, x)$ is a monoidal category. As before, there's a looping-delooping equivalence between one-object 2-categories and monoidal categories.

Example 7.8. Define a 2-category **Bimod**: the objects are rings A. The morphisms from A to B are the category of A-B-bimodules. The composition functor is tensoring: ${}_AM_B \circ_B N_C = M \otimes_B C$. The identity bimodule is the diagonal bimodule. Associators and units are given by the standard ones for the tensor product.

Let C be a 2-category and $f : x \to y$ a 1-morphism in C. There are canonical isomorphisms $\mathrm{id}_y \circ (\mathrm{id}_y \circ f) \to \mathrm{id}_y \circ f$ defined by first applying the associator and then the unit υ_y. Since $\mathrm{id}_y \circ -$ is fully faithful, it follows that there is a unique isomorphism $\lambda_f : \mathrm{id}_y \circ f \to f$ for all f. We call λ_f the **left unitor**. Similarly, there are also **right unitors** ρ_f. The assignment $f \mapsto \lambda_f$ is natural in f.

The **triangle identities** express that applying an associator turns a left unitor into a right unitor; we do not reproduce them here. Compare with the triangle identities in a monoidal category.

7.3 Composition

In a 2-category, we have both horizontal and vertical composition of 2-morphisms. Let $f, g, h : x \to y$ be 1-morphisms, and let $\gamma : f \to g$ and $\delta : g \to h$ be two 2-morphisms in a hom-category $C(x, y)$. Their **vertical composition** is their composition $\delta\gamma \in C(x, y)$. Diagramatically, this corresponds to sticking the corresponding 2-cells together end-to-end.

Now suppose that $f, g : x \to y$ and $f', g' : y \to z$ are 1-morphisms and $\gamma : f \to g$ and $\gamma' : f' \to g'$ are 2-morphisms. Their **horizontal composition** is $\gamma' \circ \gamma : f' \circ f \to g' \circ g$. Diagramatically, this glues two 2-cells along their common vertices and regards the diagram as one large 2-cell.

7.4 Opposites

Let C be a 2-category. There is an obvious notion of **opposite**: C^{op} is the 2-category with $C^{\text{op}}(x, y) := C(y, x)$. The units and associators are inherited from C. Exercise: write these down explicitly.

However, one could instead apply the opposite functor to the hom-2-categories! The **conjugate** C^c of C has $C^c(x, y) := C(x, y)^{\text{op}}$. Again, the units and associators are inherited from C - write these down explicitly if the last exercise wasn't enough for you.

Observation: The opposite of C has horizontal composition reversed but vertical composition the same. The conjugate of C has horizontal composition the same but vertical composition reversed.

Example 7.9. If C is a 1-category viewed as a 2-category, then C^{op} is the usual opposite category. The conjugate C^c is C itself.

Example 7.10. Let C be a monoidal category. Then $B(C)^{\text{op}}$ is $B(C^{\text{rev}})$, where C^{rev} is the **reverse** of C: the underlying category is the same but the monoidal product is reversed. In particular if C is symmetric monoidal then $B(C)$ is isomorphic to its opposite. Similarly, $B(C)^c$ is $B(C^{\text{op}})$, where C^{op} is endowed with the natural monoidal structure.

Exercise: convince yourself that $(C^{\text{op}})^c = (C^c)^{\text{op}}$.

7.5 Functors

If M, N are monoidal categories then there are three fundamental notions of functors between them: strict monoidal, monoidal, and lax monoidal (there are also oplax monoidal functors, but these can be intepreted in terms of opposite categories). Loosely, strict monoidal functors preserve the monoidal structure up to equality, monoidal functors up to natural isomorphism, and lax monoidal functors simply provide a natural comparison map (for oplax monoidal functors this comparison map runs in the opposite direction). Viewing 2-categories as many-object monoidal categories, it should come as no surprise that there are three different notions of functor between them.

Definition 7.11. A **strict 2-functor** between two 2-categories C, D is:

- A map F on objects.
- For each pair x, y a functor $F_{xy} : C(x, y) \to D(Fx, Fy)$.

Satisfying the identities:

- $F\upsilon = \upsilon_F$ for all units υ.
- $F\alpha = \alpha_F$ for all associators α.

Remark 7.12. This is the many-object version of a strict monoidal functor.

Remark 7.13. The identity 2-functor is a strict 2-functor.

Example 7.14. If C and D are strict 2-categories, then a strict 2-functor between them is just a **Cat**-enriched functor. In particular, if C and D are 1-categories, then strict 2-functors are the same thing as usual functors.

As before, this notion is too rigid. We want a looser definition.

Definition 7.15. A **lax 2-functor** F between two 2-categories C, D is:

- A map F on objects.
- For each pair x, y a functor $F_{xy} : C(x, y) \to D(Fx, Fy)$.
- For every $x \in C$, a 2-morphism $\epsilon : \mathrm{id}_{Fx} \to F(\mathrm{id}_x)$ which we call the **identity constraint**.
- For every composable pair (g, f) of 1-morphisms in C, a 2-morphism

$$\mu_{gf} : Fg \circ Ff \to F(g \circ f)$$

  which we call the **composition constraint**.

Satisfying the conditions:

1. μ_{gf} is natural in both g and f.
2. Compatibility between λ_{Ff} and $F\lambda_f$ via the ϵ and μ.
3. Compatibility between ρ_{Ff} and $F\rho_f$ via the ϵ and μ.
4. Compatibility between $F\alpha$ and α_F via the μ.

Exercise: spell out this definition explicitly.

Definition 7.16.

1. A **unitary lax 2-functor** of 2-categories is a lax 2-functor with all the ϵ invertible.

2. A unitary lax 2-functor is a **2-functor** if in addition all the μ are invertible.

3. A **strict unitary lax 2-functor** is a lax 2-functor with all ϵ the identity morphism. Clearly a strict unitary lax 2-functor is a unitary lax 2-functor.

We abbreviate "unitary lax" to **ulax** and "strict unitary lax" to **sulax**.

Remark 7.17. Just like strict 2-functors are many-object versions of strict monoidal functors, 2-functors generalise monoidal functors and lax 2-functors generalise lax monoidal functors. Exercise: show this!

Remark 7.18. An **oplax 2-functor** is the same as a lax 2-functor, but the unit and composition constraints go the other way. These generalise oplax monoidal functors. Exercise: show that an oplax 2-functor $C \to D$ is the same thing as a lax 2-functor $C \to D^c$ to the conjugate of D.

All of these notions of functor are closed under composition. In particular, the collection **2Cat** of 2-categories forms a 1-category when equipped with any of these collections of functors. We have obvious inclusions

$$\begin{array}{ccccccc} & & & \mathbf{2Cat} & & & \\ & & \nearrow & & \searrow & & \\ \mathbf{Cat} \hookrightarrow & \mathbf{2Cat}_{\mathrm{str}} & & & & \mathbf{2Cat}_{\mathrm{ulax}} & \hookrightarrow \mathbf{2Cat}_{\mathrm{lax}} \\ & & \searrow & & \nearrow & & \\ & & & \mathbf{2Cat}_{\mathrm{sulax}} & & & \end{array}$$

where no functor except the leftmost is fully faithful, and all functors except the leftmost are the identity on objects.

Remark 7.19. Every 2-category is equivalent to a strict 2-category; this is a many-object version of the strictification theorem for monoidal categories. However, one should think of this fact as a coincidence, rather than a desirable feature of the theory: whatever an n-category is, for $n \geq 3$ not all n-categories are equivalent to strict n-categories.

7.6 Coarse homotopy categories

If C is a strict 2-category, then by forgetting the 2-morphisms we can view it as a 1-category. However, if C is a 2-category then this does not work, since composition is no longer associative. We remedy this by introducing two different notions of homotopy category for 2-categories.

