Noncommutative Hodge Theory
Lecture 2: Hochschild Homology

Lecture by Matt Booth
Notes by Brent Pym

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Abstract

We introduce one of the basic ingredients of noncommutative Hodge theory: Hochschild homology of algebras and categories.

1 Geometric motivation

Differential forms on a variety X and their sheaf cohomology groups

$$\mathsf{H}^{p,q}(\mathsf{X}) = \mathsf{H}^q(\mathsf{X}, \Omega^p_{\mathsf{X}})$$

play an essential role in defining the Hodge structure on the cohomology of X. Thus the first step towards defining noncommutative Hodge structures is to find a good replacement for the groups $\mathsf{H}^{p,q}(\mathsf{X})$ when X is a noncommutative space.

In this lecture, we introduce an important invariant of algebras and categories: the Hochschild homology $\mathsf{HH}_\bullet(-)$. Standard references for this material include [3, 5]. In the next lecture we will see that, when applied to the algebra of functions (or category of sheaves) on a smooth variety X, the Hochschild homology recovers the spaces $\mathsf{H}^{p,q}(\mathsf{X})$.

2 Hochschild homology for algebras

Let $\mathbb{K}$ be a field. Throughout these notes, all algebras are unital $\mathbb{K}$-algebras, i.e. they contain a multiplicative identity 1. The tensor product symbol $\otimes$ denotes the tensor product over $\mathbb{K}$ unless otherwise specified.

Definition 2.1. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra. The ***Hochschild complex of*** A is

$$\mathsf{C}_\bullet(\mathcal{A}) = \left(\cdots \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \right)$$

where $\mathcal{A}$ sits in degree zero, and the differential is given by

$$\mathrm{b}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}$$

for all $a_0, \dots, a_n \in \mathcal{A}$. The ***Hochschild homology of*** $\mathcal{A}$ is the homology of this complex: $\mathsf{HH}_\bullet(\mathcal{A}) = \mathsf{H}_\bullet(\mathsf{C}_\bullet(\mathcal{A}), \mathrm{b})$.

Let us compute the differential d in the simplest case. Given $a_0, a_1 \in \mathcal{A}$ we have that

$$\mathrm{b}(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0 = [a_0, a_1],$$

the commutator of a_0 and a_1. Thus the zeroth Hochschild homology has a simple interpretation, as the ***cocentre***:

$$\mathsf{HH}_0(\mathcal{A}) = \frac{\mathsf{C}_0(\mathcal{A})}{\mathrm{d}(\mathsf{C}_1(\mathcal{A}))} = \frac{\mathcal{A}}{\mathrm{d}(\mathcal{A} \otimes \mathcal{A})} = \frac{\mathcal{A}}{[\mathcal{A}, \mathcal{A}]}$$

where $[\mathcal{A}, \mathcal{A}] \subset \mathcal{A}$ is the subspace spanned by all commutators of $\mathcal{A}$.

In general, the homology of the Hochschild complex is quite difficult to compute directly. For this reason, it is helpful to give an interpretation of $\mathsf{HH}_\bullet(\mathcal{A})$ as a derived functor, which often allows us to compute it using a simpler complex.

To this end, we recall that the ***enveloping algebra of*** $\mathcal{A}$ is the algebra

$$\mathcal{A}^{\mathsf{e}} := \mathcal{A} \otimes \mathcal{A}^{\mathsf{op}},$$

where $\mathcal{A}^{\mathsf{op}}$ denotes the opposite algebra of $\mathcal{A}$. By construction, a left (or right) $\mathcal{A}^{\mathsf{e}}$-module is the same thing as an $\mathcal{A}$-bimodule. In particular, $\mathcal{A}$ is both a left and a right $\mathcal{A}^{\mathsf{e}}$-module, so that we may define the tensor product $\mathcal{A} \otimes_{\mathcal{A}^{\mathsf{e}}} \mathcal{A}$, and its derived version, the Tor groups $\mathsf{Tor}_\bullet^{\mathcal{A}^{\mathsf{e}}}(\mathcal{A}, \mathcal{A})$.

