Elementary (ha-ha) Aspects of Topos Theory

Matt Booth

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Contents

This is an introductory talk on topos theory from a geometer's perspective[1]. From this viewpoint topos theory can be said to be a generalisation of topology - we'll see that we can put a topology not just on a set, but on a category. We'll start with the definition of a sheaf of sets on a topological space, and then extend that definition to get sheaves of sets on categories. Then we'll look at some nice properties of topoi.

In this talk, all categories will be locally small. Otherwise, we will mostly ignore any size issues.

1 Sheaves on topological spaces

1.1 Presheaves on spaces

Definition 1.1.1. Let X be a topological space. A **presheaf of sets** $\mathcal{F}$ on X is the data of:

[1] However, I won't really talk about any geometry!

i) A set $\mathcal{F}(U)$ for every open set $U \subseteq X$

ii) A map $\rho_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$ for every pair of open sets $U \subseteq V$

such that:

i) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ for all open U

ii) If $U \subseteq V \subseteq W$ is a triple of open sets, then $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$

We call ρ_{UV} the **restriction map** from V to U. If $s \in \mathcal{F}(V)$ then we'll often write $s|_U$ for $\rho_{UV}(s)$. The two conditions we require are intuitively just some obvious facts about restrictions.

We often refer to the elements $s \in \mathcal{F}(U)$ as the **sections of $\mathcal{F}$ over U**. This language is supposed to evoke the idea that the set $\mathcal{F}(U)$ lives 'above' U in some sense. Sections of $\mathcal{F}$ over X are the **global sections**. An alternate notation, common in algebraic geometry, is to write $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$. This is often used when U is considered fixed and $\mathcal{F}$ is allowed to vary.

Example 1.1.2. The **constant presheaf on X with value** A is the presheaf that assigns the set A to every open set $U \subseteq X$. The restriction maps are all identity maps.

Example 1.1.3. If Y is a space, then we can define a presheaf $\mathcal{F}$ on X by letting $\mathcal{F}(U)$ be the collection of continuous functions from U to Y. The restriction maps ρ_{UV} are the genuine restriction maps $f \mapsto f|_U$.

Definition 1.1.4. A morphism of presheaves of sets $\mathcal{F} \to \mathcal{G}$ on X is an assignment of a map $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ to every open set $U \subseteq X$ such that the maps ϕ_U are compatible with the restriction maps.

Since this is a category theory workshop, we want to try and rephrase the definition of a presheaf in a more categorical manner. This isn't too hard:

Definition 1.1.5. Given a space X, the category $\mathrm{Open}(X)$ of **opens in** X is the category whose objects are the open sets $U \subseteq X$ and whose morphisms are the inclusions.

Note that a continuous map f:$X \to Y$ induces a functor $\mathrm{Open}(Y) \to \mathrm{Open}(X)$ by sending an open set U to its preimage under f.

Proposition 1.1.6. *Let X be a space. Then a presheaf of sets on X is the same thing as a functor* $\mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}$. *A morphism of presheaves is a natural transformation between such functors.*

Definition 1.1.7. The category of **presheaves on** X is the category $\mathbf{Psh}(X) := \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathbf{Set})$.

Definition 1.1.8. A presheaf on X is **representable** if it is isomorphic to a presheaf of the form $h_U = \mathrm{Hom}(-, U)$ for some open subset U of X.

1.2 Digression on pointless topology

A natural question to ask is: how does the category $\mathrm{Open}(X)$ compare to the space X? In particular, can we recover X from $\mathrm{Open}(X)$?

Definition 1.2.1. A topological space X is **sober** if every irreducible closed subset V is the closure of exactly one point ζ of X. The point ζ is called the **generic point** of V. The category **Sob** of sober spaces has objects the sober spaces and morphisms the continuous maps.

Example 1.2.2. A Hausdorff space is sober, since the only irreducible subsets are the points and every singleton is closed. A sober space is Kolmogorov (any two points are topologically distinguishable).

Proposition 1.2.3. *If X is sober, then the category* $\mathrm{Open}(X)$ *determines the space X up to homeomorphism.*

So if we only care about sober spaces X, then we might as well deal with the categories $\mathrm{Open}(X)$ up to isomorphism. In fact, $\mathrm{Open}(X)$ is a certain kind of lattice called a **frame**. The above Proposition tells us that we can recover a sober space X from its associated frame.

