# $\infty$-categories intro talk

## Heuristic idea behind higher categories

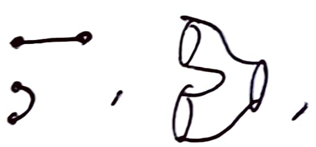
a higher category should have

- objects (= 0-morphisms) $\bullet$
- morphisms (= 1-morphisms) $\bullet \to \bullet$
- 2-morphisms 
- 3-morphisms 

etc.

## Example to keep in mind

a cobordism between two closed $n$-manifolds $M, N$ is a ~~man~~ $n+1$-mfld $X$ with $\partial X = M \sqcup N$.

eg.  , $\emptyset \to \emptyset$ .

Should have some sort of higher cobordism category with

objects 0-dim mflds

morphisms cobordisms (= 1-dim mflds with boundaries)

2-morphisms cobordisms between cobordisms

$\vdots$

$n$-morphisms $n$-dim$^{l}$ cobordisms with corners

## First attempt

a strict 1-category is a category.

A strict $n$-category is a category enriched in strict $(n-1)$-categories.

THIS DEFINITION IS BAD! ($n \leq 2$ is ok)

~~the~~ e.g. composition of cobordisms is __not__ strictly associative, so cobordisms __don't__ form a strict $n$-cat ($n > 1$)

__Another (non-) example: the path category__

$X$ a topological space. Form a category $pX$ with ← aka the fundamental groupoid.

- objects: points of $X$
- morphisms: $\mathrm{Hom}(x,y) = \{\text{paths } x \to y \text{ in } X\}$

once again, composition is __not__ strictly associative.

Usual fix: look at paths __up to homotopy__

$pX$ then becomes a groupoid, & $\mathrm{End}_{pX}(x) \simeq \pi_1(X, x)$.

General slogan: don't quotient out by data but __remember it__ ~~the~~ (eg. a $G$-set $X$ is better than the set $X/G$)

~~So~~ we should have a higher category $\pi_{\leq \infty} X$ with

- objects: points of $X$
- morphisms: paths btwn points
- 2-morphisms: homotopies btwn paths
- 3-morphisms: homotopies btwn homotopies

⋮

__GOAL OF THIS SEMINAR__

- Axiomatise __$(\infty, 1)$-categories__: these are higher categories with $k$-morphisms for each $k \in \mathbb{N}$, st. every $r$-morphism is invertible for $r > 1$
- See how they relate to some other notions (esp. dg categories)

# Homotopy hypothesis

whatever an $\infty$-category is,
we should be able to extract from
a topological space an $\infty$-category $\Pi_{\leq\infty} X$.
In fact $\Pi_{\leq\infty} X$ should be an <u>$\infty$-groupoid</u>
i.e. all $r$-morphisms are invertible for $r \geq 1$.
Grothendieck's <u>homotopy hypothesis</u> (in Pursuing Stacks, 1983)
says that $\Pi_{\leq\infty}$ should be an equivalence
between topological spaces & $\infty$-groupoids
(in an appropriate sense!)
So $\infty$-categories should be objects that
simultaneously generalise categories & topological
spaces.

<u>DIFFERENT MODELS ARE AVAILABLE!</u>

most naive def: an $\infty$-cat. is a category enriched
in Top. Just about can be made to work but
is tricky - topological spaces are tricky analytic
objects whereas we'd like something more discrete/combinatorial.

<u>Simplicial sets</u> are combinatorial models of spaces.
vague definition: a sSet has a set of $n$-simplices for
each $n$, linked by a collection of <u>face</u>
& <u>degeneracy</u> maps.

Think: 


a topological space has a <u>singular simplicial set</u> which at level $n$ is the set $\mathrm{Hom}(\Delta^n, X)$ of maps from the standard $n$-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ to $X$. (*)

(Closely related to the singular chain complex that computes homology.)

a category has a <u>nerve</u> which is a sSet which at level $n$ is the strings of $n$ composable morphisms.

A <u>quasicategory</u> (synonymous for us with $\infty$-category) will be a certain kind of simplicial set. This will generalise both spaces & categories.

<u>Other models</u>

- Simplicially enriched categories (Bergner)
- (Complete) Segal spaces (Rezk) ← Good for $(\infty,n)$-cats.
- Model categories $\Big\}$ encode certain kinds of $\infty$-cats.
- DG categories
- categories w/ weak equivalences

$\vdots$

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(*): $\Delta^n = \left\{ x \in \mathbb{R}^{n+1} : \underset{x_i \geq 0}{x_0 + \cdots + x_n = 1} \right\}$