Proposition 2.2. *We have a canonical isomorphism*

$$\mathsf{HH}_\bullet(\mathcal{A}) \cong \mathsf{Tor}_\bullet^{\mathcal{A}^{\mathsf{e}}}(\mathcal{A}, \mathcal{A}).$$

Sketch of proof. (See, e.g. [3, Section 1.1] for more details.) To compute the Tor group, we find a resolution of $\mathcal{A}$ by free $\mathcal{A}^{\mathsf{e}}$-modules. This resolution is given by the bar complex

$$\mathsf{Bar}_\bullet(\mathcal{A}) = \left(\cdots \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}'} \mathcal{A} \otimes \mathcal{A} \right)$$

where the differential is given by

$$\mathrm{b}'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

Note that this is similar to the Hochschild differential, but with less terms. If we view $\mathcal{A}^{\otimes n}$ as an $\mathcal{A}$-bimodule by multiplication on the leftmost and rightmost tensor factors, then $\mathsf{Bar}_\bullet(\mathcal{A})$ becomes a complex of free $\mathcal{A}^{\mathsf{e}}$-modules.

Since $\mathcal{A}$ is unital, multiplication gives a natural surjective $\mathcal{A}$-bimodule map

$$\mathsf{Bar}_0(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} \twoheadrightarrow \mathcal{A}$$

and one can show that this induces a quasi-isomorphism $\mathsf{Bar}_\bullet(\mathcal{A}) \cong \mathcal{A}$ of complexes of $\mathcal{A}$-bimodules. Hence we may compute $\mathsf{Tor}_\bullet^{\mathcal{A}^\mathsf{e}}(\mathcal{A},\mathcal{A})$ as the homology of the complex $\mathsf{Bar}_\bullet(\mathcal{A}) \otimes_{\mathcal{A}^\mathsf{e}} \mathcal{A}$, but this is canonically isomorphic to the Hochschild complex $\mathsf{C}_\bullet(\mathcal{A})$, which gives the result. □

Remark 2.3. When $\mathcal{A}$ is commutative, this proposition has a geometric interpretation in terms of the scheme $\mathsf{X} = \mathsf{Spec}(\mathcal{A})$. Indeed $\mathcal{A}^\mathsf{e} = \mathcal{O}(\mathsf{X}) \otimes \mathcal{O}(\mathsf{X})$ can be viewed as the algebra $\mathcal{O}(\mathsf{X} \times \mathsf{X})$ of functions on the product $\mathsf{X} \times \mathsf{X}$. The multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ used in the bar resolution is dual to the diagonal inclusion $\mathsf{X} \hookrightarrow \mathsf{X} \times \mathsf{X}$. The tensor product $\mathcal{A} \otimes_{\mathcal{A}^\mathsf{e}} \mathcal{A} \cong \mathcal{A}$ is therefore naturally interpreted as the algebra of functions on the self-intersection $\mathsf{X} \cap \mathsf{X} = \mathsf{X}$ of the diagonal inside $\mathsf{X} \times \mathsf{X}$. This intersection is not transverse, and the failure of transversality is measured by the higher Tor groups $\mathsf{Tor}_\bullet^{\mathcal{A}^\mathsf{e}}(\mathcal{A},\mathcal{A}) = \mathsf{HH}_\bullet(\mathcal{A})$. Indeed, one of the key ideas of derived algebraic geometry (originating from an intersection multiplicity formula due to Serre) is that the higher Tor groups should be considered part of the *definition* of $\mathsf{X} \cap \mathsf{X}$ as a "derived scheme". □

Using Proposition 2.2, we can determine the Hochschild homology of some simple algebras.