Since the functor assigning a space X to its associated frame is contravariant, we really should consider the opposite category of the category of frames:

Definition 1.2.4. The category **Loc** of **locales** is defined to be the opposite of the category of frames.

In particular we can define a presheaf on a locale L as a functor $L^{\mathrm{op}} \to \mathbf{Set}$. Locales are 'pointless' versions of topological spaces, since they just remember the structure of the lattice of open sets:

Theorem 1.2.5. *There is an embedding* $\mathbf{Sob} \hookrightarrow \mathbf{Loc}$*, defined by sending a sober space to its associated locale.*

Remark 1.2.6. This is a weak form of **Stone duality**. In fact, there is an adjunction between **Top** and **Loc** restricting to an equivalence between the full subcategories **Sob** and **sLoc**, the category of 'spatial locales'. The unit of this adjunction is the 'soberification' map $\mathbf{Top} \to \mathbf{Sob}$ and the counit is the 'pointification' map $\mathbf{Loc} \to \mathbf{sLoc}$.

1.3 Sheaves on spaces

Presheaves are all well and good, but some presheaves have undesirable properties. For example we'd like presheaves to be 'local': if $\{U_i : i \in I\}$ is an open cover of an open subset U, then we'd like the sections of a presheaf on U to be determined by the sections on the U_i.

Definition 1.3.1. A presheaf $\mathcal{F}$ on a space X is **separated** if, for any open set U of X and any open cover $\{U_i : i \in I\}$ of U, if s, t are two sections of $\mathcal{F}(U)$ with $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

We'd also like to be able to 'glue' sections together. If $\{U_i : i \in I\}$ is a cover of U then we'll often write U_{ij} for $U_i \cap U_j$.

Definition 1.3.2. A separated presheaf $\mathcal{F}$ on a space X is a **sheaf** if, for any open set U of X and any open cover $\{U_i : i \in I\}$ of U, for any collection $s_i \in \mathcal{F}(U_i)$ of sections on the U_i such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, there exists a section s of $\mathcal{F}$ over U, called the **gluing** of the s_i, such that $s_{U_i} = s_i$. Since $\mathcal{F}$ is separated this gluing will be unique.

Remark 1.3.3. It's worth commenting on the sections of a sheaf $\mathcal{F}$ over the empty set $\emptyset$. Note that $\emptyset$ is an open subset of X that is covered by the empty covering. Hence the 'collection of sections' is an element of the empty product in **Set**, which is a final object of **Set**, a.k.a. a singleton. So $\mathcal{F}(\emptyset)$ is always a one-element set.

The condition that a presheaf is a sheaf can be phrased simply as 'compatible sections can be glued together uniquely'. A sheaf can hence be used to track locally defined data on a space.

Example 1.3.4. Let $f : Y \to X$ be a continuous map. The **sheaf of sections of** f is the sheaf $\Gamma(Y/X)$ whose value at U is the set of maps $s : U \to Y$ with $f \circ s = \mathrm{id}_U$. The restriction maps are just restriction of functions. This is the prototypical example of a sheaf.

Example 1.3.5. If X is a smooth manifold then it comes equipped with many sheaves: the sheaf of n-times differentiable functions $\mathcal{O}_X^n$, sheaf of smooth functions $\mathcal{O}_X$ (also called the **structure sheaf**), and the cotangent sheaves Ω_X^p of differential p-forms on X. If X is a complex manifold then it also comes equipped with a sheaf of holomorphic functions.

Example 1.3.6. Consider the presheaf $\mathcal{F}$ on $\mathbb{R}$ sending an open subset U to the collection of bounded real-valued functions on U. Then $\mathcal{F}$ is a separated presheaf but not a sheaf: locally bounded functions do not necessarily glue together to give a globally bounded function.

