Higher Categories, Complete Segal Spaces and the Cobordism Hypothesis

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Contents

A note on notation I try to denote usual categories with italic letters, and higher categories (2-categories, (∞, n)-categories, etc.) with calligraphic letters.

1 Extending Cob(n)

1.1 Extending down

We've seen that the cobordism categories **Cob**(n) should really have more structure than just categories. In particular we should have, for every integer $k \leq n$, a k-category[1] $\mathbf{Cob}_k(n)$ with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{closed oriented } (n-k)\text{-manifolds} \\ \text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\ \text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms} \\ &\vdots \\ k\text{-morphisms} &\longleftrightarrow \text{(diffeomorphism classes of) } n\text{-manifolds with corners} \end{aligned}$$

We'd like to have a nice definition of k-category that includes $\mathbf{Cob}_k(n)$. Here's an obvious definition to make:

Definition 1.1.1. A **strict 1-category** is a category. A **strict k-category** is defined inductively as a category enriched over strict $(k-1)$-categories.

This definition is not the correct one. In particular $\mathbf{Cob}_k(n)$ is not a strict k-category since composition is not strictly associative, only associative up to isomorphism. We could adjust the definition of $\mathbf{Cob}_k(n)$ so that composition does become strictly associative, but this quickly gets messy.

Moral 1.1.2. We're going to need a better notion of k-category, where composition need only be associative up to isomorphism.

1.2 Extending up

Let's suppose we have a good definition of what a k-category is. Then we can define a (k,n)-category to be a k-category where all of the i-morphisms are invertible for $n < i \leq k$.

Example 1.2.1. A $(1,0)$-category should just be a groupoid.

It's also often useful to allow $k = \infty$; in fact we're going to define (∞,n)-categories later. This gives us a definition of (k,n)-categories simply by ignoring the morphisms above level k.

Example 1.2.2. An $(\infty,0)$-category is an **∞-groupoid**. Given a topological space X, we can form an ∞-groupoid $\pi_{\leq\infty}(X)$ called the **fundamental ∞-**

[1] In this section we'll treat higher categories at an informal level. Note that the concept of a "k-category" has not been defined!

groupoid of X, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X \\ \text{1-morphisms} &\longleftrightarrow \text{paths between points} \\ \text{2-morphisms} &\longleftrightarrow \text{homotopies between paths} \\ \text{3-morphisms} &\longleftrightarrow \text{homotopies between homotopies} \\ &\vdots \end{aligned}$$

The fundamental groupoid of X remembers all of X up to weak homotopy equivalence. More formally, the fundamental groupoid construction is an equivalence between topological spaces (up to weak homotopy equivalence) and ∞-groupoids (up to equivalence). This assertion is known as the **homotopy hypothesis**[2]. This allows us to think of $(\infty, 0)$-categories as really being topological spaces. So as well as generalising category theory, higher category theory should also generalise topology.

Recall that in defining $\mathbf{Cob}(n)$, we defined a morphism $M \to N$ to be a diffeomorphism class of (oriented) cobordisms $M \to N$. Instead of considering two diffeomorphic cobordisms to be the same map, we could say that they differ by an invertible 2-morphism. Hence we should have an $(\infty, 1)$-category $\mathbf{Cob}^t(n)$ with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{closed oriented } (n-1)\text{-manifolds} \\ \text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\ \text{2-morphisms} &\longleftrightarrow \text{diffeomorphisms between cobordisms} \\ \text{3-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms} \\ &\vdots \end{aligned}$$

Note that this definition allows us to keep track of the diffeomorphism groups of our cobordisms.

We can combine our two higher-categorical versions of $\mathbf{Cob}(n)$ into a single (∞, n)-category $\mathbf{Bord}_n$ with

[2] This is not a theorem yet, since we don't have a definition of ∞-groupoid. We could either **define** an ∞-groupoid to be a topological space, or we could regard the homotopy hypothesis as being a condition that our models of higher categories need to satisfy.

$$
\begin{aligned}
\text{objects} &\longleftrightarrow \text{0-manifolds} \\
\text{1-morphisms} &\longleftrightarrow \text{cobordisms between 0-manifolds} \\
\text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms} \\
&\vdots \\
n\text{-morphisms} &\longleftrightarrow n\text{-manifolds with corners} \\
(n+1)\text{-morphisms} &\longleftrightarrow \text{diffeomorphisms} \\
(n+2)\text{-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms} \\
&\vdots
\end{aligned}
$$

Moral 1.2.3. We're going to need a good definition of (∞, n)-categories. Note that the disjoint union operation on 0-manifolds should turn $\mathbf{Bord}_n$into a symmetric monoidal (∞, n)-category.

1.3 Intuitive statement of the cobordism hypothesis

The cobordism hypothesis is stated in terms of **framed** cobordisms. This is a technical point and won't really concern us. Denote the (∞, n)-category of framed cobordisms by $\mathbf{Bord}_n^{\mathrm{fr}}$.

If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then consider the category of $\mathcal{C}$-valued fully extended framed TFTs: we can identify this category with the category $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ of symmetric monoidal functors from $\mathbf{Bord}_n^{\mathrm{fr}}$ to $\mathcal{C}$.

The cobordism hypothesis more or less says that the evaluation functor $Z \mapsto Z(*)$ determines a bijection between isomorphism classes of $\mathcal{C}$-valued fully extended framed TFTs and isomorphism classes of objects in $\mathcal{C}$ satisfying suitable finiteness conditions.[3]

Remark 1.3.1. This specialises to a statement about $\mathbf{Cob}(n)$ by taking homotopy n-categories.

2 $(\infty, 1)$-categories as complete Segal spaces

We'll first define $(\infty, 1)$-categories and then soup up our definition in § 3 to get to (∞, n)-categories.

[3]By 'suitable finiteness conditions' we mean **full dualisability**, which we'll see a definition of in § 5. In $\mathbf{Vect}_k$ the fully dualisable objects are exactly the finite-dimensional vector spaces.

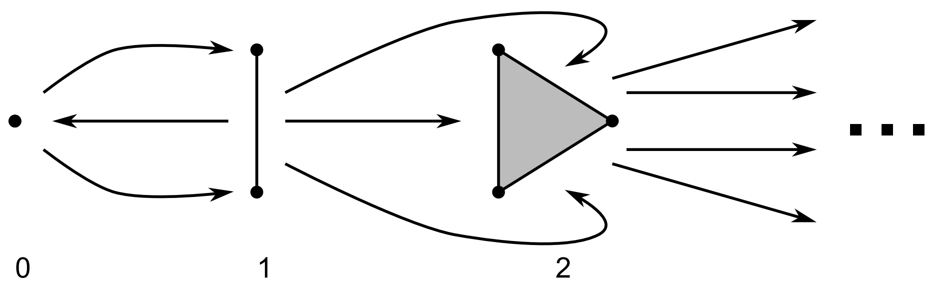
Intuitively, an $(\infty, 1)$-category should be a **topological category**; one where the hom-sets have the structure of topological spaces.[4] Higher morphisms are homotopies, homotopies between homotopies, and so on. However, while intuitive, this definition is very difficult to work with.