Example 2.4. Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring. Then the enveloping algebra is also a polynomial ring, whose generators we give different names.

$$\mathcal{A}^\mathsf{e} = \mathcal{A} \otimes \mathcal{A}^\mathsf{op} \cong \mathbb{K}[y_1, \ldots, y_n, z_1, \ldots, z_n]$$

Under this identification, the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is given by the ring homomorphism that sends $y_i \mapsto x_i$ and $z_i \mapsto x_i$. Evidently the elements $r_i = y_i - z_i$ are annihilated by this map, and in fact they generate the kernel, so that we have a presentation

$$\mathcal{A} = \mathcal{A}^\mathsf{e}/(r_1, \ldots, r_n)$$

In other words, if $\mathcal{M} := (\mathcal{A}^\mathsf{e})^{\oplus n}$, we have a map

$$\mathcal{M} \xrightarrow{(r_1,\ldots,r_n)} \mathcal{A}^\mathsf{e}$$

whose cokernel is isomorphic to $\mathcal{A}$. This extends to the Koszul complex

$$(\wedge^\bullet \mathcal{M}, \mathrm{d}) = \Big(\cdots \longrightarrow \wedge^3 \mathcal{M} \longrightarrow \wedge^2 \mathcal{M} \longrightarrow \mathcal{M} \longrightarrow \mathcal{A}^\mathsf{e} \Big)$$

where $\wedge^k \mathcal{M}$ denotes the kth exterior power of $\mathcal{M}$ as an $\mathcal{A}^\mathsf{e}$-module, and where the differential is given in terms of the basis elements $e_i \in \mathcal{M}$ by the formula

$$\mathrm{d}(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{k=1}^{p} (-1)^k r_{i_k} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}.$$

One can show that the Koszul complex gives a free resolution of $\mathcal{A}$ as an $\mathcal{A}^\mathsf{e}$-module; this uses the fact that the elements $r_1, \ldots, r_n$ form a regular sequence (see, e.g. [1, Chapter 17]).

If we let $\mathcal{N} = \mathcal{M} \otimes_{\mathcal{A}^e} \mathcal{A} \cong \mathcal{A}^{\oplus n}$ then by Proposition 2.2, the Hochschild homology of $\mathcal{A}$ is the homology of the complex $(\wedge^\bullet \mathcal{M}, \mathrm{d}) \otimes_{\mathcal{A}^e} \mathcal{A} \cong (\wedge^\bullet \mathcal{N}, \mathrm{d} = 0)$. Therefore

$$\mathsf{HH}_\bullet(\mathcal{A}) \cong \wedge^\bullet \mathcal{N}$$

where $\mathcal{N}$ is a free $\mathcal{A}$-module of rank n. In the next lecture, we will see that $\mathcal{N}$ is best interpreted as the differential one-forms on the affine space $\mathbb{A}^n = \mathsf{Spec}(\mathcal{A})$, and view the result above as a special case of the Hochschild–Kostant–Rosenberg theorem. □

Example 2.5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{K}$, and let $\mathcal{U}(\mathfrak{g})$ be its universal enveloping algebra. Thus $\mathcal{U}(\mathfrak{g})$ is the quotient

$$\mathcal{U}(\mathfrak{g}) = \frac{\mathsf{T}(\mathfrak{g})}{(x \otimes y - y \otimes x - [x, y])}$$

where $\mathsf{T}(\mathfrak{g}) = \mathbb{K} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$ is the tensor algebra of $\mathfrak{g}$. By construction, a left $\mathcal{U}(\mathfrak{g})$-module is the same data as a representation of the Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}_t$ denote the Lie algebra whose bracket is given by rescaling the bracket on $\mathfrak{g}$ by the constant t, i.e. $[-,-]_t = t[-,-]$ for $t \in \mathbb{K}$. When $t = 0$, the Lie algebra $\mathfrak{g}_t$ is abelian and its universal enveloping algebra is simply the symmetric algebra $\mathcal{U}(\mathfrak{g}_t) = \mathcal{S}ym(\mathfrak{g})$, i.e. a polynomial ring. We can view it as the algebra of functions on the affine space $\mathfrak{g}^\vee$ (the dual vector space of $\mathfrak{g}$). If $[-,-] \neq 0$, then $\mathcal{U}(\mathfrak{g}_t)$ is noncommutative for $t \neq 0$, but the Poincaré–Birkhoff–Witt theorem states that there is a natural isomorphism of vector spaces $\mathcal{S}ym(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$, given by the symmetrization of the product on $\mathcal{U}(\mathfrak{g})$. Thus we can view the product on $\mathcal{U}(\mathfrak{g})$ as a noncommutative deformation (quantization) of the product on the commutative ring $\mathcal{S}ym(\mathfrak{g})$.