Example 1.3.7. Let X be a space, $x \in X$ a point, and A a set. The **skyscraper sheaf** $\mathrm{sky}_\mathrm{x}^\mathrm{A}$ is the sheaf whose value on an open set U is

$$\mathrm{sky}_\mathrm{x}^\mathrm{A}(U) = \begin{cases} A & \text{if } x \in U \\ * & \text{else} \end{cases}$$

Again, we'd like a categorical characterisation of sheaves:

Proposition 1.3.8. *Let $\mathcal{F} \in \mathbf{Psh}(X)$. Then $\mathcal{F}$ is a sheaf if and only if, for any open set U of X and any open cover $\{U_i : i \in I\}$ of U, the diagram*

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_i \mathcal{F}(U_i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_{ij})$$

where $\rho = \prod_i \rho_{U_i U}$, $\alpha = \prod_{i,j} \rho_{U_{ij} U_i}$ and $\beta = \prod_{i,j} \rho_{U_{ij} U_j}$, is an equaliser.

A morphism of sheaves is a morphism of presheaves. The collection of sheaves on a space X forms a category $\mathbf{Sh}(X)$.

1.4 Sheafification

We'd like a method for turning a presheaf into a sheaf. Luckily this is always possible:

Proposition 1.4.1. *The forgetful functor* $\mathbf{Sh}(X) \to \mathbf{Psh}(X)$ *admits a left adjoint, which we call the* ***sheafification****.*

There are many ways of proving this. One way is to use the **éspace étalé** construction.

Example 1.4.2. The **constant sheaf with value** A is the sheafification of the constant presheaf.

Proposition 1.4.3. *Sheafification preserves finite limits.*

Remark 1.4.4. Since it is a left adjoint, sheafification preserves all colimits.

1.5 Operations on sheaves

We can push and pull sheaves around by maps:

Definition 1.5.1. Let $f : X \to Y$ be a map and let $\mathcal{F}$ be a sheaf on X. The **direct image sheaf** $f_*\mathcal{F}$ is the sheaf on Y with $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. This operation is functorial.

Inverse image sheaves are more complicated, since the image of an open set under a continuous function need not be open.

Definition 1.5.2. Let $f : X \to Y$ be a map and let $\mathcal{F}$ be a sheaf on Y. The **inverse image sheaf** $f^{-1}\mathcal{F}$ is the sheaf on X defined as the sheafification of the presheaf $U \mapsto \mathrm{colim}_{V \supseteq f(U)} \mathcal{F}(V)$. This operation is also functorial.

2 Sheaves on categories

2.1 Grothendieck topologies

We have a categorical description of what it means to be a presheaf on a category:

Definition 2.1.1. Let $\mathcal{C}$ be any category. The category $\mathbf{Psh}(\mathcal{C})$ of **presheaves on** $\mathcal{C}$ is the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$.

However, the notion of a sheaf required us to have the concept of a topology. We're going to need to put a topology on a category. Grothendieck's realisation was that the correct thing to axiomatise is the notion of covering family:

Definition 2.1.2. Let $\mathcal{C}$ be a category. A **Grothendieck topology** on $\mathcal{C}$ is an assignment of, to every $U \in \mathcal{C}$, a collection of sets of maps $\{U_\alpha \to U\}$, called the **covering families of** U, such that the following conditions are satisfied:

i) *(Base change)* If $\{U_\alpha \to U\}$ is a covering family and $V \to U$ is any arrow, then the fibre products $U_\alpha \times_U V$ exist, and the collection of projections $\{U_\alpha \times_U V \to V\}$ is a covering family.

ii) *(Locality)* If $\{U_\alpha \to U\}$ is a covering family, and for each index α we have a covering family $\{U_{\alpha\beta} \to U_\alpha\}$, then the composite family $\{U_{\alpha\beta} \to U\}$ is a covering. Informally, this states that 'a cover of a cover is again a cover'.

iii) *(Isomorphisms)* If $V \to U$ is an isomorphism, then the one-element family $\{V \to U\}$ is a covering family.

A category together with a choice of Grothendieck topology is called a **site**.

Strictly this is the notion of a **Grothendieck pretopology**; in this talk we can do without the more general notion of a Grothendieck topology. Grothendieck pretopologies require $\mathcal{C}$ to have certain fibre products, whereas Grothendieck topologies do not. Moreover, different pretopologies may give the same topology.

Example 2.1.3. If X is a topological space, then $\mathrm{Open}(X)$ admits the structure of a site where the covering families of U are the open covers of U.