Note that there is a natural inclusion functor $i : \mathbf{Cat} \to \mathbf{2Cat}$ which regards a category as a strict 2-category with discrete hom-categories.

Definition 7.20. Let C be a 2-category. A 2-functor $\eta : C \to iD$ exhibits D as the **coarse homotopy category** of C if it induces a bijection $\mathbf{Cat}(D, E) = \mathbf{2Cat}(C, iE)$ via postcomposition.

In particular, if one can naturally choose such 2-functors η then the assignment of coarse homotopy categories defines a functor h from **2Cat** to **Cat** which is left adjoint to i. In particular $D = hC$ is then unique up to unique isomorphism.

Theorem 7.21. *Coarse homotopy categories exist.*

Proof. We begin with the construction of hC. The objects of hC are the same as the objects of C. The homsets are given by $hC(x, y) \coloneqq \pi_0 N_\bullet C(x, y)$. Composition is inherited from C using that π_0 and $N_\bullet$ preserve finite products. Associativity follows because the nerve functor sends components to components: in $C(x, y)$, if two 1-morphisms f, g are linked by a 2-morphism $\phi : f \to g$, then they lie in the same connected component of $N_\bullet C(x, y)$.

Now we need to define the comparison 2-functor $\eta : C \to hC$. It's the identity on objects. On 1-morphisms, η sends an object $f \in C(x, y)$ to its homotopy class $[f] \in \pi_0 N_\bullet(x, y)$. Given a 2-morphism $f \to g$, we send it to $\mathrm{id}_{[f]}$; as before this is well-defined since the nerve preserves connected components. The identity and composition constraints are given by identities. One can easily check that this defines a 2-functor.

Finally we need to check that η is universal. Let E be a 1-category and suppose that we are given a 2-functor $F : C \to iE$. We need to show that there exists a unique $\bar{F} : hC \to E$ such that $\bar{F}\eta = F$. For existence, set $\bar{F}$ to be the identity on objects. On morphisms, put $\bar{F}[f] \coloneqq [Ff]$; again this is well-defined by the same connected components argument. Clearly $\bar{F}$ lifts F. For uniqueness, suppose we have another functor G lifting F. Certainly G must agree with $\bar{F}$ on objects, and moreover we have $G[f] = [Ff] = \bar{F}[f]$, so that $G = \bar{F}$. □

Observe that the coarse homotopy category of a general 2-category kills too much information: in particular it identifies any two 1-morphisms that are linked by any zigzag of 2-morphisms. We'd like to first apply the core functor to our hom-categories in order to discard non-invertible 2-morphisms.

7.7 (2,1)-categories and the pith

A **(2,1)-category** is a 2-category where all hom-categories are groupoids. Clearly a 1-category is a (2,1)-category.

Definition 7.22. For a 1-category C, the **core** of C is the maximal subgroupoid $C^{\simeq}$; i.e. one takes C and throws away all of the non-invertible morphisms. The core is functorial, and is the right adjoint to the inclusion of groupoids into categories (the left adjoint is given by freely adding inverses).

Definition 7.23. Let C be a 2-category. The **pith** of C is the (2,1)-category $\mathrm{Pith}C$ which has the same objects as C and hom-groupoids given by

$$\mathrm{Pith}C(x, y) \coloneqq C(x, y)^{\simeq}.$$

Composition and units are inherited from C via the functoriality of the core; one can check that Pith$C$ remains a 2-category. As before, the pith is functorial, and moreover is right adjoint to the inclusion of $(2,1)$-categories into 2-categories.

7.8 Homotopy categories

Let C be a 2-category. The **homotopy category** (or **fine homotopy category** when we need to distinguish it from the coarse one) of C is the 1-category $\mathfrak{h}C := h\mathrm{Pith}C$. Note that $\mathfrak{h}$ is functorial, but not an adjoint - it is the composition of a right and a left adjoint. It is possible to give a more down-to-earth definition of $\mathfrak{h}C$: one sets

$$\mathfrak{h}C(x,y) := \{\text{isoclasses of } C(x,y)\}.$$

This alternate characterisation follows since, for G a groupoid, $\pi_0 N_\bullet G$ is precisely the set of isoclasses of G: two objects of G live in the same connected component of the nerve if and only if they are isomorphic.

7.9 Isomorphisms

Now we have the (correct) notion of the homotopy category of a 2-category, we can say what it means for a 1-morphism of a 2-category to be an isomorphism:

Definition 7.24. Let C be a 2-category and $f : x \to y$ a 1-morphism. Say that f is an **isomorphism** precisely when $[f]$ is an isomorphism in the homotopy category $\mathfrak{h}C$.

Exercise: $f : x \to y$ is an isomorphism if and only if there exists $g : y \to x$ and invertible 2-morphisms $fg \to \mathrm{id}_y$ and $gf \to \mathrm{id}_x$.

Example 7.25. **Cat** is a 2-category. A 1-morphism is an isomorphism in the above sense if and only if it is an equivalence of categories.

7.10 The Duskin nerve

We now expect that a $(2,1)$-category should give an ∞-category via some sort of nerve construction. Then one can give a nerve for any 2-category by first taking the pith.

To do this, we will define for **any** 2-category C a simplicial set $N_\bullet^D C$, the **Duskin nerve** of C (first introduced by Street for strict 2-categories and Duskin for all 2-categories).

We will later show that when C is a $(2,1)$-category, its Duskin nerve is an ∞-category. We will also show that the formation of Duskin nerves gives a fully faithful functor $\mathbf{2Cat}_{\mathrm{sulax}} \to \mathbf{sSet}$, just like in the 1-categorical case.

Recall the standard cosimplicial category $\tilde{\Delta}^\bullet$ with $\tilde{\Delta}^n := [n]$. Composing $\tilde{\Delta}$ with the embedding $\mathbf{Cat} \hookrightarrow \mathbf{2Cat}_{\mathrm{sulax}}$, we obtain a cosimplicial object in 2-categories, which we will still refer to by $\tilde{\Delta}^\bullet$.

Definition 7.26. The **Duskin nerve** functor $N^D_\bullet : \mathbf{2Cat}_{\mathrm{sulax}} \to \mathbf{sSet}$ is the nerve with respect to $\tilde{\Delta}$. Concretely, the n-simplices of $N^D_\bullet C$ are in bijection with the sulax 2-functors $[n] \to C$.

Exercise: if C is a 1-category, then its Duskin nerve is canonically isomorphic to $N_\bullet C$.

Remark 7.27. The Duskin nerve sends opposites to opposites, but does not have any simple relationship to conjugates (since taking the conjugate turns lax functors into oplax functors).

Remark 7.28. The Duskin nerve does not admit a left adjoint: the problem is that $\mathbf{2Cat}_{\mathrm{sulax}}$ does not have enough colimits to allow realisation with respect to $\tilde{\Delta}^\bullet$.

7.11 Low-dimensional simplices of the Duskin nerve

One can write down concretely what an n-simplex of the Duskin nerve of a 2-category is. We refrain from doing this in generality, but give some low-dimensional examples.

Let C be a 2-category.

Example 7.29 (0-simplices). There is a natural bijection $N^D_0 C \cong \mathbf{Ob} C$.

Example 7.30 (1-simplices). There is a natural bijection $N^D_1 C \cong \mathbf{Mor}_1 C$. Moreover, the face and degeneracy maps linking 0- and 1-simplices are exactly the expected ones from the 1-categorical nerve.

Example 7.31 (2-simplices). A 2-simplex $\sigma \in N^D_2 C$ can be identified with the following data:

1. Three objects X, Y, Z.
2. Three 1-morphisms $f : X \to Y$, $g : Y \to Z$ and $h : X \to Z$.
3. A 2-morphism $\phi : gf \to h$.