Correspondingly, the Hochschild complex of $\mathcal{U}(\mathfrak{g})$ can be computed using a deformation of the Koszul complex from Example 2.4. More precisely, we have a complex $(\mathcal{U}(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g}, \mathrm{d})$, where the differential is given by

$$\begin{aligned}\mathrm{d}(u \otimes x_1 \wedge \cdots \wedge x_n) = &\sum_{i=1}^n (-1)^i (x_i u - u x_i) \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n\end{aligned}$$

This is the complex that computes the Chevalley–Eilenberg Lie algebra homology of $\mathcal{U}(\mathfrak{g})$ as a right $\mathfrak{g}$-module, where the action of $x \in \mathfrak{g}$ on $u \in \mathcal{U}(\mathfrak{g})$ is given by $u \cdot x = ux - xu$ for $x \in \mathfrak{g}$ and $u \in \mathcal{U}(\mathfrak{g})$; in other words, $\mathfrak{g}$ is acting via the adjoint representation on the bimodule $\mathcal{U}(\mathfrak{g})$.

One can check that the antisymmetrization map

$$\begin{aligned}\mathcal{U}(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g} &\to \mathsf{C}_\bullet(\mathcal{U}(\mathfrak{g})) \\ u \otimes g_1 \wedge \cdots \wedge g_p &\mapsto \sum_{\sigma \in S_p} \mathrm{sign}(\sigma) u \otimes g_{\sigma_1} \otimes \cdots \otimes g_{\sigma_p}\end{aligned}$$

is a quasi-isomorphism (see [3, Section 3.3.1] for details), so that we have an isomorphism

$$\mathsf{HH}_\bullet(\mathcal{U}(\mathfrak{g})) \cong \mathsf{H}_\bullet(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$$

between the Hochschild homology and the Lie algebra homology. □

3 Hochschild homology of dg algebras

We now briefly explain how to extend the definition of Hochschild homology to the more general setting of differential graded algebras.

Definition 3.1. A (unital) ***differential graded (dg) algebra*** is a cochain complex $(\mathcal{A}, \delta)$ with a maps of complexes

$$\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

that gives an associative product on $\mathcal{A}$, and a cocycle $1 \in \mathcal{A}$ that is the multiplicative unit.

A dg algebra is, in particular, an algebra, so we can form the usual Hochschild differential

$$\cdots \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A}$$

but now since $\mathcal{A}$ is itself a complex, this is really a double complex, where the vertical arrows are the usual differentials on the tensor product of complexes:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
\cdots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^1 & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^1 & \xrightarrow{\mathrm{b}} & \mathcal{A}^1 \\
 & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
\cdots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^0 & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^0 & \xrightarrow{\mathrm{b}} & \mathcal{A}^0 \\
 & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
\cdots & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})^{-1} & \xrightarrow{\mathrm{b}} & (\mathcal{A} \otimes \mathcal{A})^{-1} & \xrightarrow{\mathrm{b}} & \mathcal{A}^{-1} \\
 & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

The ***Hochschild complex of*** $\mathcal{A}$ is then the direct sum total complex of this double complex:

$$\mathsf{C}_\bullet(\mathcal{A}) = \mathsf{Tot}^{\oplus}\left(\cdots \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mathrm{b}} \mathcal{A} \right)$$

If $\mathcal{A}$ is an ordinary algebra, viewed as a dg algebra concentrated in degree zero, this clearly recovers the previous definition.