Example 2.1.4. The category **Top** admits a Grothendieck topology, the **global classical topology**, where a covering of U is a jointly surjective collection of open injections $\{U_\alpha \to U\}$. **Top** also admits another Grothendieck topology (the **global étale topology**) where we require the covers to be local homeomorphisms, not just open injections.

Example 2.1.5. Any category admits the **indiscrete topology**, where the covering families are the isomorphisms.

2.2 Sheaves on a site

If $U_i \to U$ and $U_j \to U$ are members of a covering family of U, then we'll denote the fibre product $U_i \times_U U_j$ by U_{ij}. This agrees with our earlier notation for topological spaces. We can now easily state what it means to be a sheaf:

Definition 2.2.1. Let $\mathcal{C}$ be a site. Then a presheaf $\mathcal{F}$ on $\mathcal{C}$ is a sheaf if and only if, for every object U of $\mathcal{C}$ and every covering family $\{U_i \to U\}$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_{ij})$$

is an equaliser. A morphism of sheaves is a morphism of presheaves. If $\mathcal{C}$ is a site, then $\mathbf{Sh}(\mathcal{C})$ is the **category of sheaves of sets on** $\mathcal{C}$.

Definition 2.2.2. A **topos**[2] is a category equivalent to the category of sheaves of sets on a site.

[2]Every time I say 'topos' in this talk I will mean 'Grothendieck topos'.

Example 2.2.3. The category **Set** is a topos; it can be identified with the category of sheaves of sets on a point.

Proposition 2.2.4. *Let $\mathcal{C}$ be a category. The forgetful functor $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Psh}(\mathcal{C})$ admits a left adjoint, which we call the* ***sheafification****.*

Remark 2.2.5. This is harder to prove than in the case of sheaves of sets on spaces.

Definition 2.2.6. The **canonical topology** on a category is the largest topology for which all representable functors are sheaves. A **subcanonical topology** is any subtopology of the **canonical topology**. We can also characterise subcanonical topologies as the topologies for which all representable functors are sheaves.

2.3 Limits and colimits

If $\mathcal{C}$ is any category, then $\mathbf{Psh}(\mathcal{C})$ is both complete and cocomplete. Moreover, there is an easy way to compute limits and colimits:

Proposition 2.3.1. *[Limits and colimits are computed pointwise] Let $i \mapsto \mathcal{F}_i$ be a diagram of presheaves in a category $\mathcal{C}$. Let X be an object of $\mathcal{C}$. Then we have isomorphisms*

$$(\lim \mathcal{F}_i)(X) \cong \lim(\mathcal{F}_i(X)) \qquad (\operatorname{colim} \mathcal{F}_i)(X) \cong \operatorname{colim}(\mathcal{F}_i(X))$$

Moreover, if every $\mathcal{F}_i$ is a sheaf, then so is $\lim \mathcal{F}_i$.

Remark 2.3.2. The assertion that the limit of sheaves is a sheaf is easy to see, since an equaliser is a limit and limits commute with limits. Colimits of sheaves are not computed pointwise! We have to sheafify, since a colimit of sheaves is not necessarily a sheaf.

Recall that the Yoneda Lemma tells us that we have a fully faithful embedding (the **Yoneda embedding**) $\mathcal{C} \to \mathbf{Psh}(\mathcal{C})$ defined by the assignment $U \mapsto h_U$. Since $\mathbf{Psh}(\mathcal{C})$ is a cocomplete category, we can regard the Yoneda embedding as a cocompletion. In fact, it is the **free cocompletion**:

Proposition 2.3.3. *The Yoneda embedding $\mathcal{C} \to \mathbf{Psh}(\mathcal{C})$ is universal among functors from $\mathcal{C}$ into cocomplete categories, i.e. any functor $F : \mathcal{C} \to \mathcal{D}$ to a cocomplete catgeory factors through $\mathbf{Psh}(\mathcal{C})$, and the map $\mathbf{Psh}(\mathcal{C}) \to \mathcal{D}$ preserves colimits.*

We have also seen the co-Yoneda lemma, which tells us that every presheaf is a canonical colimit of representables.

2.4 Giraud's Theorem

We'd like an intrinsic definition of what it means to be a topos. First we need to set up some terminology.

