An Introduction to Morita Theory

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My main reference for these notes was Chapter II of Bass's book *Algebraic K-Theory* (1968); you can find a more detailed exposition there.

1 Motivation

When we're doing representation theory we want to study the structure of the category **Mod-**A for some ring A. So we want to know if and when two different rings A and B give us the same category. With this in mind, two rings are said to be Morita equivalent when their module categories are equivalent. In many cases, we often only care about rings up to Morita equivalence. If this is the case, then given a ring A, we'd like to find some particularly nice representative of the Morita equivalence class of A.

2 Morita Equivalence

First some notation: Let R be a ring. Write **Mod-**R for the catgeory of right R-modules and R-module homomorphisms. Write **mod-**R for the (full) subcategory of finitely generated R-modules. Write **Proj-**R for the subcategory of projective modules, and **proj-**R for the subcategory of finitely generated projective modules.

Two rings R, S are defined to be **Morita equivalent** if the categories **Mod-**R and **Mod-**S are equivalent. In fact, any such equivalence will be additive: this is a general fact about equivalences between abelian categories, since the sum of two morphisms $f, g : X \to Y$ can be recovered as the composition

$$f + g : \ X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y$$

of $f \oplus g$ with the diagonal and fold maps. Note that any property defined categorically will be preserved under equivalence: for example if
$F : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$ is an equivalence, then F will take projective modules to projective modules and hence induce an equivalence $\mathbf{Proj}\text{-}R \longrightarrow \mathbf{Proj}\text{-}S$.

Moreover, an equivalence F as above wil induce an equivalence between **proj-**R and **proj-**S, since the finitely generated projective modules are precisely those projective modules P for which the functor $\mathrm{Hom}_R(P, -)$ distributes over direct sums.

Example If R is a division ring, then R is Morita equivalent to all of its matrix rings $M_n(R)$. We'll see a proof of a special case of this later on, using quivers.

Remark One can show that $Z(R) \cong \mathrm{End}_{[\mathbf{Mod}\text{-}R, \mathbf{Mod}\text{-}R]}(\mathrm{id})$, the endomorphism ring of the identity functor of **Mod-**R (sometimes called the centre of **Mod-**R): first note that id is just $\mathrm{Hom}_A(A, -)$ where A is regarded as an $A-A$-bimodule. So by an appropriate version of the Yoneda lemma (we need to be careful about A-linearity), we get that $\mathrm{End}_{[\mathbf{Mod}\text{-}R, \mathbf{Mod}\text{-}R]}(\mathrm{id}) \cong \mathrm{End}_{A^e}(A, A) \cong Z(A)$, where $A^e := A \otimes A^{\mathrm{op}}$ is the enveloping algebra. So if R and S are Morita equivalent, then $Z(R) \cong Z(S)$. In particular two commutative rings are Morita equivalent if and only if they are isomorphic. So Morita equivalence is only interesting for noncommutative rings!

A **generator** for a category C is an object G such that for any two parallel morphisms $f, g : X \to Y$ with $f \neq g$, then there is some morphism $h : G \to X$ such that $fh \neq gh$. Note that generators are preserved under equivalence. If C is **Mod-**R, then a generator for C is the same thing as a module G such that every R-module is a quotient of a (possibly infinite) direct sum of copies of G.

A **progenerator** in **Mod-**R is a finitely generated projective generator. Progenerators always exist: as an easy example, R is a progenerator for **Mod-**R. We care about progenerators because of the following result:

Theorem (Morita)**.** *Two rings R and S are Morita equivalent if and only if there exists a progenerator P of* **Mod-**R *such that $S \cong \mathrm{End}_R(P)$.*

Note that we're considering R as a right R-module. In particular, $R \cong \mathrm{End}_R(R)$.

We can easily prove one direction of this theorem now: If R and S are Morita equivalent, then take an equivalence $F : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$. Since R is a progenerator for **Mod-**R, $F(R)$ is a progenerator for **Mod-**S. We have isomorphisms $R \cong \mathrm{End}_R(R) \cong \mathrm{End}_S(F(R))$.