There are several other models[5] but we're going to use **complete Segal spaces** as our models for $(\infty, 1)$-categories since they generalise easily to (∞, n)-categories.

2.1 Preliminary: simplicial sets

Definition 2.1.1. The **simplex category** Δ has objects $[n] = \{0, 1, \dots, n\}$ and morphisms the weakly order-preserving maps.

It looks like



where we've omitted the maps from [2] to [1]. The maps going to the right are the **face maps** and the maps going to the left are the **degeneracy maps**.

Definition 2.1.2. A **simplicial object** in a category C is a functor $\Delta^{\text{op}} \to C$. More concretely a simplicial object is a collection of objects X_n indexed by the positive integers together with various face and degeneracy maps.

Definition 2.1.3. A **morphism** between two simplicial objects $F : \Delta^{\text{op}} \to C$ and $G : \Delta^{\text{op}} \to C$ is a natural transformation $F \to G$. Concretely, a morphism of simplicial objects is a collection of maps $X_n \to Y_n$ commuting with the face and degeneracy maps.

Proposition 2.1.4. *The collection of simplicial objects in a category C and their morphisms itself forms a category, which we denote sC.*

A simplicial object $X_\bullet$ looks like

$$X_0 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; X_1 \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \; X_2 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \cdots$$

[4] One way to think about this is that an $(\infty, 1)$-category should be enriched in $(\infty, 0)$-categories, which are the same thing as topological spaces.

[5] A good account of these are given in [8].

We'll be interested in **simplicial sets**; simplicial objects in **Set**.[6] Later we'll be interested in simplicial topological spaces.

Example 2.1.5. Given a topological space X, we can define (functorially) a simplicial set $\mathrm{Sing}(X)$ that at level n is the set $\mathrm{Hom}(\Delta^n, X)$ of maps from the n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ to X. We also have a geometric realisation functor $|\cdot| : s\mathbf{Set} \to \mathbf{Top}$ and in fact $|\mathrm{Sing}(X)|$ is weakly homotopy equivalent to X. Simplicial sets are good combinatorial models of topological spaces.[7]

Example 2.1.6. Given a category C, the **nerve** is a simplicial set $N(C)$ which at level n consists of the strings $C_0 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n$ of n composable morphisms. It's possible to recover C up to isomorphism from its nerve $N(C)$.

We might wonder what simplicial sets are the nerves of categories.

Proposition 2.1.7 (the Nerve Theorem)**.** *A simplicial set X is isomorphic to the nerve of a category if and only if for all $m, n \geq 0$ the diagram*

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps

$$\begin{array}{ccccccc} & & 0<1<\cdots<m & \longleftarrow\!\shortmid & 0<1<\cdots<m & & \\ m<m+1<\cdots<m+n & & [m+n] & \longleftarrow & [m] & & m \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0<1<\cdots<n & & [n] & \longleftarrow & [0] & & 0 \\ & & 0 & \longleftarrow\!\shortmid & 0 & & \end{array}$$

is Cartesian (i.e. a pullback diagram).

Whenever this diagram appears, we will fix the convention that the maps featuring are the maps described above.

2.2 Homotopy theory

Our philosophy is that $(\infty, 1)$-category theory should be category theory, but done in a homotopy-theoretic manner. This is because an $(\infty, 1)$-category is just a topological category where the higher morphisms are given by homotopies.

[6] A simplicial set is the same thing as a presheaf on Δ.

[7] Technically $s\mathbf{Set}$ and **Top** are Quillen equivalent (via these two functors), so they have the same homotopy theory.

Since Proposition 2.1.7 tells us that we can recover a category from its nerve, we'll try to code up the concept of a nerve in homotopy theory. We'll see that a Segal space is precisely this concept of 'homotopy nerve'. However we'll see that a Segal space alone won't quite be enough to recover an $(\infty, 1)$-category: we'll need some more conditions.

Definition 2.2.1. Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a diagram of topological spaces[8] and continuous maps. The **homotopy fibre product** $X \times_Z^h Y$ of X and Y along f and g is the space $X \times_Z Z^{[0,1]} \times_Z Y$ whose points are triples (x, y, p) with $x \in X$, $y \in Y$ and $p : [0, 1] \to Z$ a path in Z from $f(x)$ to $g(y)$.

Remark 2.2.2. There is a canonical map from $X \times_Z Y$ to $X \times_Z^h Y$ given by $(x, y) \mapsto (x, y, p)$ where p is the constant path from $f(x) = g(y)$ to itself.

Example 2.2.3. Let X be a space and $p : * \to X$ be the inclusion of a basepoint. Then the homotopy fibre product of $* \xrightarrow{p} X \xleftarrow{p} *$ is the space ΩX of loops in X based at p. The usual fibre product is the one point space $*$.

Example 2.2.4. The homotopy fibre product of $* \xrightarrow{p} Y \xleftarrow{f} X$ is the homotopy fibre of f over the basepoint p in Y.

The usual fibre product of topological spaces does not respect homotopy equivalences. The homotopy fibre product is invariant under homotopy equivalence: if we replace f and g by homotopic maps then the weak homotopy type of $X \times_Z^h Y$ does not change.

Remark 2.2.5. Another nice property of the homotopy fibre product is that we have a long exact sequence of homotopy groups

$$\cdots \to \pi_n(X \times_Z^h Y) \to \pi_n(X) \times \pi_n(Y) \to \pi_n(Z) \to \cdots \to \pi_0(X) \times \pi_0(Y)$$

Proposition 2.2.6. *The homotopy fibre product $X \times_Z^h Y$ comes with two canonical projection maps to X and Y making the diagram*

$$\begin{array}{ccc} X \times_Z^h Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

commute up to canonical homotopy. Moreover if the square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

*is homotopy commutative then there is a unique up to homotopy map $W \to X \times_Z^h Y$ making the two triangles obtained strictly commutative. For this reason we often call $X \times_Z^h Y$ the **homotopy pullback** of X and Y along f and g.*

[8]For technical reasons we need to work with a 'convenient category' of spaces. For example we can use CGH (compactly generated Hausdorff) spaces as in [5] or CGWH (compactly generated weak Hausdorff) spaces as in [6]. For concreteness we may suppose that all topological spaces are CGWH.