Definition 2.4.1. An **equivalence relation** R on an object X of a category $\mathcal{C}$ is a map $R \to X \times X$ such that for all objects Y, the image of the map $\mathrm{Hom}(Y, R) \to \mathrm{Hom}(Y, X) \times \mathrm{Hom}(Y, X)$ is an equivalence relation on the set $\mathrm{Hom}(Y, X)$.

Example 2.4.2. Let $X \to Y$ be a morphism. The limit, if it exists, of the diagram $X \to Y \leftarrow X$ is an equivalence relation. This limit is the **kernel pair** of the morphism $X \to Y$.

Definition 2.4.3. Let R be an equivalence relation on an object X. Let X/R be the coequaliser of the diagram $R \to X \times X$, called the **quotient** of the relation. Say that R is **effective** if the induced map $R \to X \times_{X/R} X$ is an isomorphism.

Example 2.4.4. Let G be a group and H a normal subgroup. Then the quotient group G/H is isomorphic to the quotient of the relation $G \times H \to G \times G$, where the map sends (g, h) to (g, gh).

Definition 2.4.5. The following statements about a category $\mathcal{C}$ are referred to as **Giraud's axioms**:

i) $\mathcal{C}$ has a set of generators.

ii) $\mathcal{C}$ has all colimits, and colimits commute with fibre products.

iii) Sums are disjoint, i.e. $X \times_{X \coprod Y} Y$ is the initial object of $\mathcal{C}$ for all objects X, Y of $\mathcal{C}$.

iv) Every equivalence relation in $\mathcal{C}$ is effective.

Example 2.4.6. A topos satisfies Giraud's axioms.

In fact more can be said:

Theorem 2.4.7 (Giraud)**.** *Let $\mathcal{C}$ be a category. The following are equivalent:*

i) $\mathcal{C}$ is a topos.

ii) $\mathcal{C}$ satisfies Giraud's axioms.

iii) There is a small category $\mathcal{D}$ and an inclusion $\mathcal{C} \hookrightarrow \mathbf{Psh}(\mathcal{D})$ that admits a left adjoint which preserves finite limits.

3 Exercises

Examples of sheaves

i) Give an example of a presheaf that is not separated.

ii) Give an example of a separated presheaf that is not a sheaf.

iii) ([3]) Let U be a subset of $\mathbb{C}$ and let $\mathcal{F}(U)$ be the set of homolomorphic functions f on U such that $\frac{\mathrm{d}f}{\mathrm{d}z} = 1/z$. Show that $\mathcal{F}$ is a sheaf under restriction of functions.

Sheaves on some small spaces

i) Let X be the space $\{0,1\}$ where we declare the subsets $\emptyset, \{0\}$, and $\{0,1\}$ to be open (this space is known as the **Sierpiński space**). Describe explicitly the presheaves on X. Describe explicitly the sheaves on X.

ii) Same question, but where X now has the discrete topology.

iii) Same question, but where X now is any set with the indiscrete topology.

Stalks

The **stalk** of a sheaf $\mathcal{F}$ on X at the point $x \in X$ is defined to be the set $\mathcal{F}_x := \varinjlim_U \mathcal{F}(U)$, where the direct limit is indexed over all open sets U containing x.

i) Let x be a point of X and let $j : \{x\} \hookrightarrow X$ be the inclusion map. Show that the stalk $\mathcal{F}_x$ is the same as the inverse image sheaf $j^{-1}\mathcal{F}$.

ii) Show that a morphism of sheaves is an epimorphism (resp. monomorphism, isomorphism) if and only if the induced map on the stalks is an epimorphism (resp. monomorphism, isomorphism).

iii) Compute the stalks of the sheaf from *Examples of sheaves, iii)*

iv) Compute the stalks of a skyscraper sheaf.

Limits and colimits

i) Prove Proposition 2.3.1.

ii) Give an example of a topological space X and a direct system of sheaves $\mathcal{F}_i$ on X such that the limit presheaf $\varinjlim_i \mathcal{F}_i$ is not a sheaf.

References

[1] Saunders MacLane, Ieke Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, Springer, 1994

[2] Angelo Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, available at `https://arxiv.org/abs/math/0412512`

[3] George Kempf, *Algebraic Varieties*, Cambridge University Press, 1993