The converse is harder to prove: we need to consider an equivalent characterisation of Morita equivalence in terms of tensor products of bimodules.

3 Bimodules

An $R - S$ **bimodule** is an abelian group M which is both a left R-module and a right S-module, such that the actions are compatible: $(rm)s = r(ms)$. If M is an $R - S$ bimodule and N is an $S - T$ bimodule then it makes sense to form the tensor product and get an $R - T$ bimodule $M \otimes_S N$.

Theorem.
Two rings R and S are Morita equivalent if and only if there exists an $S - R$ bimodule P and an $R - S$ bimodule Q such that $P \otimes_R Q \cong S$ and $Q \otimes_S P \cong R$.

Note that these are isomorphisms of $S - S$ bimodules and $R - R$ bimodules respectively.

Proof. To prove the 'if' direction, suppose we have P and Q such that $P \otimes_R Q \cong S$ and $Q \otimes_S P \cong R$. Then setting $F := - \otimes_R Q$ and $G := - \otimes_S P$, we have $GF = (- \otimes_R Q) \otimes_S P \cong - \otimes_R (Q \otimes_S P) \cong - \otimes_R R \cong \mathrm{id}_R$ and similarly $FG \cong \mathrm{id}_S$.

To prove the 'only if' direction we need a version of the Eilenberg-Watts Theorem, which tells us that if $F : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Mod}\text{-}S : G$ is an equivalence, then there exists an $R - S$ bimodule Q such that $F \cong - \otimes_R Q$ (and furthermore, Q is a progenerator for $\mathbf{Mod}\text{-}S$). So if we have an equivalence as above, applying Eilenberg-Watts twice we get bimodules P and Q such that $F \cong - \otimes_R Q$ and $G \cong - \otimes_S P$. Using $\mathrm{id}_R \cong GF$ and $\mathrm{id}_S \cong FG$ it's not hard to check that $P \otimes_R Q \cong S$ and $Q \otimes_S P \cong R$. □

4 Morita's Theorem

Armed with this new characterisation of Morita equivalence we can prove the other half of Morita's Theorem. So suppose we have a progenerator P of $\mathbf{Mod}\text{-}R$ with $S \cong \mathrm{End}_R(P)$. The left action of $\mathrm{End}_R(P)$ on P gives us a left action of S on P making P into an $S - R$ bimodule.

Set $Q := \mathrm{Hom}_R(P, R)$. Q has a right action by $S \cong \mathrm{End}_R(P)$ where we precompose a morphism $P \to R$ with an endomorphism of P, and a left action by $R \cong \mathrm{End}_R(R)$ where we compose with an endomorphism of R. This turns Q into an $R - S$ bimodule.

If we can prove that $P \otimes_R Q \cong \mathrm{End}_R(P)$ and $Q \otimes_S P \cong R$ then we're done. Define maps $\phi : Q \otimes_S P \to R$ and $\psi : P \otimes_R Q \to \mathrm{End}_R(P)$ by $\phi(f \otimes p) = f(p)$ and $\psi(p \otimes f) = pf$. The map ϕ is onto since P is a generator. The map ψ is onto since P is a direct summand of R^n and so any endomorphism of P is a sum of endomorphisms factoring through R. Note that we have identities $\psi(x \otimes f)(y) = x\phi(f \otimes y)$ and $g \circ \psi(x \otimes f) = \phi(g \otimes x)f$, for $x, y \in P$ and $f, g \in Q$.

We just need to prove that ϕ and ψ are injective maps. We'll only show that ψ is injective since the argument for ϕ is similar. Since ψ is surjective, first find an element $\sum_i x_i \otimes f_i$ with $\psi(\sum_i x_i \otimes f_i) = 1$. Let $\sum_i y_i \otimes g_i$ be any element of $P \otimes_R Q$. Then:

$\sum_i y_i \otimes g_i$

$= \sum_{i,j} (y_i \otimes g_i)\psi(x_j \otimes f_j)$ (multiplying by 1 and using linearity of ψ)

$= \sum_{i,j} y_i \otimes \phi(g_i \otimes x_j) f_j$ (using the second identity)

$= \sum_{i,j} y_i \phi(g_i \otimes x_j) \otimes f_j$ (pulling an element of R through the tensor product)

$= \sum_{i,j} \psi(y_i \otimes g_i)(x_j \otimes f_j)$ (using the first identity)

So if $\psi(\sum_i y_i \otimes g_i) = 0$ then $\sum_i y_i \otimes g_i = 0$, and so ψ is injective.

