The derived contraction algebra

Matt Booth
The University of Edinburgh

Birational geometry and derived noncommutative geometry

- Threefold flopping contraction $X \to \operatorname{Spec} R$ over $\mathbb{C}$. $\exists$ a **flop** $X \dashrightarrow X^+$ over R. MMP: flops link minimal models.
- Bridgeland, Chen: $X \dashrightarrow X^+$ induces $D^b(X) \xrightarrow{\simeq} D^b(X^+)$.
- Van den Bergh: $\exists\ A = \operatorname{End}_R(R \oplus M)$ with $D^b(X) \xrightarrow{\simeq} D^b(A)$. Flop equivalences $\rightsquigarrow$ derived Morita equivalences $A \to A^+$.
- $X^{++} \cong X$ induces a **nontrivial** $FF : D^b(X) \xrightarrow{\simeq} D^b(X)$.
- C_i flopping curves. VdB's equivalence sends $\mathcal{O}_{C_i}(-1) \mapsto S_i$ 1-dimensional simple A-modules, with $FF(S_i) \cong S_i[-2]$.

The contraction algebra

- Donovan-Wemyss: $A_{\mathrm{con}} := \underline{\mathrm{End}}_R(M) \cong A/AeA$ with $e = \mathrm{id}_R$. It's NC Artinian m-pointed ($m = \#$flopping curves).
- e.g. Pagoda flop with base $uv - (x + y^k)(x - y^k) = 0$ and 1 curve above singular point has $A_{\mathrm{con}} \cong \mathbb{C}[\xi]/\xi^k$.
- A_{con} represents the pointed NC deformation theory of the S_i: $\mathrm{Def}(\{S_i\}_i) \cong \mathrm{Hom}_{\mathbf{Art}_{\mathbb{C}^m}}(A_{\mathrm{con}}, -)$.
- $FF \cong \mathbb{R}\,\mathrm{Hom}_A(\ker(A \to A_{\mathrm{con}}), -)$. So A_{con} also controls FF.
- A_{con} recovers all known invariants of flops. Conjecture: A_{con} classifies $\mathrm{Spec}\, R$ complete locally.

Surfaces

- Cut $X \to \operatorname{Spec} R$ along a generic hyperplane $g \in R$ to get $X' \to \operatorname{Spec}(R/g)$ partial resolution of ADE surface singularity.
- e.g. cut Atiyah flop (Pagoda $k = 1$) along $u = v^n$ to get a 1-curve partial resolution of an A_n singularity $zw - v^{n+1} = 0$.
- Can still define A_{con}; in the A_n example $A_{\mathrm{con}} \cong \mathbb{C}$ independent of n. So D-W conjecture fails.
- In the example, FF is represented by a 2-term complex. So A_{con} does not control FF for surfaces.

The derived contraction algebra

- $A_{\text{con}}^{\text{der}} := A/^{\mathbb{L}}AeA$ (construction of Braun-Chuang-Lazarev). Nonpositive (cohom. graded) DGA with $H^0(A_{\text{con}}^{\text{der}}) \cong A_{\text{con}}$.
- derived NC defm theory: $\mathrm{Def}^{\mathrm{fr}}(S) \cong \mathrm{Hom}_{\mathbf{pro(dgArt_{\mathbb{C}})}}(A_{\text{con}}^{\text{der}}, -)$. Lets us do explicit computations via Koszul duality.
- $A_{\text{con}}^{\text{der}}$ recovers $\langle M \rangle \subseteq D_{\text{sg}}^{\text{dg}}(R)$, the DG singularity category. $D_{\text{sg}}^{\text{dg}}(R)$ recovers R: derived D-W conj. holds if M generates.
- R is a hypersurface: Eisenbud $\implies$ 2-periodicity in $D_{\text{sg}}^{\text{dg}}(R)$. Get a periodicity element $\eta \in A_{\text{con}}^{\text{der}}$ of degree -2. Conjecture: $A_{\text{con}}^{\text{der}}/\eta$ controls FF. True for 3folds and the A_n example.
- A_n example: $A_{\text{con}}^{\text{der}} \cong \mathbb{C}[\eta]\langle\zeta\rangle$ with $|\zeta| = -1$, $m_{n+1}(\zeta^{n+1}) = \eta^n$. Pagoda: $A_{\text{con}}^{\text{der}} \cong \frac{\mathbb{C}[\xi]}{\xi^k}[\eta]$ with $|\xi| = 0$ and higher m_{2r}.