7.12 Thin simplices

In this part we aim to give (an idea of) the proof of the following theorem:

Theorem 7.32 (Duskin). *Let C be a 2-category. Then the Duskin nerve $N^D_\bullet C$ is an ∞-category if and only if C is a $(2,1)$-category.*

We will prove it using the auxiliary concept of a **thin** simplex.

Definition 7.33. Let X be a simplicial set. Say that a 2-simplex τ of X is **thin** if the following condition holds: for every $n \geq 3$ and for every $0 < i < n$, let $\sigma : \Lambda_i^n \to X$ be an inner i-horn containing τ as the 2-simplex $\{i-1, i, i+1\}$. Then σ admits a lift to Δ^n.

It is clear that every 2-simplex in an ∞-category is thin. Conversely, if every 2-simplex of a simplicial set X is thin, then X is an ∞-category if and only if it admits fillers for inner 2-horns.

Proposition 7.34. *Let C be a 2-category and σ a 2-simplex of the Duskin nerve, corresponding to a 2-morphism $\phi : gf \to h$. Then σ is thin if and only if ϕ is an isomorphism.*

We will not give the proof of this as it is rather involved. However, it can be reduced to checking only that 3- and 4-simplices lift, using the following 2-categorical version of 3.6:

Lemma 7.35. *Let C be a 2-category. Then the restriction map*

$$\mathbf{sSet}(\Delta^n, N_\bullet^D C) \to \mathbf{sSet}(\partial\Delta^n, N_\bullet^D C)$$

is a bijection for $n \geq 4$ and an injection for $n = 3$.

Proof of 7.32. If C is any 2-category then $N_\bullet^D C$ admits fillers for inner 2-horns using the composition in C. So $N_\bullet^D C$ is an ∞-category if and only if every 2-simplex is thin. By 7.34, this is equivalent to C being a $(2,1)$-category. □

7.13 Fully faithfulness

Theorem 7.36 (Duskin)**.** *The Duskin nerve is a fully faithful functor*

$$N_\bullet^D : \mathbf{2Cat}_{\text{sulax}} \hookrightarrow \mathbf{sSet}.$$

Again, the proof is rather involved, so we omit it. Clearly one can recover objects and 1-morphisms from the Duskin nerve. Moreover one can recover 2-morphisms from degenerate 2-simplices. The hard part is checking that one can recover the associators and units.

Corollary 7.37. *The Duskin nerve is a fully faithful functor*

$$N_\bullet^D : \mathbf{(2,1)Cat}_{\text{sulax}} \hookrightarrow \infty\mathbf{Cat}.$$

Corollary 7.38. *Let C be a 2-category. Then the coarse homotopy category hC is $hN_\bullet^D C$. In particular, if C is a $(2,1)$-category then $\mathfrak{h}C \simeq hN_\bullet^D C$.*

7.14 Strict 2-categories

What sort of structure does the Duskin nerve of a strict 2-category have? We show that it has an alternate description solely in terms of strict 2-functors.

Definition 7.39. Let P be a poset. We define a strict 2-category, the **path 2-category** $\mathrm{Path}_2[P]$, as follows.

- The objects of $\mathrm{Path}_2[P]$ are the elements of P.
- The hom-category between two elements x, y is the poset of chains from x to y (i.e. linearly ordered subsets with minimal element x and maximal element y), ordered by *reverse* inclusion. Note that this hom-category has a maximal element $\{x \leq y\}$.
- The identity 1-morphism id_x is the poset $\{x\}$.
- Associators are given by concatenation: if S is a chain from x to y and T is a chain from y to z then $S \cup T$ is a chain from x to z.

Remark 7.40. If P is a poset, regarded as a quiver, then the underlying 1-category of $\mathrm{Path}_2[P]$ is the path category $\mathrm{Path}[P]$.

Since the assignment $n \mapsto [n]$ defines a cosimplicial category, applying the path 2-functor yields a cosimplicial strict 2-category $\mathrm{Path}_2[\bullet]$. We can then take the nerve with respect to this cosimplicial strict 2-category, which turns out to be precisely the Duskin nerve:

Theorem 7.41. *The nerve with respect to the cosimplicial strict 2-category* $\mathrm{Path}_2[\bullet]$ *is canonically isomorphic to the functor*

$$\mathbf{2Cat}_{\mathrm{str}} \hookrightarrow \mathbf{2Cat}_{\mathrm{sulax}} \xrightarrow{N^D_\bullet} \mathbf{sSet}.$$

Proof sketch. For any poset P we will first construct a natural sulax 2-functor $T_P : P \to \mathrm{Path}_2[P]$. On objects it is the identity. On 1-morphisms, if $x \leq y$ then the unique map $x \to y$ of P is sent to the poset $\{x < y\}$. If $x < y < z$ then the composition constraint is the inclusion $\{x < z\} \hookrightarrow \{x < y < z\}$. One can check that this defines a sulax functor T_P. Composition with T_P then induces, for every strict 2-category C, a natural map

$$\mathbf{2Cat}_{\mathrm{str}}(\mathrm{Path}_2[P], C) \to \mathbf{2Cat}_{\mathrm{sulax}}(P, C)$$

and one checks that this is a bijection. In particular, the natural map of cosimplicial 2-categories $T_\bullet : [\bullet] \to \mathrm{Path}_2[\bullet]$ yields a map of nerves

$$N_\bullet^{\mathrm{Path}_2[\bullet]}(C) \to N^D_\bullet(C)$$

which is an isomorphism. □

Corollary 7.42. *If C is a strict 2-category, then there is a natural bijection*

$$N^D_n(C) \cong \mathbf{2Cat}_{\mathrm{str}}(\mathrm{Path}_2[n], C).$$

Remark 7.43. The category of strict 2-categories has all colimits, and hence the restricted Duskin nerve functor $N^D_\bullet : \mathbf{2Cat}_{\mathrm{str}} \to \mathbf{sSet}$ admits a left adjoint, which geometrically realises a simplicial set as a strict 2-category: the standard n-simplex is sent to the path category $\mathrm{Path}_2[n]$ and then extended via colimits. More generally, a poset P, viewed as a 1-dimensional simplicial set, is realised as the strict 2-category $\mathrm{Path}_2[P]$.

8 Simplicially enriched categories

In this section we consider categories enriched in simplicial sets, which we will think of as alternate models for ∞-categories. More precisely, only certain simplicially enriched categories will model ∞-categories: just like 2-categories, these will be the simplicially enriched categories with only invertible higher morphisms.

We'll describe a nerve functor, the **homotopy coherent nerve** N^{hc}, which takes a simplicially enriched category to a simplicial set; just like the Duskin nerve, the image of the homotopy coherent nerve includes simplicial sets that are not ∞-categories.

8.1 First definitions

Definition 8.1. A **simplicial category** C is a category enriched in simplicial sets. More concretely, a simplicial category consists of:

- A class of objects.
- A simplicial set $C(x, y)$ for every pair of objects x, y.
- Composition maps which are morphisms in **sSet**.
- Distinguished identity vertices $\mathrm{id}_x \in C(x, x)_0$.
- Coherence axioms for composition and identities.

Definition 8.2. A **simplicial functor** $F : C \to D$ between simplicial categories consists of a map F on objects, and for every pair of objects (x, y) a morphism $F_{xy} : C(x, y) \to D(Fx, Fy)$ of simplicial sets. We also demand that the coherence axioms $F_{xx}(\mathrm{id}_x) = \mathrm{id}_{Fx}$ and $F_{yz}F_{xy} = F_{xz}$ are satisfied for all x, y, z.