5 Induced equivalences between subcategories

We know that an equivalence $F : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$ induces equivalences between the subcategories $\mathbf{Proj}\text{-}R$ and $\mathbf{Proj}\text{-}S$, as well as an equivalence between $\mathbf{proj}\text{-}R$ and $\mathbf{proj}\text{-}S$. What about $\mathbf{mod}\text{-}R$ and $\mathbf{mod}\text{-}S$? It turns out that we have the following very nice theorem:

Theorem (Morita)**.** *$\mathbf{Mod}\text{-}R$ is equivalent to $\mathbf{Mod}\text{-}S$ if and only if $\mathbf{mod}\text{-}R$ is equivalent to $\mathbf{mod}\text{-}S$.*

Proof. If $F : \mathbf{Mod}\text{-}R \longrightarrow \mathbf{Mod}\text{-}S$ is an equivalence, then by the Eilenberg-Watts theorem $F \cong - \otimes_R P$ for some $R - S$ bimodule P that's a progenerator for $\mathbf{Mod}\text{-}S$. Since P is a finitely generated $S-$module, F restricts to a functor $\mathbf{mod}\text{-}R \longrightarrow \mathbf{mod}\text{-}S$, and furthermore this is an equivalence since the inverse of F restricts in the same way.

Conversely if $\mathbf{mod}\text{-}R$ and $\mathbf{mod}\text{-}S$ are equivalent, then we can run the proof at the end of Section 2 again to obtain a progenerator P of $\mathbf{mod}\text{-}R$ such that $S \cong \mathrm{End}_R(P)$. But a progenerator for $\mathbf{mod}\text{-}R$ is a progenerator for $\mathbf{Mod}\text{-}R$, and hence, applying Morita's Theorem, R and S are Morita equivalent. $\square$

6 Basic algebras

We are interested in finite dimensional algebras over a field. Therefore for any such algebra A we want to find a 'nice' algebra A' Morita equivalent to A. Then we can prove results about **Mod-**A' and get results about **Mod-**A. Here we describe how to build one such A', the basic algebra of A.

Let A be a finite dimensional algebra over a field. Then A admits a decomposition $A \cong \bigoplus_{i=0}^{n} e_i A$ as right A-modules, where the e_i are a complete set of primitive orthogonal idempotents. We call A **basic** if for every $i \neq j$, the two A-modules $e_i A$ and $e_j A$ are not isomorphic.

In general, A need not be basic (see the example below). But there is an easy way to construct a basic algebra from A. Suppose we have a decomposition of A as above. Choose a subset $\{e_{a_1}, \ldots, e_{a_m}\}$ of $\{e_1, \ldots, e_n\}$ maximal with respect to the condition that $e_{a_i} A \ncong e_{a_j} A$ whenever $i \neq j$. In particular every $e_i A$ is isomorphic to some $e_{a_j} A$.

Set $e := e_{a_1} + \cdots + e_{a_m}$, and put $A^b := eAe$. Then A^b is a basic algebra. Up to isomorphism, A^b will not depend on the choice of $\{e_{a_1}, \ldots, e_{a_m}\}$. Call A^b the **basic algebra associated to** A. Then A and A^b are Morita equivalent: one can show that eA is a progenerator for **Mod-**A, and its endomorphism ring is precisely A^b.

Example Set $A = M_n(k)$, and let e_i be the matrix with a 1 in position (i, i) and zeroes elsewhere. Then the e_i are a complete set of primitive orthogonal idempotents, and for all i and j, $e_i A \cong k^n \cong e_j A$. So $A^b \cong e_1 A e_1$ is a copy of k. This provides an alternate proof that $M_n(k)$ is Morita equivalent to k.