Definition 2.2.7. A homotopy commutative square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is **homotopy Cartesian** (or just **h-Cartesian**) if there is a weak homotopy equivalence $W \to X \times^h_Z Y$ such that the triangles obtained are strictly commutative.

Neither

$$\text{h-Cartesian} \implies \text{Cartesian}$$

nor

$$\text{Cartesian} \implies \text{h-Cartesian}$$

is true in general! If our maps are sufficiently nice (e.g. if $X \to Z$ is a fibration) then a Cartesian square is homotopy Cartesian. In this situation we can compute the homotopy fibre product by computing the usual fibre product.

2.3 Segal spaces

Definition 2.3.1 (Rezk)**.** A simplicial topological space $X_\bullet$ is a **Segal space** if for all $m, n \geq 0$ the diagram

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

is h-Cartesian. We can equivalently specify that for all n the **Segal maps**

$$X_n \to \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{n}$$

are weak homotopy equivalences.

Remark 2.3.2. This is not a universally accepted definition. Some authors, for example [2], specify in addition that $X_\bullet$ should be **Reedy fibrant**, a 'niceness' condition on simplicial spaces that ensures that the homotopy pullback $X_m \times^h_{X_0} X_n$ is the usual pullback $X_m \times_{X_0} X_n$. In this case it's enough to demand that $X_{m+n} \to X_m \times_{X_0} X_n$ is a weak homotopy equivalence. Reedy fibrancy is a technical condition that can always be satisfied. For more on model category theory and the definition of Reedy fibrancy, the reader can consult e.g. Appendix A.2 of [7].

What do Segal spaces have to do with $(\infty, 1)$-categories? Let's suppose for the moment that we already have a good theory of $(\infty, 1)$-categories. Just like a 1-category has an underlying groupoid, obtained by throwing away all of the noninvertible morphisms, an $(\infty, 1)$-category should have an underlying ∞-groupoid obtained in the same way:

Idea 2.3.3. Let $\mathcal{C}$ be any $(\infty, 1)$-category. We can loosely define the **underlying ∞-groupoid** of $\mathcal{C}$, which I will denote $\pi_{\leq\infty}(\mathcal{C})$, to be the ∞-groupoid with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{invertible 1-morphisms in } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{2-morphisms between invertible 1-morphisms of } \mathcal{C} \\ &\vdots \end{aligned}$$

Since we can identify ∞-groupoids with topological spaces, we may think of $\pi_{\leq\infty}(\mathcal{C})$ as a topological space $B_0\mathcal{C}$ which we refer to as the **classifying space for objects of $\mathcal{C}$**. Note that by definition the fundamental ∞-groupoid of $B_0\mathcal{C}$ is the underlying ∞-groupoid of $\mathcal{C}$.

Clearly $B_0\mathcal{C}$ should not in general encode all of the information about $\mathcal{C}$. For example it doesn't know about noninvertible morphisms or how composition works. We can extend the above definition to get classifying spaces for n-morphisms of $\mathcal{C}$ (since we can think of an object as a 0-morphism), and hopefully this collection of classifying spaces should allow us to recover $\mathcal{C}$.

Idea 2.3.4. Let $[n]$ be the 1-category associated to the ordered set $\{0, 1, 2 \ldots, n\}$. Let $\mathcal{C}$ be an $(\infty, 1)$-category. We can think of an n-morphism in $\mathcal{C}$ as a functor $[n] \to \mathcal{C}$. The collection $\mathrm{Fun}([n], \mathcal{C})$ of functors $[n] \to \mathcal{C}$ itself naturally has the structure of an $(\infty, 1)$-category, so it has an underlying ∞-groupoid $\pi_{\leq\infty}(\mathrm{Fun}([n], \mathcal{C}))$. Let $B_n\mathcal{C}$ be the topological space associated to this ∞-groupoid. We call $B_n\mathcal{C}$ the **classifying space for n-morphisms in $\mathcal{C}$**. Again, by definition the fundamental ∞-groupoid of $B_n\mathcal{C}$ is the underlying ∞-groupoid of $\mathrm{Fun}([n], \mathcal{C})$.

What kind of object should the collection $B_\bullet\mathcal{C}$ be? Moreover, to what extent does it determine $\mathcal{C}$? The answer to the first question is that $B_\bullet\mathcal{C}$ should be a simplicial space, and moreover a Segal space. The Segal conditions formalise the idea that giving a chain

$$A_0 \to A_1 \to \cdots \to A_{n+m}$$

of composable morphisms should be equivalent to giving two chains

$$A_0 \to \cdots \to A_n \qquad\qquad A_n \to \cdots \to A_{n+m}$$

and moreover that it should not matter where we break the chains. To answer the second question, we can try to define an 'inverse' to the operation $\mathcal{C} \to B_\bullet\mathcal{C}$ and see if we need to add any extra data to a general Segal space in order to extract an ∞-category.

Idea 2.3.5. Given a Segal space $X_\bullet$ we should be able to construct an $(\infty, 1)$-category $\mathcal{C}(X_\bullet)$ which has

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X_0 \\ \text{Mapping spaces } \mathrm{Map}(x,y) &\longleftrightarrow \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\} \\ \text{composition law} &\longleftrightarrow \text{given by } X_2 \\ \text{higher associativity information} &\longleftrightarrow \text{given by } X_3, X_4 \ldots \end{aligned}$$

Observe that the connected components of the space $\mathrm{Map}(x, y)$ should be precisely the homotopy classes of 1-morphisms in $\mathcal{C}(X_\bullet)$. With this in mind, we can construct a 1-category from a Segal space:

Definition 2.3.6. The **homotopy category** $hX_\bullet$ of a Segal space $X_\bullet$ is the category whose objects are the points of X_0, and whose homsets are

$$\begin{aligned} \mathrm{Hom}_{hX_\bullet}(x,y) :&= \pi_0(\mathrm{Map}(x,y)) \\ &= \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\}) \end{aligned}$$

Remark 2.3.7. The homotopy category of $X_\bullet$ records some of the basic information about $\mathcal{C}(X_\bullet)$ - it knows what the objects should be, for example - but it forgets all of the homotopical information by identifying all homotopic maps. It can be thought of as a 1-categorical 'flattening' of the $(\infty, 1)$-category $\mathcal{C}(X_\bullet)$.