We denote the category of (small) simplicial categories by **sCat**.

Example 8.3. **sSet** is itself a simplicial category, with hom-spaces given by the usual closed monoidal structure $\mathbf{sSet}(X, Y)_n = \mathbf{sSet}(X \times \Delta^n, Y)$.

Example 8.4. Similarly, the category of topological spaces is a simplicial category, with $\mathbf{Top}(X, Y)_n \coloneqq \mathbf{Top}(X \times \underline{\Delta}^n, Y)$.

Example 8.5. If C is a category, then the **constant simplicial category on** C has the same objects as C, and hom-spaces $C(x, y)_n = C(x, y)$ for all n. This gives a fully faithful embedding from **Cat** to **sCat**.

8.2 Simplicial objects in categories

Warning: our choice of terminology "simplicial categories" is potentially confusing! There is a difference between **sCat** and the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ of simplicial objects in the category of categories. However, one can compare the two:

Lemma 8.6. *There is a fully faithful embedding* $\mathbf{sCat} \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ *whose image consists of those functors* X *which are constant on objects, i.e.* $\mathrm{Obj}(X[n])$ *is a constant simplicial set.*

Proof sketch. Let C be a simplicial category. Then for each $n \geq 0$ we associate a category C_n with the same objects, and $C_n(x, y) \coloneqq C(x, y)_n$. Composition is inherited from C; recall that products of simplicial sets are defined levelwise. Identities are id_x, viewed as a degenerate n-simplex. This construction defines a functor from **sCat** to $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat})$ which sends C to the functor $[n] \mapsto C_n$. Fullness and faithfullness of this functor come from unwinding the definition of simplicial functors. It is clear that the image consists of those functors which are constant on objects. □

Proposition 8.7. sCat *is complete and cocomplete.*

Proof sketch. Since **Cat** is bicomplete, so is $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat})$. In particular, any (small) diagram of simplicial categories has a (co)limit in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat})$. But (co)limits in this category are computed levelwise, and in particular (co)limits of diagrams which are constant on objects remain constant on objects (since **Set** is closed under colimits in **sSet**). In other words, the (co)limit of a diagram in **sCat** remains in **sCat**. □

8.3 Homotopies and homotopy categories

Definition 8.8. Let C be a simplicial category and $f, g : x \to y$ two vertices of $C(x, y)$. A **homotopy** $f \to g$ is an edge h of $C(x, y)$ such that $d^1 h = f$ and $d^0 h = g$.

Example 8.9. Homotopy in **Top** is precisely the usual notion of homotopy.

Example 8.10. Homotopy in **sSet** is the usual notion of simplicial homotopy.

Just like simplicial homotopy is badly behaved for arbitrary simplicial sets, the notion of homotopy between two morphisms of a simplicial category is also ill-behaved in general; we'd like to restrict to a nice subclass of simplicial categories. Unsurprisingly, these are the categories enriched in Kan complexes:

Definition 8.11. A simplicial category is **locally Kan** if all mapping spaces are Kan complexes.

Example 8.12. **Top** is locally Kan. If C is a category, then the constant simplicial category on C is locally Kan.

Proposition 8.13. *If* C *is locally Kan, then for all* $x, y \in C$ *homotopy is an equivalence relation on* $C(x, y)_0$.

Proof. If K is a Kan complex (more generally a weak Kan complex), homotopy is an equivalence relation on K_0. □

Definition 8.14. Let C be a simplicial category. The **homotopy category** of C is the category $\pi_0 C$ obtained by taking connected components of the morphisms spaces of C.

Proposition 8.15. *If C is locally Kan, then $\pi_0 C(x,y) \simeq C(x,y)_0/$(homotopy).*

Proof. Clear from the fact that if X is a Kan complex then $\pi_0 X$ is the quotient of X_0 by homotopies. □

Remark 8.16. There are several particularly nice functors $E : \mathbf{sSet} \to \mathbf{Kan}$ that one can view as 'Kanification'; for example Kan's Ex^∞ functor is the canonical choice but one could also use $\mathrm{Sing}_\bullet|-|$. Both of these functors preserve finite products and hence applying E levelwise gives a functor $\mathcal{E}$ that sends an arbitrary simplicial category to a locally Kan simplicial category.

Remark 8.17. The functor E above is a fibrant replacement functor for the usual Kan-Quillen model structure on **sSet**. The category **sCat** admits an enriched model structure, the **Bergner model structure**, where the weak equivalences are the Dwyer–Kan-equivalences and the fibrations are the enriched fibrations. Then the functor $\mathcal{E}$ above is a fibrant replacement functor in the Bergner model structure.

8.4 The homotopy coherent nerve

We're going to define a nerve functor from simplicial categories to simplicial sets. In particular, we want to cook up a cosimplicial object in simplicial categories. The following definition is the simplicial category version of 7.39, and is originally due to Cordier.

Definition 8.18. Let P be a poset. For $x, y \in P$ consider the poset $P_{x,y}$ whose elements are the chains from x to y, ordered by reverse inclusion. Using this we can define a simplicial category $\underline{\mathrm{Path}}(P)$, whose objects are the elements of P, and whose hom-spaces are $\underline{\mathrm{Path}}(P)(x,y) \coloneqq N_\bullet P_{x,y}$. Identities are given by the poset $\{x\}$ and composition is given by concatenation (i.e. union).

Remark 8.19. The choice of reverse ordering is a convention; one must make a choice since categories have a $\mathbb{Z}/2$-symmetry (given by the opposite). Cordier's original definition used the opposite convention; we choose this convention for compatibility with the Duskin nerve.

Example 8.20. The path category of $[n]$ has hom-spaces given by $\underline{\mathrm{Path}}[n](i,j) = (\Delta^1)^{\times(j-i-1)}$ if $i < j$, the terminal simplicial set if $i = j$, and empty otherwise.

Just like with path 2-categories, $\underline{\mathrm{Path}}$ is a functor, and hence applying it to the standard cosimplicial poset we obtain a cosimplicial simplicial category $[n] \mapsto \mathrm{Path}[n]$. The nerve with respect to this is the **homotopy coherent nerve** N^{hc}.

Remark 8.21. Let C be a simplicial category. The vertices of $N^{\mathrm{hc}}C$ are the objects of C. The edges of $N^{\mathrm{hc}}C$ are the morphisms of C, and the face and degeneracy maps are as expected.

Proposition 8.22. *If C is a category, then there is a natural isomorphism $N_\bullet C \simeq N^{\mathrm{hc}}C$.*

Theorem 8.23 (Cordier–Porter)**.** *Let C be a locally Kan simplicial category. Then $N^{\mathrm{hc}}C$ is a quasi-category.*

The above is rather difficult to prove, so we make no attempt at doing so.

Remark 8.24. Be warned that the homotopy coherent nerve is not fully faithful; loosely this is because higher simplices of the nerve mix the categorical information with the simplicial information. Heuristically, there is no way to tell whether a given n-simplex of $N^{\mathrm{hc}}C$ is obtained by composing simplices of lower dimension, or whether it arises as an n-simplex in some mapping space of C.

Comparing the two definitions of homotopy category yields:

Proposition 8.25. *If C is a locally Kan simplicial category, then there is a natural equivalence $hN^{\mathrm{hc}}(C) \simeq \pi_0 C$.*

Remark 8.26. The levelwise nerve functor yields a functor $N : \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Cat}) \to$ **ssSet**, which sends a simplicial object X in categories to the bisimplicial set NX with $N_{n,m}X \coloneqq N_m(X(n))$. Note that NX is in fact a simplicial object in quasi-categories. If C is a simplicial category, taking the diagonal of NC yields a simplicial set $\mathcal{N}C$ with $\mathcal{N}_n C = N_n C_n$. Then $\mathcal{N}C$ is weakly homotopy equivalent to $N^{\mathrm{hc}}(C)$. The same remains true if one replaces the diagonal by the totalisation.