4 Differential graded categories

Finally, we extend to the setting of differential graded categories, which is where many of the most interesting examples are situated. Standard references include [2, 4].

Definition 4.1. A $\mathbb{K}$-linear ***differential graded category*** (***dg category*** for short) is a category $\mathcal{C}$ enriched in complexes of $\mathbb{K}$-vector spaces. More concretely, it consists of

- a collection of objects $\mathcal{M}, \mathcal{N}, \ldots \in \mathcal{C}$.
- a complex $\mathcal{C}(\mathcal{M}, N) = \mathsf{Hom}^{\bullet}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ of morphisms for every pair of objects x, y
- composition maps

$$\mathcal{C}(\mathcal{L}, \mathcal{M}) \otimes \mathcal{C}(\mathcal{M}, \mathcal{N}) \to \mathcal{C}(\mathcal{L}, \mathcal{N})$$

  that are morphisms of complexes, and satisfy the usual associative law from compositions: $(f \circ g) \circ h = f \circ (g \circ h)$.
- a cocycle $1 \in \mathcal{C}(\mathcal{M}, \mathcal{M})$ for every object $\mathcal{M}$, that acts as the identity for the composition law

If $\mathcal{C}$ is a dg category, then we obtain a $\mathbb{K}$-linear category $\mathsf{H}^0(\mathcal{C})$ by taking zeroth cohomology of all the morphism complexes. This is called the ***homotopy category of*** $\mathcal{C}$.

Example 4.2. Any $\mathbb{K}$-linear category is a dg category $\mathcal{C}$ where the morphism complex is concentrated in degree zero. In this case $\mathsf{H}^0(\mathcal{C}) = \mathcal{C}$. □

Example 4.3. A dg algebra $\mathcal{A}$ can be viewed as a dg category with a single object $* \in \mathcal{C}$ whose endomorphisms are given by $\mathcal{C}(*, *) = \mathcal{A}$. In this case $\mathsf{H}^0(\mathcal{C})$ is the category with one object $*$ and endomorphism algebra $\mathsf{H}^0(\mathcal{A})$. □

Example 4.4. If $\mathcal{A}$ is an algebra (or more generally, a dg algebra), then there is a dg category $\mathsf{Cplx}(\mathcal{A})$ whose objects are complexes of $\mathcal{A}$-modules (resp. dg modules over $\mathcal{A}$). Give a pair of complexes $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ and $(\mathcal{N}, \mathrm{d}_{\mathcal{N}})$, the complex of morphisms $\mathsf{Hom}^{\bullet}_{\mathsf{Cplx}(\mathcal{A})}(\mathcal{M}, N)$ has degree-n piece given by

$$\mathsf{Hom}^{n}_{\mathsf{Cplx}(\mathcal{A})}(\mathcal{M}, \mathcal{N}) = \prod_{i=-\infty}^{\infty} \mathsf{Hom}_{\mathcal{A}}(\mathcal{M}^i, \mathcal{N}^{i+n})$$

the space of degree-n maps of graded $\mathcal{A}$-modules. The differential is given by $\mathrm{d}f = \mathrm{d}_{\mathcal{N}} f - (-1)^n f \mathrm{d}_{\mathcal{M}}$ where $f \in \mathsf{Hom}^{n}_{\mathsf{Cplx}(\mathcal{A})}(\mathcal{M}, \mathcal{N})$. Note that the degree zero cocycles are precisely the cochain maps, and the degree-zero coboundaries are the null-homotopic cochain maps. Hence $\mathsf{H}^0(\mathsf{Cplx}(\mathcal{A}))$ is the homotopy category of complexes of $\mathcal{A}$-modules. □

Example 4.5. The category of ***perfect complexes of $\mathcal{A}$-modules*** is the full subcategory

$$\mathsf{Perf}(\mathcal{A}) \subset \mathsf{Cplx}(\mathcal{A})$$

consisting of complexes that are quasi-isomorphic to a bounded complex of projective $\mathcal{A}$-modules of finite rank. □