2.4 Completeness

If we start with a general Segal space $X_\bullet$, how does it compare to the Segal space $Y_\bullet := B_\bullet(\mathcal{C}(X_\bullet))$? The fundamental groupoid of Y_0 is the classifying space for 0-morphisms of $\mathcal{C}(X_\bullet)$. This receives a map from the fundamental groupoid of X_0 but this map is not necessarily an equivalence, since there may be invertible 1-morphisms in $\mathcal{C}(X_\bullet)$ which do not come from paths in X_0. We'd like to impose an extra condition on our Segal spaces which ensures that every invertible 1-morphism in $\mathcal{C}(X_\bullet)$ comes from an essentially unique path in X_0.

Definition 2.4.1. Let p_i be the map from $[0]$ to $[1]$ given by $0 \mapsto i$. For any Segal space $X_\bullet$ write $p_i^* : X_1 \to X_0$ for the map induced by p_i. If $f \in X_1$ then write $x := p_0^*(f)$ and $y := p_1^*(f)$ so that we can think of f as a path from x to y. The map $\{f\} \to \{x\} \times_{X_0} X_1 \times_{x_0} \{y\} \to \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\}$ determines an element $[f]$ of $\mathrm{Hom}_{hX_\bullet}(x, y) = \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\})$. Say that f is **invertible** if $[f]$ is an isomorphism.

Example 2.4.2. If $X_\bullet$ is a Segal space let $\delta : X_0 \to X_1$ be the map induced by the unique map $[1] \to [0]$. Then for every $x \in X_0$, the map $[\delta(x)]$ is the identity map id_x in the homotopy category. So $\delta(x)$ is invertible for every x.

Definition 2.4.3. If $Z \subseteq X_1$ is the subspace of invertible elements of a Segal space $X_\bullet$, then say that $X_\bullet$ is **complete** if $\delta : X_0 \to Z$ is a weak homotopy equivalence.

So a complete Segal space is one where every isomorphism in $\mathcal{C}(X_\bullet)$arises from an essentially unique path in X_0. In fact, if $X_\bullet$ is complete then we should have an equivalence $X_\bullet \cong B_\bullet(\mathcal{C}(X_\bullet))$.

Remark 2.4.4. If one is more careful and starts with a rigorous axiomatisation of $(\infty, 1)$-categories then the above assertions and intuitive ideas can be turned into theorems. This was done by Toën in [9].

We've seen that the well-defined theory of complete Segal spaces should correspond to the as-yet-undefined theory of $(\infty, 1)$-categories. With this in mind, we make the following rather bold definition:

Definition 2.4.5. An $(\infty, 1)$**-category** is a complete Segal space.

Proposition 2.4.6 (Rezk)**.** *Any Segal space $X_\bullet$ has a completion; i.e. admits a homotopy universal morphism*[9] *$X_\bullet \to Y_\bullet$ where $Y_\bullet$ is complete. In general $Y_\bullet$ is unique up to homotopy and we refer to it as the* ***completion*** *of $X_\bullet$, denoted $\hat{X}_\bullet$. The map $X_\bullet \to \hat{X}_\bullet$ is functorial.*

Remark 2.4.7. Complete Segal spaces are the fibrant objects of a suitable model structure on the category of simplicial spaces, just as quasicategories are the fibrant objects of the Joyal model structure on the category of simplicial sets.

3 (∞, n)-categories as n-fold complete Segal spaces

Now we have a definition of $(\infty, 1)$-category as a certain functor $\Delta^{\mathrm{op}} \to \mathbf{Top}$, we're going to generalise this and define an (∞, n)-category as a certain functor $(\Delta^{\mathrm{op}})^{\times n} \to \mathbf{Top}$.

Definition 3.1.1. An n**-fold simplicial object** in a category C is a functor

$$\underbrace{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}}_{n} \to C$$

Example 3.1.2. A 0-fold simplicial object is an object. A 1-fold simplicial object is just a simplicial object in the usual sense.

In general an n-fold simplicial object in a category C is a collection $X_{i_1 \cdots i_n}$ of objects of C indexed by n-tuples of nonnegative integers $\underline{i} = (i_1, \ldots, i_n)$ along with a collection of face and degeneracy maps. We'll always use an underbar to denote multiindices in this manner. We think of n-fold simplicial objects as having n 'directions' in which to compose.

Definition 3.1.3. An n**-fold simplicial space** is an n-fold simplicial object in the category $\mathbf{Top}$.

[9] A morphism of Segal spaces is a morphism of the underlying simplicial spaces.

Definition 3.1.4. A map $X \to Y$ of n-fold simplicial spaces is a **weak homotopy equivalence** if all of the maps $X_{\underline{i}} \to Y_{\underline{i}}$ are weak homotopy equivalences.

Definition 3.1.5. A diagram

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

of n-fold simplicial spaces is **homotopy Cartesian** if for all multiindices $\underline{i}$ the square

$$\begin{array}{ccc} W_{\underline{i}} & \longrightarrow & Y_{\underline{i}} \\ \downarrow & & \downarrow \\ X_{\underline{i}} & \longrightarrow & Z_{\underline{i}} \end{array}$$

is homotopy Cartesian.

Definition 3.1.6. An n-fold simplicial space X is **essentially constant** if it's weakly homotopy equivalent to a constant n-fold simplicial space.

Via currying, whenever $n > 0$ we can always think of an n-fold simplicial object in C as a simplicial object in the category of $(n-1)$-fold simplicial objects in C. This idea will form the basis of our inductive definition of an n-fold complete Segal space.

Definition 3.1.7. For $n > 0$ an n-fold simplicial space X, thought of as a simplicial object in the category of $(n-1)$-fold simplicial spaces, is said to be an **n-fold Segal space** if the following conditions are met:

i) Every X_k is an $(n-1)$-fold Segal space.

ii) For all m and l the diagram

$$\begin{array}{ccc} X_{m+l} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_l & \longrightarrow & X_0 \end{array}$$

is a homotopy Cartesian square of $(n-1)$-fold simplicial spaces.

iii) X_0 is an essentially constant $(n-1)$-fold simplicial space.

Moreover, we say that an n-fold Segal space is **complete** if

iv) Each X_k is a complete $(n-1)$-fold Segal space.

v) The Segal space $Y_\bullet = X_{\bullet,0,0,\ldots,0}$ is complete.

Definition 3.1.8. An (∞, n)**-category** is a complete n-fold Segal space.

Proposition 3.1.9. *Any n-fold Segal space has a completion.*

Loosely, an n-fold complete Segal space is a 'fattened' or 'spread out' version of an (∞, n)-category. Some illuminating diagrams are given in §2.2.1 of [2].