Since **sCat** is cocomplete, the general nerve-realisation machinery yields a left adjoint to N^{hc}; we call this the **rigidification** functor and denote it by $\mathfrak{C}$. We think of $\mathfrak{C}$ as 'rigidifying' a quasi-category by replacing it with a category in which composition is strictly associative. If P is a poset, then $\mathfrak{C}(N_\bullet P)$ is the simplicial category $\underline{\mathrm{Path}}(P)$; in general the rigidification is defined by taking colimits of such simplicial categories.

Remark 8.27. The adjunction $\mathfrak{C} \dashv N^{\mathrm{hc}}$ is a Quillen equivalence when **sSet** is given the Joyal model structure, so that quasi-categories and locally Kan simplicial categories have the same homotopy theory. In particular if C is locally Kan then $\mathfrak{C}N^{\mathrm{hc}}C \to C$ is a DK-equivalence, and if D is an ∞-category then $D \to N^{\mathrm{hc}}\mathfrak{C}D$ is a categorical equivalence.

9 DG categories

As another source of examples of ∞-categories, we will consider dg categories, i.e. categories enriched over chain complexes. We will construct a **dg nerve** for dg categories, which will allow us to view any dg category as an ∞-category. We will then describe the **Dold–Kan correspondence**, which gives a way of turning a dg category C into a simplicially enriched category $\tilde{C}$. We will compare the dg nerve of C with the homotopy coherent nerve of $\tilde{C}$.

9.1 Basic definitions

Fix a commutative ring k. A **dg category** is a $\mathrm{Ch}(k)$-enriched category; i.e. for each pair of objects x, y we have a chain complex $C(x, y)$, together with unit maps $k \to C(x,x)$ and composition maps $C(y,z) \otimes C(x,y) \to C(x,z)$ satisfying the usual unitality and associativity axioms.

A **dg functor** is a Ch(k)-enriched functor; i.e. a map F on objects and a collection of maps $F_{xy} : C(x,y) \to D(Fx, Fy)$ satisfying the usual compatibilities.

Remark 9.1. The forgetful functor $\mathrm{Ch}(k) \to \mathbf{Set}$ is lax monoidal, and hence every dg category has an underlying category.

Examples of dg categories include:

- Ch(k) itself, with the usual mapping complexes.
- More generally $\mathrm{Ch}(\mathcal{A})$ for $\mathcal{A}$ any k-linear additive category.
- dg categories with one object are the same thing as dg algebras, since both can be described as monoids in Ch(k).
- If A is a dg k-algebra, then the category $\mathbf{Mod} - A$ of left dg A-modules is a dg category.
- If C is a dg k-coalgebra, then the category $\mathbf{Comod} - C$ of left dg C-comodules is a dg category.
- Various kinds of derived categories: if A is a (dg) k-algebra then $D(A)$, $D^b(A)$, $D^{\pm}(A)$, $\mathbf{per}\,A$ are all dg categories.
- Various kinds of categories of sheaves on a ringed space: if X is a space and $\mathcal{O}_X$ a sheaf of k-algebras, then the category $\mathbf{Mod} - \mathcal{O}_X$ (as well as various associated derived categories) are dg categories.

9.2 The DG nerve

For a dg category C we will construct a simplicial set $N^{\mathrm{dg}}(C)$, its **dg nerve**. Before we begin we give some terminology. Let I be a nonempty subset of $[n]$. We write bI for the bottom (i.e. minimal with respect to $<$) element of I, and tI for the top (i.e. maximal with respect to $<$) element of I. If $i \in I$ then we let $i- \subseteq I$ denote the set of all $j \in I$ with $j \leq i$, and similarly $i+ := \{j \in I : j \geq i\}$. An n-simplex of $N^{\mathrm{dg}}(C)$ is then defined to be the following collection of data:

1. Objects $X_0, \ldots, X_n$ of C.
2. For each $I \subseteq [n]$ with $|I| \geq 2$, an element $f_I \in C(X_{bI}, X_{tI})_{|I|-2}$.

satisfying the condition that, for each such I, we have

$$df_I = \sum_{i \in I} (-1)^i \left(f_{i+} \circ f_{i-} - f_{I - \{i\}} \right).$$

Loosely, this condition says that for all paths p in $[n]$, the map assigned to p gives a homotopy between all possible ways of breaking up p into two subpaths, together with some higher coherencies.

One can equip the collection $N_n^{\mathrm{dg}}(C)$ with face and degeneracy maps making it into a simplicial set. We obtain a functor $N^{\mathrm{dg}} : \mathbf{dgCat} \to \mathbf{sSet}$ which we call the **dg nerve**.

Theorem 9.2. *The dg nerve of a dg category is an ∞-category.*

The proof is a rather technical combinatorial exercise, so we omit it.

Remark 9.3. There is a natural monomorphism $N(C) \to N_n^{\mathrm{dg}}(C)$: if $f_1, \dots, f_n$ is a string of n composable 0-cycles, then composing them in all possible ways yields a simplex of the dg nerve.

Remark 9.4. DG categories are generalised by A_∞-categories, which drop the strict associativity of composition in exchange for a system of coherent homotopies witnessing homotopy associativity. Faonte constructed a natural A_∞-nerve with respect to a certain cosimplicial A_∞-category, whose restriction to dg categories agrees with our dg nerve. Rivera and Zeinalian also constructed a dg nerve with respect to a certain cosimplicial dg category; however the cosimplicial dg category in question is rather complicated (it arises from a cocubical dg category).

Remark 9.5. Given a simplicial set X, there is a dg category ΩX defined as follows: the objects of ΩX are the vertices of X, and the mapping spaces are defined via the dg algebra $\Omega C_* X$, where Ω denotes the cobar construction. Up to equivalence, the dg nerve is the right adjoint to Ω.

9.3 Low-dimensional simplices of the DG nerve

Let C be a dg category.

- The set $N_0^{\mathrm{dg}}(C)$ is in bijection with the objects of C.
- The set $N_1^{\mathrm{dg}}(C)$ is in bijection with the 0-cycles of C; i.e. the triples (x, y, f) consisting of two objects x, y and a morphism $f \in C(x, y)_0$ with $df = 0$. The face and degeneracy maps are the expected ones.
- There is a bijection between N_2^{dg} and the set of triangles of degree 0 cycles together with a degree 1 morphism witnessing homotopy commutativity of the triangle. More accurately, giving a degree 2 simplex of the dg nerve is equivalent to giving a septuple $(X, Y, Z, f, g, h, \theta)$ such that $f \in C(X, Y)_0$, $g \in C(Y, Z)_0$, $h \in C(X, Z)_0$ such that $df = 0$, $dg = 0$, $dh = 0$ and with $\theta \in C(X, Z)_1$ satisfying $d\theta = gf - h$.

9.4 Homotopy categories

A dg category C has a **homotopy category** $H_0 C$ defined by having the same objects as C, and homsets defined by $(H_0 C)(x, y) = H_0(C(x, y))$. Composition is inherited from C. Note that $H_0(C)$ is actually a k-linear category, but we forget the extra k-linear structure on the hom-spaces. The following proposition can be proved by unwinding the definition of the left-hand side:

Proposition 9.6. *There is a natural equivalence $hN^{\mathrm{dg}}(C) \simeq H_0(C)$.*

9.5 Dold–Kan

Let $\mathcal{A}$ be any abelian category. Recall that $\mathrm{Ch}(\mathcal{A})_{\geq 0}$ denotes the category of **connective chain complexes** in $\mathcal{A}$, i.e. those chain complexes V_* with $V_i \cong 0$ for $i < 0$.