Example 4.6. Let X be a quasi-compact quasi-separated scheme (we need this technical condition to ensure that $\mathsf{QCoh}(\mathsf{X})$ is Grothendieck abelian, i.e. a category we can do homological algebra with). Then there are dg categories

$$\mathsf{Perf}(\mathsf{X}) \subset \mathsf{Coh}_{\mathsf{dg}}(\mathsf{X}) \subset \mathsf{QCoh}_{\mathsf{dg}}(\mathsf{X})$$

whose homotopy categories are the bounded derived categories

$$\mathsf{D}^{\mathsf{b}}\mathsf{Perf}(\mathsf{X}) \subset \mathsf{D}^{\mathsf{b}}\mathsf{Coh}(\mathsf{X}) \subset \mathsf{D}^{\mathsf{b}}\mathsf{QCoh}(\mathsf{X})$$

of perfect/coherent/quasicoherent sheaves on X. The construction of these dg categories is slightly subtle because $\mathsf{QCoh}(\mathsf{X})$ does not have enough projectives. One approach is to use take the subcategory of the dg category of chain complexes in $\mathsf{QCoh}(\mathsf{X})$, consisting of complexes that are injective as $\mathcal{O}_{\mathsf{X}}$-modules and locally quasi-isomorphic to perfect/coherent/quasi-coherent complexes. □

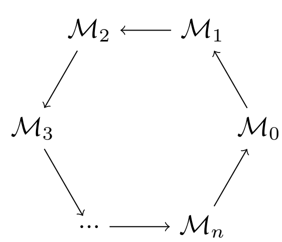
It is straightforward to extend the definition of Hochschild homology to arbitrary (small) dg categories (c.f. [2], Section 5.3). Indeed, suppose that $\mathcal{C}$ is a dg category. Then we may define a double complex

$$\cdots \xrightarrow{\mathrm{b}} \bigoplus_{\mathcal{M}_0,\mathcal{M}_1\in\mathcal{C}} \mathcal{C}(\mathcal{M}_0,\mathcal{M}_1)\otimes\mathcal{C}(\mathcal{M}_1,\mathcal{M}_0) \xrightarrow{\mathrm{b}} \bigoplus_{\mathcal{M}_0\in\mathcal{C}} \mathcal{C}(\mathcal{M}_0,\mathcal{M}_0)$$

where the nth column is given by the space

$$\bigoplus_{\mathcal{M}_0,\mathcal{M}_1,\ldots,\mathcal{M}_n\in\mathcal{C}} \mathcal{C}(\mathcal{M}_0,\mathcal{M}_1)\otimes\mathcal{C}(\mathcal{M}_1,\mathcal{M}_2)\otimes\cdots\otimes\mathcal{C}(\mathcal{M}_n,\mathcal{M}_0)$$

formed from morphisms in $\mathcal{C}$ that compose in a cycle:



The horizontal differential is given by

$$\begin{aligned}\mathrm{b}(f_0\otimes\cdots\otimes f_n) = &\sum_{i=0}^{n-1}(-1)^{i+|f_i||f_{i+1}|} f_0\otimes\cdots\otimes(f_{i+1}\circ f_i)\otimes\cdots\otimes f_n \\ &+(-1)^{|f_n|(|f_{n-1}|+\cdots+|f_0|)} f_n\circ f_0\otimes\cdots\otimes f_{n-1}.\end{aligned}$$

and the vertical differential is the usual one on the tensor product. Denoting by $\mathsf{C}_\bullet(\mathcal{C})$ the direct sum total complex of this double complex, the ***Hochschild homology*** of $\mathcal{C}$ is the cohomology $\mathsf{HH}_\bullet(\mathcal{C}) := \mathsf{H}_\bullet(\mathsf{C}_\bullet(\mathcal{C}), \mathrm{b})$. It's not hard to see that if $\mathcal{C}$ is a dga, then this agrees with the Hochschild homology defined earlier.

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