4 The (∞, n)-category $\mathbf{Bord}_n$

In this section we'll code up our ideas about $\mathbf{Bord}_n$ to define an n-fold simplicial space $\mathbf{PBord}_n$. We'll indicate how this is an n-fold Segal space that in general is not complete. Then we can define the (∞, n)-category $\mathbf{Bord}_n$ to be the completion $\widehat{\mathbf{PBord}}_n$ of $\mathbf{PBord}_n$. Our exposition will be fairly informal; for a more rigorous explanation see §2 of [2].

4.1 The level sets $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

We want to think of $(\mathbf{PBord}_n)_{(k_1,\ldots,k_n)}$ as a collection of $k_1 k_2 \cdots k_n$ composed cobordisms, with k_i cobordisms in the i^{th} direction.

Idea 4.1.1. Cobordisms are easier to deal with when we consider them as submanifolds of some large $\mathbb{R}^m$. So we'll define sets of cobordisms living in $\mathbb{R}^m$ for varying m, and then take a limit over m. Whitney's embedding theorem will ensure that we get all of the cobordisms, since every l-dimensional manifold can be embedded in $\mathbb{R}^{2l}$.

Definition 4.1.2. Let V be a finite-dimensional real vector space and fix a multiindex $\underline{k} = (k_1, \ldots, k_n)$. Define $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ to be the set of tuples

$$(M, (t_0^i, \ldots, t_{k_i}^i)_{i=1\ldots n})$$

satisfying the following:

i) For each $1 \leq i \leq n$, $t_0^i \leq \cdots \leq t_{k_i}^i$ is an ordered tuple of $k_i + 1$ real numbers.

ii) M is a closed n-dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is proper[10].

iii) For a subset S of $\{1, \ldots, n\}$ let $p_S : M \to \mathbb{R}^S$ denote the composition $M \xrightarrow{\pi} \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$. Then we require that for every $1 \leq i \leq n$ and every $0 \leq j \leq k_i$, that for all $x \in p_{\{i\}}^{-1}\left(t_j^i\right)$, the map $p_{\{i,\ldots,n\}}$ is submersive[11] at x.

Remark 4.1.3. What's the motivation behind this definition? If we want to think of M as being a collection of composed cobordisms, the numbers t_j^i record the 'cutting points' where we glue two cobordisms together. So the region of M between the hyperplanes corresponding to t_j^i and t_{j+1}^i should be the $(j+1)^{\text{st}}$ cobordism glued in the i^{th} direction.

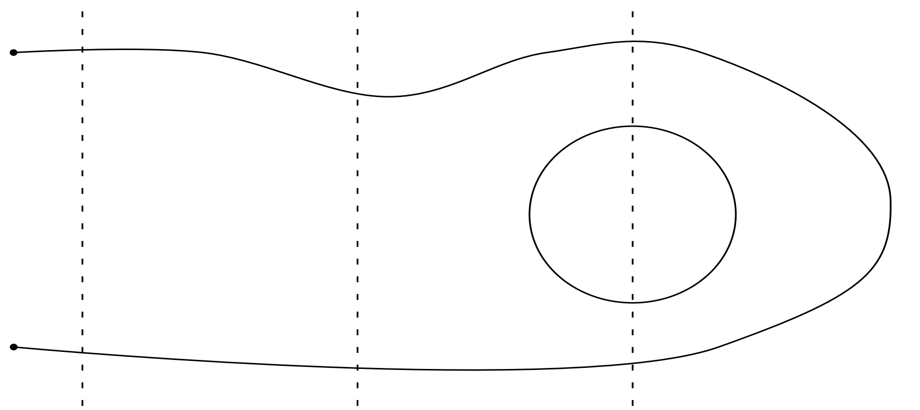
[10] A map is **proper** if preimages of compact sets are compact.

[11] A map $f : M \to N$ is **submersive at** $m \in M$ if the differential $\mathrm{d}f_x : T_xM \to T_xN$ is surjective. A map is **submersive** if it's submersive at every point of its domain.

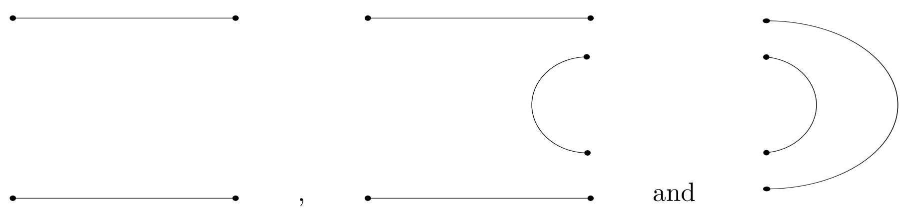
Condition iii) says that in particular the set $p_{\{n\}}^{-1}\left(t_j^n\right)$ is an $(n-1)$-dimensional submanifold that we can think of as one of the $(n-1)$-cobordisms that we glue together to get M.

Furthermore the set $p_{\{n-1,n\}}^{-1}\left\{t_{j_{n-1}}^{n-1}, t_{j_n}^{n}\right\}$ is an $(n-2)$-dimensional manifold that is one of the $(n-2)$-cobordisms joined by an $(n-1)$-cobordism. Similarly, the preimage $p_{\{m,\ldots,n\}}^{-1}\left\{t_{j_m}^{m}, \ldots, t_{j_n}^{n}\right\}$ is an $(m-1)$-dimensional manifold that we can loosely think of as one of our $(m-1)$-morphisms.

Example 4.1.4. Here is an element of $\mathbf{PBord}_1^{\mathbb{R}}$:



The cutting points indicated by the dotted lines allow us to view this as a composition of the three 1-cobordisms



4.2 The topological spaces $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

Fact 4.2.1 ([10], Chapter II). The set $\mathrm{Emb}(X,Y)$ of smooth embeddings of a smooth manifold X into a smooth manifold Y has a topology, the **Whitney** C^∞ **topology**.

Theorem 4.2.2 ([11]). *The space* $\mathrm{Sub}(V \times \mathbb{R}^n)$ *of closed n-dimensional submanifolds of* $V \times \mathbb{R}^n$ *can be identified with the space*

$$\bigsqcup_{[L]} \mathrm{Emb}(L, V \times \mathbb{R}^n)/\mathrm{Diff}(L)$$

where the disjoint union is taken over diffeomorphism classes of n-dimensional manifolds L. Moreover the topology on $\mathrm{Sub}(V \times \mathbb{R}^n)$ *has neighbourhood basis at* $M \subseteq V \times \mathbb{R}^n$ *the sets*

$$\{N \subseteq V \times \mathbb{R}^n : N \cap K = f(M) \cap K \text{ for all } f \in W\}$$

where K is a compact subset of $V \times \mathbb{R}^n$ and $W \subseteq \mathrm{Emb}(M, V \times \mathbb{R}^n)$ *is a neighbourhood (in the Whitney C^∞ topology) of the inclusion $M \hookrightarrow V \times \mathbb{R}^n$.*

Remark 4.2.3. The space $\mathrm{Sub}(V \times \mathbb{R}^n)$ is sometimes denoted by $\Psi(V \times \mathbb{R}^n)$.