Proposition 9.7 (Dold–Kan correspondence)**.** *There exists an equivalence of categories* $\mathbf{s}\mathcal{A} \simeq \mathrm{Ch}(\mathcal{A})_{\geq 0}$.

We will restrict to the case where $\mathcal{A}$ is the category of modules over some commutative ring k, as this simplifies matters. One direction of the correspondence is constructed using the **Moore complex** of a simplicial k-module. Given a simplicial k-module M, let C_*M be the chain complex with $C_nM = M_n$, and with differential given by the alternating sum of the face maps. The simplicial identities imply that this is a chain complex.

Let D_*M be the subcomplex of C_*M generated by the degenerate simplices, and put $\Gamma_*M \coloneqq C_*M/D_*M$. Then Γ_*M is known as the **normalised Moore complex**.

Remark 9.8. Since D_*M is acyclic, the natural surjection $C_*M \twoheadrightarrow \Gamma_*M$ is a quasi-isomorphism.

Remark 9.9. If M is a semisimplicial set (i.e. a simplicial set without degeneracy maps), then one can still form its normalised Moore complex. Indeed there is a naturally defined subcomplex $\Gamma'_*M \hookrightarrow C_*M$, which in degree n is defined to be the intersection of $\ker d_i$ for $i = 1, 2, \dots, n$. The natural composition

$$\Gamma'_*M \hookrightarrow C_*M \twoheadrightarrow \Gamma_*M$$

is an isomorphism. This implies that the short exact sequence defining Γ_*M splits.

If X is a simplicial set, one can form the free simplicial k-module kX by taking free k-modules levelwise. Hence one can consider the chain complex $\Gamma_*(kX)$, known as the **complex of reduced chains on X with coefficients in k**. The homology of $\Gamma_*(kX)$ is known as the **reduced homology of X with coefficients in k**.

In particular, the standard cosimplicial simplicial set $\Delta^\bullet$ yields a cosimplicial chain complex $\Gamma_*(k\Delta^\bullet)$. This allows us to define a functor $K : \mathrm{Ch}(\mathcal{A})_{\geq 0} \to \mathbf{s}\mathcal{A}$ by setting $K(L) \coloneqq \mathrm{Hom}_k(\Gamma_*(k\Delta^\bullet), L)$.

Proposition 9.10 (Dold, Puppe, Kan)**.** *K and Γ_* are inverses.*

Remark 9.11. K and Γ_* are the enriched nerve and realisation with respect to $\Gamma_*(k\Delta^\bullet)$. Automatically it follows that K is right adjoint to Γ_*.

9.6 The Alexander–Whitney map

Our aim in this section will be to show that K is a lax monoidal functor. This is equivalent to proving that its inverse Γ_* is oplax monoidal. In other words, we want to exhibit natural maps $\Gamma_*(A \times B) \to \Gamma_*(A) \otimes_k \Gamma_*(B)$. How do we do this? The idea is that for each $0 \leq p \leq n$ we can break $[n]$ into two intervals $[p] + [n-p]$. This determines two maps $\alpha_p : [p] \to [n]$ which sends $i \mapsto i$ and $\beta_p : [n-p] \to [n]$ which sends $j \mapsto j+p$. The Alexander–Whitney map is then defined on n-dimensional generators by sending $a \times b$ to the sum $\sum_{p=0}^{p=n} \alpha_p^*(a) \otimes \beta_p^*(b)$. This gives a well-defined chain map which one can check restricts to the normalised Moore complex $\Gamma_*(A \times B)$.

Proposition 9.12. *The Alexander–Whitney map makes Γ_* into an oplax monoidal functor.*

Corollary 9.13. *The Dold–Kan functor K is a lax monoidal functor.*

In fact, one can write down the lax monoidal structure on K: if X and Y are two complexes then the natural map $K(X) \times K(Y) \to K(X \otimes Y)$ sends an n-simplex of the product $(\sigma : \Gamma_*(k\Delta^n) \to X, \quad \tau : \Gamma_*(k\Delta^n) \to Y)$ to the composition

$$\Gamma_*(k\Delta^n) \xrightarrow{\text{diag}} \Gamma_*(k\Delta^n \times k\Delta^n) \xrightarrow{\text{AW}} \Gamma_*(k\Delta^n) \otimes \Gamma_*(k\Delta^n) \xrightarrow{\sigma\otimes\tau} X \otimes Y.$$

Remark 9.14. Γ_* is not strong monoidal; in particular the Alexander–Whitney maps are not isomorphisms. They are however quasi-isomorphisms - this is the Künneth theorem.

Remark 9.15. The Alexander–Whitney map is what gives the singular chain complex of a topological space its coalgebra structure.

9.7 From dg to simplicial categories

Definition 9.16. Let $\tilde{K} : \mathrm{Ch}(k) \to \mathbf{sSet}$ be the functor defined by the following composition:

$$\mathrm{Ch}(k) \xrightarrow{\tau_{\geq 0}} \mathrm{Ch}(k)_{\geq 0} \xrightarrow{K} \mathbf{sMod}k \xrightarrow{U} \mathbf{sSet}$$

where $\tau_{\geq 0}$ is the truncation functor and U is the forgetful functor.

Lemma 9.17. *The functor $\tilde{K}$ is lax monoidal.*

Proof. It suffices to show that all of $\tau_{\geq 0}$, K, and U are lax monoidal. The Alexander–Whitney map shows that K is lax monoidal, and we leave the proof for the truncation and forgetful functors as an exercise. □

Hence, if C is a dg category, applying $\tilde{K}$ to the hom-complexes will yield a simplicial category which we denote $\tilde{C}$. Our final aim is to give an idea of the following theorem:

Theorem 9.18. *If C is a dg category then there is a natural equivalence of ∞-categories*

$$\mathcal{E}_C : N^{\mathrm{hc}}(\tilde{C}) \to N^{\mathrm{dg}}(C)$$

and in particular the two nerves are weakly homotopy equivalent.

Proof sketch. Recall that an n-simplex of the homotopy coherent nerve is a simplicial functor from the n-path category: we have

$$\mathrm{N}_n^{hc}(\tilde{C}) \coloneqq \mathbf{sCat}(\underline{\mathrm{Path}}[n], \tilde{C}).$$

Fix a subset $I = \{a < i_1 < i_2 < \cdots < i_m < b\} \subseteq [n]$ of size $m+2$, with $m \geq 0$. For each permutation $\sigma \in S_m$ we obtain an m-simplex $\tau_{I,\sigma} \in \underline{\mathrm{Path}}[n](a,b)_m$ as the chain

$$\{a < b\} \subseteq \{a < i_{\sigma 1} < b\} \subseteq \cdots \subseteq \{a < i_{\sigma 1} < \cdots < i_{\sigma m} < b\} = I$$

Given a simplicial functor $F : \underline{\mathrm{Path}}[n] \to \tilde{C}$, for every I and σ as above we hence obtain an element

$$F_{I,\sigma} \coloneqq F(\tau_{I,\sigma}) \in \tilde{K}(C(Fa, Fb))_m$$

which corresponds across Dold–Kan to an element $f_{I,\sigma} \in C(Fa, Fb)_m$. We can then put

$$f_I \coloneqq \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) f_{I,\sigma} \in C(Fa, Fb)_m$$

and the collection of all such f_I, as I varies, form an n-simplex of the dg nerve of C, with associated collection of objects $X_i = Fi$. Moreover, this assignment extends to a map of simplicial sets $\mathcal{E}_C : N^{\mathrm{hc}}(\tilde{C}) \to N^{\mathrm{dg}}(C)$ which is natural in C. It's easy to see that $\mathcal{E}_C$ is bijective on objects, so to prove that it is an equivalence one needs only to compare the mapping spaces in both categories. □

10 Vistas

We skim over some of the more advanced things one can do with ∞-categories.

10.1 Mapping spaces

If x, y are two objects of an ∞-category C, then one can construct a **mapping space** $\mathrm{Map}_C(x, y) \in \mathbf{sSet}$. This is constructed as the fibre of the natural map

$$\pi : \mathrm{Fun}(\Delta^1, C) \to C \times C$$

above the point (x, y). In fact, π is a Kan fibration, so that $\mathrm{Map}_C(x, y)$ is a Kan complex - one can regard it as a topological space. Proving this takes some significant work and a detailed analysis of Kan fibrations.

Remark 10.1. If C is a locally Kan simplicial category, then $\mathrm{Map}_{N^{\mathrm{dg}}(C)}(x, y)$ is naturally homotopy equivalent to $C(x, y)$.

If **hKan** denotes the homotopy category of Kan complexes, then there are well-defined composition maps

$$\mathrm{Map}_C(y,z) \times \mathrm{Map}_C(x,y) \to \mathrm{Map}_C(x,z)$$

in **hKan** that are associative and unital in **hKan**. This makes precise the intuition that ∞-categories should be 'enriched in topological spaces', although we caution that this is not a literal enrichment.

Mapping complexes are functorial up to homotopy: if $F : C \to D$ is a functor then there are well-defined maps $F_{xy} : \mathrm{Map}_C(x,y) \to \mathrm{Map}_D(Fx,Fy)$ in **hKan** that interact appropriately with composition. Say that F is **fully faithful** if each F_{xy} is a homotopy equivalence (i.e. an isomorphism in **hKan**). Say that F is **essentially surjective** if the 1-functor hF is essentially surjective.

Theorem 10.2. *Let $F : C \to D$ be a functor of ∞-categories. Then the following are equivalent:*

- *F is an equivalence, i.e. there exists a $G : D \to C$ such that $FG \simeq \mathrm{id}$ and $GF \simeq \mathrm{id}$ as objects of the appropriate functor categories.*
- *F is fully faithful and essentially surjective.*

10.2 Localisation

Let C be a 1-category equipped with a class of weak equivalences W. We assume that W contains all isomorphisms and satisfies the two-out-of-three property (for some applications stronger properties are required, e.g. two-out-of-six). Then one can localise at W to obtain a simplicial category $L_W C$. We will describe the **hammock localisation**, which is one such construction originally due to Dwyer and Kan.

A W-**zigzag** from x to y is a diagram of the form

$$x - a_1 - a_2 - a_3 - \cdots - a_m - y$$

where the adjacent morphisms go in opposite directions and all left pointing morphisms are in W. A **hammock** from x to y is a morphism between identically oriented W-zigzags from x to y, where all vertical components are in W. An n**-hammock** is then n hammocks pasted together along compatible zigzags, with all hammock morphisms running in the same direction. The n-simplices of $L_W C(x,y)$ are then given by the n-hammocks, and the face and degeneracy maps are composition and insertion of identities in the hammock direction.

Theorem 10.3. *There is a natural equivalence $h(L_W C) \simeq C[W^{-1}]$.*

We hence view a category with weak equivalences as a *presentation* of an ∞-category.

Remark 10.4. In particular, a model category is a certain kind of category with weak equivalences. However, there are easier ways of passing from a model category to its associated ∞-category. If C is a simplicial model category, then the category C_{cf} of bifibrant objects in C is a locally Kan simplicial category, and its homotopy coherent nerve is equivalent to $L_W C$. More generally, a theorem of Dugger says that every combinatorial model category is Quillen equivalent to a simplicial model category (in fact a localisation of a category of simplicial presheaves). In fact, cominatorial model categories correspond precisely to the **presentable** (cocomplete and accessible) ∞-categories - a combinatorial model category presents a presentable ∞-category, and every presentable ∞-category has a presentation by a combinatorial model category.

10.3 Stabilisation and spectra

Informally, a **spectrum** is a sequence of pointed topological spaces X_n together with structure maps $\Sigma X_n \to X_{n+1}$. These are supposed to model the **stable** homotopy theory of topological spaces, i.e. the homotopical properties that are stable under repeated applications of the suspension functor.

Here is one easy construction of spectra in the ∞-categorical setting. Let C be an ∞-category with finite limits and a zero object. Then one defines the **loop space** of an object X as the ∞-categorical pullback of the diagram $0 \to X \leftarrow 0$.

Example 10.5. If C denotes the ∞-category of pointed topological spaces, then ΩX is weakly homotopy equivalent to the based loop space of X.

Say that C is **stable** if Ω is an autoequivalence. In particular, C then necessarily has finite colimits and the inverse of Ω is given by the **suspension** functor Σ, given on an object X by the pushout of $0 \leftarrow X \to 0$.

Loosely, a stable ∞-category is supposed to behave like an ∞-categorical version of a triangulated or abelian category:

Example 10.6. If A is a ring, then the derived category $D(A)$, viewed as an ∞-category, is stable: suspension is given by $[1]$ and loops by $[-1]$.

An ∞-category C with a zero object and finite limits has a **stabilisation** $\mathbf{Stab}C$, which can be defined as the inverse limit of the tower

$$\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} C$$

The stabilisation of $\mathbf{sSet}_*$ is then known as the ∞-category of spectra, denoted by $\mathbf{Sp} \coloneqq \mathbf{Stab}(\mathbf{sSet}_*)$. This is then a monoidal ∞-category, with monoidal unit given by $\mathbb{S}$, the sphere spectrum (whose homotopy groups are the stable homotopy groups of spheres). Moreover $\mathbf{Sp}$ is the universal stable ∞-category: any stable ∞-category is canonically enriched up to homotopy in $\mathbf{Sp}$, just like any ∞-category is canonically enriched up to homotopy in $\mathbf{sSet}$.

Since any chain complex V can be recovered from its truncations $\tau_{\geq -n}V$ as $n \to \infty$, it follows that for a ring A we have an equivalence of ∞-categories

$$\mathrm{Ch}(A) \simeq \mathbf{Stab}\left(\mathrm{Ch}(A)_{\geq 0}\right)$$

and the Dold–Kan correspondence then yields an equivalence

$$\mathrm{Ch}(A) \simeq \mathbf{Stab}\left(\mathbf{sMod} - A\right).$$

Applying the forgetful functor from simplicial A-modules to simplicial sets then yields a functor $\mathrm{Ch}(A) \to \mathbf{Sp}$; the objects in the image of this functor are known as **Eilenberg–Mac Lane spectra**. Moreover, this 'spectral Dold–Kan correspondence' is appropriately monoidal, and hence allows one to obtain spectrally enriched categories from dg categories.

10.4 Segal models for (∞, n)-categories

We've already seen two 'models' for ∞-categories - namely, quasicategories (weak Kan complexes) and locally Kan simplicial categories. These are 'equivalent', in the sense that both are the bifibrant objects of two Quillen equivalent model structures (the **Joyal model structure** on **sSet** and the **Bergner model structure** on **sCat**, respectively).

One other useful model to use is that of **complete Segal spaces**. In particular, complete Segal spaces are useful since they generalise readily to give models for (∞, n)-categories. A **Segal space** is a bisimplicial set X (viewed as a simplicial object in simplicial sets) satisfying the conditions

- Each X_n is a Kan complex.
- The **Segal maps** $X_n \to X_1 \times \cdots \times X_n$ are all homotopy equivalences.