Since we can view $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ as a subset of $\mathrm{Sub}(V \times \mathbb{R}^n) \times \mathbb{R}^k$, for some $k \in \mathbb{N}$ depending only on $\underline{k}$, we can give $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ the subspace topology.

4.3 The n-fold simplicial space $\mathbf{PBord}_n$

Proposition 4.3.1. *There are face and degeneracy maps making the collection of spaces*

$$\left\{\left(\mathbf{PBord}_n^V\right)_{\underline{k}} : \underline{k} \in \mathbb{N}^n\right\}$$

into an n-fold simplicial space.

Call this n-fold simplicial space $\mathbf{PBord}_n^V$. Loosely, the face maps forget a number t_j^i whereas the degeneracy maps repeat a number t_j^i.

Now we can remove the dependence on the vector space V. Let $\mathbb{R}^\infty$ be the unique real vector space of countably infinite dimension. We define the n-fold simplicial space $\mathbf{PBord}_n$ to be the limit

$$\mathbf{PBord}_n := \varinjlim_{V \subseteq \mathbb{R}^\infty} \mathbf{PBord}_n^V$$

4.4 The n-fold Segal spaces $\mathbf{PBord}_n$ and $\mathbf{Bord}_n$

We need to verify that the n-fold simplicial space $\mathbf{PBord}_n$is in fact an n-fold Segal space. The first point is to prove that the Segal map

$$\begin{array}{c} (\mathbf{PBord}_n)_{k_1,\ldots,k_i+k_i',\ldots,k_n} \\ \downarrow \\ (\mathbf{PBord}_n)_{k_1,\ldots,k_i,\ldots,k_n} \underset{(\mathbf{PBord}_n)_{k_1,\ldots,0,\ldots,k_n}}{\times^h} (\mathbf{PBord}_n)_{k_1,\ldots,k_i',\ldots,k_n} \end{array}$$

is a weak homotopy equivalence; loosely, this is true since one can glue submanifolds along compatible intersections in an appropriate homotopical sense. The second point is to prove that the degeneracy maps are weak equivalences. In fact, they are deformation retracts, essentially because one can homotope the cut points around. This proves:

Theorem 4.4.1. $\mathbf{PBord}_n$ *is an n-fold Segal space.*

The n-fold Segal space $\mathbf{PBord}_n$ is not in general complete. We define $\mathbf{Bord}_n := \widehat{\mathbf{PBord}_n}$ to be its completion. Then $\mathbf{Bord}_n$ is an (∞, n)-category.

Remark 4.4.2. The spaces $\mathbf{PBord}_1$ and $\mathbf{PBord}_2$ **are** complete. However, for $n \geq 6$, $\mathbf{PBord}_n$ is **not** complete; this is because not all invertible cobordisms $M \to N$ arise from diffeomorphisms $M \to N$. The **s-cobordism theorem** says that for $n \geq 6$, this statement is equivalent to the vanishing of an invariant of the cobordism known as the **Whitehead torsion**. It's known that for $n \geq 6$ that there are n-bordisms which have nontrivial Whitehead torsion, and hence that $\mathbf{PBord}_n$ is not complete.

4.5 Extra structure on $\mathbf{Bord}_n$

Most importantly, $\mathbf{Bord}_n$ is a **symmetric monoidal** (∞, n)**-category**, which means that it has a symmetric monoidal structure (given by the disjoint union) compatible with the (∞, n) structure.

We can also restrict to cobordisms with certain properties: for example there is an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ of framed cobordisms, and an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{or}}$ of oriented cobordisms. Both of these categories also carry a symmetric monoidal structure.

The (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ of framed cobordisms will be our focus from now on, since the Cobordism Hypothesis is stated in terms of framed cobordisms.

Note on the constructions The construction of $\mathbf{Bord}_n$ outlined above is similar to Lurie's definition in [1]. Lurie's original definition contained an error, and this was corrected by Calaque and Scheimbauer in [2] (which consists mainly of material from [3]) who construct their spaces differently.

They first construct a Segal space of intervals in $\mathbb{R}^n$ and then lift this Segal space structure to $\mathbf{PBord}_n$. The definitions in [2] correspond roughly to the definitions here by taking our t_j^i to be points in their intervals.

5 Adjoints and dualisablity

Given a topological field theory $Z : \mathbf{Bord}_n^{\mathrm{fr}} \to \mathcal{C}$, we'd like to classify the kind of objects of $\mathcal{C}$ that could be the image of the 0-manifold $*$ under Z. Such objects should satisfy some finiteness condition: for example when $n = 1$ and

$\mathcal{C} = \mathbf{Vect}_k$ we saw that $Z(*)$ had to be finite-dimensional, and conversely that any finite-dimensional vector space can be obtained as the image of $*$ under some TFT.

The correct analogue of finite-dimensionality in the ∞-categorical setting is **full dualisability**, and to define this is the goal of the current section.

It turns out that requiring dualisability for objects is not enough: we'll also need a notion of dualisability for k-morphisms as well. In the 2-category **Cat** we already have a reasonable notion of dualisability for 1-morphisms: a left dual (if it exists) for a functor $F : C \to D$ should be its left adjoint $G : D \to C$. We extend this definition to general 2-categories and then to general (∞, n)-categories. Adjoints and duals are very closely related in higher categories.

All of this section is from [1].

5.1 Duals for objects

Recall the following 1-categorical definition:

Definition 5.1.1. Let C be a monoidal category. Let V be an object of C. A **right dual** for V is the data of an object $V^\vee$ and maps

$$\begin{aligned} \mathrm{ev} : V \otimes V^\vee &\to 1 \qquad \text{the **evaluation map**} \\ \mathrm{coev} : 1 &\to V^\vee \otimes V \qquad \text{the **coevaluation map**} \end{aligned}$$

such that the triangles[12]

$$\begin{array}{ccc} V & & \\ \downarrow{\scriptstyle \mathrm{id}_V \otimes \mathrm{coev}} & \searrow^{\mathrm{id}_V} & \\ V \otimes V^\vee \otimes V & \xrightarrow[\mathrm{ev} \otimes \mathrm{id}_V]{} & V \end{array} \qquad \begin{array}{ccc} V^\vee & & \\ \downarrow{\scriptstyle \mathrm{coev} \otimes \mathrm{id}_{V^\vee}} & \searrow^{\mathrm{id}_{V^\vee}} & \\ V^\vee \otimes V \otimes V^\vee & \xrightarrow[\mathrm{id}_{V^\vee} \otimes \mathrm{ev}]{} & V^\vee \end{array} \tag{1}$$

commute. We also say in this situation that V is a **left dual** of $V^\vee$.