One should think of X_0 as the space of objects, X_1 as the space of morphisms, X_2 as the space of compositions, et cetera. A Segal space has a **homotopy category** hX defined analogously. A morphism in X_1 is said to be a **homotopy equivalence** if is becomes an isomorphism in hX. A Segal space is then said to be **complete** if $s_0 : X_0 \to X_{\mathrm{hoeq}}$ is a homotopy equivalence: in other words, the degenerate 1-morphisms should (up to homotopy!) be the homotopy equivalences. There's a completion functor that produces a complete Segal space from a general one.

Complete Segal spaces are then models for ∞-categories, in the model-categorical sense above.

One can similarly define an n**-fold Segal space** inductively as a certain kind of simplicial object in $(n-1)$-fold Segal spaces, and completeness in a similar manner. These are models for (∞, n)-categories, and often appear in particular in treatments of the Cobordism Hypothesis.

Remark 10.7. Rezk defined the notion of Θ_n-spaces, which also model (∞, n)-categories. Very loosely a Θ_n-space is a certain sort of presheaf of simplicial sets on a certain small category Θ_n of combinatorial shapes, just like an n-fold complete Segal space is a certain sort of presheaf of simplicial sets on $(\Delta^{\mathrm{op}})^n$.

A Nerve and realisation

We'll meet nerves in generality several times. My reference here was the nLab page of the same name.

A.1 Nerve

Here's a simple observation:

Proposition A.1. *Let $X : S \to C$ be a functor. There is an associated functor $N_X : C \to \mathrm{Fun}(S^{\mathrm{op}}, \mathbf{Set})$ which sends c to the functor*

$$s \mapsto \mathrm{Hom}_C(X(s), c)$$

We call N_X the **nerve** associated to the functor X.

Example A.2. Take S to be the simplex category Δ. Then X is a cosimplicial object in C, and the nerve functor has codomain **sSet**.

The slogan is: 'cosimplicial objects in C give nerve functors $C \to$ **sSet**'. Let's see some examples of this.

Example A.3. The identity functor on Δ can be viewed as a cosimplicial object in Δ, with n-cosimplices given by the object $[n]$. The corresponding nerve functor N_{id} sends $[m]$ to the simplicial set Δ^m, the standard m-simplex.

Example A.4. The previous example gives a functor $\Delta \to$ **sSet**, which we can think of as a cosimplicial simplicial set $\Delta^\bullet$: the n-cosimplices are the simplicial set Δ^n. This is known as the **standard cosimplicial simplicial set**. The corresponding nerve functor $N_{\Delta^\bullet}$ is the identity functor on **sSet**, since for any simplicial set K, the set of n-simplices K_n is naturally in bijection with $\mathrm{Hom}(\Delta^n, K)$.

Example A.5. Let $\underline{\Delta}^\bullet : \Delta \to \mathbf{Top}$ denote the **standard cosimplicial space**; the n-cosimplices are the standard n-simplex $\underline{\Delta}^n \subseteq \mathbb{R}^{n+1}$. The corresponding nerve is the **singular simplicial set** functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$.

Example A.6. Recall that Δ is defined to be the category with elements the posets $[n]$ and morphisms the monotone maps. But a poset can be viewed as a category, and a monotone map between posets is then the same thing as a functor. Hence we obtain a functor $\tilde{\Delta} : \Delta \to \mathbf{Cat}$ which sends the poset $[n]$ to the category $[n]$. We view $\tilde{\Delta}$ as a cosimplicial category and refer to it as the **standard cosimplicial category**. The corresponding nerve functor is the usual nerve $N : \mathbf{Cat} \to \mathbf{sSet}$.

Example A.7. A 2-category has a **Duskin nerve**, which is the nerve with respect to $\tilde{\Delta}$, viewed as a cosimplicial 2-category.

Example A.8. A simplicially enriched category has a **homotopy coherent nerve**, which is the nerve with respect to a certain cosimplicial simplicially enriched category, the **path category** $\underline{\mathrm{Path}}[\bullet]$. This functor gives the comparison between simplicially enriched categories and quasicategories. An equivalent construction views a simplicially enriched category C as a simplicial object in categories (whose simplicial set of objects is constant), applies the nerve levelwise to obtain a bisimplicial set, and then totalises.

Example A.9. A dg category has a **dg nerve**, which gives the comparison between dg categories and quasicategories. It is possible to view this in the above framework although the specifics are tricky: Faonte has a nice construction using a cosimplicial A_∞-category. There are several other ways to convert dg categories to quasicategories: given a dg category C one can truncate the hom-complexes and then apply Dold–Kan to them to end up with a category enriched in simplicial vector spaces. One then forgets the linear structure to obtain a simplicially enriched category, and one can then convert this into a quasicategory as above.

A.2 Realisation

Let $X : S \to C$ be a functor, so that we can define its associated nerve N_X.

Theorem A.10. *If C is cocomplete, the nerve functor N_X has a left adjoint, the* ***realisation*** *functor $|-|_X$.*

Proof. We'll sketch a proof in the case $S = \Delta$; the general case is no harder. First define $|\Delta^n|_X \coloneqq X^n$. Since every simplicial set is generated by the standard n-simplices under colimits, we simply extend the above formula by colimits to define it for all simplicial sets K. In other words, for $K \cong \operatorname{colim}_{\mathrm{el}(K)} \Delta^n$ we put $|K|_X \coloneqq \operatorname{colim}_{\mathrm{el}(K)} X^n$.

We then have

$$\begin{aligned}\mathrm{Hom}(|K|_X, c) &\simeq \mathrm{Hom}(|\operatorname{colim} \Delta^n|_X, c)\\ &\simeq \mathrm{Hom}(\operatorname{colim} X^n, c)\\ &\simeq \lim \mathrm{Hom}(X^n, c)\\ &\simeq \lim N_X(c)_n\\ &\simeq \lim \mathrm{Hom}(\Delta^n, N_X(c))\\ &\simeq \mathrm{Hom}(\operatorname{colim} \Delta^n, N_X(c))\\ &\simeq \mathrm{Hom}(K, N_X(c)).\end{aligned}$$

as required. □

Remark. More formally, the realisation functor is constructed as a left Kan extension or as a coend $|K|_X \coloneqq \int^n X^n \cdot K_n$; this formalises the phrase "extend via colimits".

Example A.11. Realisation with respect to the standard cosimplicial simplicial set $\Delta^\bullet$ is the identity functor (exercise: convince yourself of this directly from the definition!).

Example A.12. Realisation with respect to the standard cosimplicial space $\underline{\Delta}^\bullet$ is the geometric realisation functor.

Example A.13. Realisation with respect to the standard cosimplicial category $\tilde{\Delta}^\bullet$ is the homotopy category of a simplicial set.

A.3 Enriched nerves

One can also carry out the nerve construction in the presence of an enrichment. Let V be a closed symmetric monoidal category and let C be a V-category. Given a cosimplicial object $X \in C$, one obtains a V-enriched nerve $N_X : C \to \mathbf{s}V$ via the usual formula. If C has enough V-colimits then N_X admits a left adjoint, which realises a simplicial V-object as an object of C.

Example A.14. If $V = \mathbf{Set}$ we recover the usual nerve-realisation construction.

Example A.15. If A is a commutative ring, take V to be the category of A-modules. Then the category $\mathrm{Ch}(A)$ is V-enriched, and has enough V-colimits. Applying the functor of normalised chains to the standard cosimplicial simplicial set $\Delta \to \mathbf{sSet}$ yields a cosimplicial chain complex X which at level n is the complex of normalised chains on Δ^n. The V-enriched nerve with respect to X then yields a functor $K : \mathrm{Ch}(A) \to \mathbf{sMod} - A$ which is the K appearing in the Dold-Kan correspondence. The inverse of K is then the realisation functor $| - |_X$.