Remark 5.1.2. If C is symmetric monoidal, then the notions of right dual and left dual coincide and we refer simply to the **dual**.

Example 5.1.3. If C is the symmetric monoidal category $\mathbf{Vect}_k$ (with monoidal structure given by the usual tensor product over k) then a vector space V has a dual if and only if it is finite-dimensional. More specifically, we can always define a space $V^* = \mathrm{Hom}(V, k)$ and an evaluation map $V \otimes_k V^* \to k$, but we can only define a compatible coevaluation map if V is finite-dimensional.

Proposition 5.1.4. *Left and right duals (if they exist) are unique up to unique isomorphism.*

[12] These triangles are usually presented as pentagons; here we have ignored the associators and the isomorphisms $X \otimes 1 \xrightarrow{\sim} X \xleftarrow{\sim} 1 \otimes X$.

We can easily extend the definition of a dualisable object to higher categories, by taking the homotopy category.

Definition 5.1.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Say that an object X of $\mathcal{C}$ is **dualisable** if it admits a dual when considered as an object of the homotopy category $h\mathcal{C}$.

If Z is an oriented or framed topological field theory with target $\mathcal{C}$, then any object X of $\mathcal{C}$ with $X = Z(*)$ must be dualisable since we can obtain $X^\vee$ by evaluating Z on a point with the opposite orientation to that of $*$. In general the condition that X be dualisable is not strong enough for such a TFT to exist. However, for $n = 1$ it turns out that dualisability is sufficient, so this problem will only manifest itself in higher dimensions.

In general we should require that the morphisms in $\mathcal{C}$ should also have duals, which leads us to the notion of adjoints.

5.2 Adjoints in 2-categories

Recall the unit-counit definition of an adjunction:

Definition 5.2.1. Let C, D be two categories and $F : C \to D$ and $G : D \to C$ two functors. An **adjunction** between F and G consists of two natural transformations

$$\begin{aligned} u &: \mathrm{id}_C \Rightarrow GF && \text{the } \textbf{unit} \\ v &: FG \Rightarrow \mathrm{id}_D && \text{the } \textbf{counit} \end{aligned}$$

such that the following two triangles[13] commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_C & & \\ {\scriptstyle \mathrm{id}_F \times u}\downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \times \mathrm{id}_F]{} & \mathrm{id}_D \circ F \end{array} \qquad \begin{array}{ccc} \mathrm{id}_C \circ G & & \\ {\scriptstyle u \times \mathrm{id}_G}\downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \times v]{} & G \circ \mathrm{id}_D \end{array} \tag{2}$$

In this situation we say that F is a **left adjoint** of G and that G is a **right adjoint** of F.

Note that the expression $\eta \times \theta$ means the horizontal composition of the natural transformations η and θ rather than the vertical composition.

Remark 5.2.2. Observe the formal similarity of the triangles of equation (2) to the ones of equation (1). This is a good indication that adjoints are 'higher duals'.

[13] Once again, these triangles are really pentagons. If we think of **Cat** as a strict 2-category, then they are squares since we don't need any associators.

Proposition 5.2.3. *Adjoints, if they exist, are unique up to unique isomorphism.*

The category **Cat** is the prototypical example of a 2-category: the objects of **Cat** are all (small) categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations. Definition 5.2.1 didn't really rely on any of the properties of **Cat**, and so we can immediately generalise it to any 2-category:

Definition 5.2.4. Let $\mathcal{C}$ be any 2-category. Let X, Y be objects of $\mathcal{C}$ and let $F : X \to Y$ and $G : Y \to X$ be two 1-morphisms. Say that a 2-morphism $u : \mathrm{id}_X \to G \circ F$ is the **unit of an adjunction between** F **and** G if there exists another 2-morphism $v : F \circ G \to \mathrm{id}_Y$ such that the following two triangles commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_X & & \\ {\scriptstyle \mathrm{id}_F \times u}\downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \times \mathrm{id}_F]{} & \mathrm{id}_Y \circ F \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{id}_X \circ G & & \\ {\scriptstyle u \times \mathrm{id}_G}\downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \times v]{} & G \circ \mathrm{id}_Y \end{array}$$

In this case we say that v is the **counit**, and that F (resp. G) is **left** (resp. **right**) adjoint to G (resp. F).

Remark 5.2.5. If u is the unit of an adjunction, then a compatible counit v is uniquely determined, and vice versa. So it's enough to specify the existence of either u or v.

Example 5.2.6. A category with a single object is the same thing as a monoid. Similarly if $\mathcal{C}$ is a 2-category with a single object $*$ then the category $\mathrm{Hom}_{\mathcal{C}}(*, *)$ is a monoidal category.

Conversely if M is a monoidal category then we can build a 2-category $\mathcal{B}M$ with a single object *, hom-category $\mathrm{Hom}_{\mathcal{B}M}(*, *) \cong M$ and composition law for 1-morphisms given by the tensor product on M.

Then an object X of M is right dual to an object Y of M if and only if it is right adjoint to Y when both are considered as 1-morphisms of $\mathcal{B}M$. We often call $\mathcal{B}M$ the **delooping** of M.

Adjoints are closely related to invertibility:

Proposition 5.2.7. *Let $\mathcal{C}$ be a 2-category in which every 2-morphism is invertible. Let f be a 1-morphism of $\mathcal{C}$. Then the following are equivalent:*

i) f is invertible.

ii) f admits a left adjoint.

iii) f admits a right adjoint.

Definition 5.2.8. Say that a 2-category $\mathcal{C}$ **has adjoints for 1-morphisms** if every 1-morphism has both a left and a right adjoint.

5.3 Adjoints in higher categories

We'd like to generalise Definition 5.2.4 from 2-categories to higher categories.

Definition 5.3.1. Let $n \geq 2$ and let $\mathcal{C}$ be an (∞, n)-category. Let $h_2\mathcal{C}$ be the **homotopy 2-category** of $\mathcal{C}$, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{1-morphisms of } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{isomorphism classes of 2-morphisms of } \mathcal{C} \end{aligned}$$

Remark 5.3.2. **Homotopy n-categories** are defined similarly.

Definition 5.3.3. Let $\mathcal{C}$ be an (∞, n)-category. Say that $\mathcal{C}$ **has adjoints for 1-morphisms** if $h_2\mathcal{C}$ has adjoints for 1-morphisms. More generally, for $1 < k < n$ say that $\mathcal{C}$ **has adjoints for k-morphisms** if for any two objects X, Y of $\mathcal{C}$ the $(\infty, n-1)$-category $\mathrm{Map}(X, Y)$ has adjoints for $(k-1)$-morphisms. Say that $\mathcal{C}$ **has adjoints** if it has adjoints for k-morphisms for all $0 < k < n$.

Remark 5.3.4. If every k-morphism in $\mathcal{C}$ is invertible then $\mathcal{C}$ has adjoints for k-morphisms. The converse is true provided that all $(k+1)$-morphisms are invertible - this is a consequence of Proposition 5.2.7.

Remark 5.3.5. The condition that $\mathcal{C}$ have adjoints depends on the choice of n. If we regard $\mathcal{C}$ as an $(\infty, n+1)$-category with all $(n+1)$-morphisms invertible then in general $\mathcal{C}$ does not have adjoints for n-morphisms unless it is an ∞-groupoid.

If $\mathcal{C}$ is monoidal then we can ask for a bit more:

Definition 5.3.6. Let $\mathcal{C}$ be a monoidal (∞, n)-category. Say that $\mathcal{C}$ **has duals** if the following two conditions are satisfied:

i) Every object X has both a left and a right dual when considered as an object of the homotopy category $h\mathcal{C}$.[14]

ii) $\mathcal{C}$ has adjoints.

Remark 5.3.7. We can generalise our earlier construction of Example 5.2.6. If $\mathcal{C}$ is a monoidal (∞, n)-category then it is possible to build an $(\infty, n+1)$-category $\mathcal{BC}$ (the **delooping** of $\mathcal{C}$) with a single object $*$, recovering $\mathcal{C}$ as the mapping object $\mathrm{Hom}_{\mathcal{BC}}(*, *)$. Then $\mathcal{C}$ has duals if and only if $\mathcal{BC}$ has adjoints.

5.4 Full dualisability

Given a symmetric monoidal (∞, n)-category we'd like to pick out the largest subcategory with duals.

[14] Note that $h\mathcal{C}$ inherits its monoidal structure from $\mathcal{C}$. If $\mathcal{C}$ is symmetric monoidal then this condition is the condition that every object be dualisable.

Theorem 5.4.1. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then there exists another symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ with duals, and a symmetric monoidal functor $i : \mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$, universal among symmetric monoidal functors $j : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ has duals.*

Remark 5.4.2. $\mathcal{C}^{\mathrm{fd}}$ is determined up to equivalence by the above properties. In general we can obtain $\mathcal{C}^{\mathrm{fd}}$ from $\mathcal{C}$ by repeatedly discarding morphisms that don't admit adjoints (and objects that don't admit duals).

Example 5.4.3. If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category then $\mathcal{C}^{\mathrm{fd}}$ is equivalent to the full subcategory of $\mathcal{C}$ spanned by the dualisable objects.

Definition 5.4.4. Say that an object X of $\mathcal{C}$ is **fully dualisable** if it belongs to the essential image[15] of the functor i.

Example 5.4.5. For each $n > 0$, the (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ has duals. Every k-morphism can be identified with an oriented manifold M; the morphism $\bar{M}$ (M with the opposite orientation) is both a left and a right adjoint to M.

Example 5.4.6. If $\mathcal{C}$ is the $(\infty, 1)$-category $\mathbf{Vect}_k$, then an object of $\mathcal{C}$ is fully dualisable if and only if it is finite-dimensional.

This generalises to the following:

Proposition 5.4.7. *An object of a symmetric monoidal $(\infty, 1)$-category is fully dualisable if and only if it is dualisable.*

In general full dualisability is a much stronger condition than dualisability! In dimension 2, there are some simple criteria for testing whether or not an object is fully dualisable:

Proposition 5.4.8. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Let X be an object of $\mathcal{C}$. Then X is fully dualisable if and only if it admits a dual $X^\vee$ and the evaluation map $\mathrm{ev} : X \otimes X^\vee \to 1$ has both a left and a right adjoint.*

6 The Cobordism Hypothesis

In this short section we rigourously state the Cobordism Hypothesis. We begin with some bookkeeping.

6.1 Terminology

Definition 6.1.1. An (∞, n)**-functor** between two (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ is a map of the underlying simplicial spaces (which is itself a natural transformation between the defining functors).

Theorem 6.1.2. *The collection $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of (∞, n)-functors between two (∞, n)-categories itself forms an (∞, n)-category.*

[15] Recall that the **essential image** of a functor $F : \mathcal{D} \to \mathcal{E}$ is the smallest isomorphism-closed subcategory of $\mathcal{E}$ containing the image of F.

Remark 6.1.3. The collection of all (small) (∞, n)-categories naturally forms an $(\infty, n+1)$-category with mapping objects $\mathrm{Map}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proposition 6.1.4. *We can also define **symmetric monoidal** (∞, n)**-functors** between symmetric monoidal (∞, n)-categories. The collection of symmetric monoidal (∞, n)-functors between two symmetric monoidal (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ itself forms an (∞, n)-category, which we refer to as* $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

Definition 6.1.5. A **fully extended framed n-dimensional topological field theory** is a symmetric monoidal (∞, n)-functor with source $\mathbf{Bord}_n^{\mathrm{fr}}$. The collection of all fully extended framed n-TFTs with target $\mathcal{C}$ is the (∞, n)-category $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$.

Definition 6.1.6. Given an (∞, n)-category $\mathcal{C}$, I will denote the underlying $(\infty, 0)$-category[16] of $\mathcal{C}$ by $\pi_{\leq\infty}(\mathcal{C})$. This notation is not standard.

6.2 A Precise Statement

Claim 6.2.1 (the Cobordism Hypothesis)**.** *If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \pi_{\leq\infty}(\mathcal{C}^{\mathrm{fd}})$$

In particular, the Cobordism Hypothesis states that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid, and hence a topological space. We can think of it as a classifying space for fully dualisable objects in $\mathcal{C}$.

It is not too difficult to prove that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid. The hard part of proving the Cobordism Hypothesis is proving that the induced functor is an equivalence. A sketch proof of this is given by Lurie in [1].

[16] a.k.a. ∞-groupoid

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