Singularity categories seminar

Matt Booth

March 28, 2023

This is an informal set of notes for the singularity categories seminar, run in Lancaster in late 2022 and early 2023. I typed these notes, as opposed to the speakers, so all errors introduced are mine.

Contents

1 Introduction

Let A be a ring (not necessarily commutative, but noetherian[1]). Associated to A are many interesting invariants, which tell us about the geometry, homological algebra, representation theory etc. of A. Often one thinks of an invariant as being a number, or maybe a vector space, or at least something reasonably concrete. But one can attach categories (with various kinds of structure on top of them) to A, and consider these as invariants. For example:

- The (abelian) categories **mod**-A and **Mod**-A. These are rather strong invariants: Morita theory tells us that when A and A' have equivalent module categories, then $Z(A) \cong Z(A')$. In the non-affine setting, one has the Gabriel-Rosenberg theorem, which says that if X is a quasi-separated (e.g. noetherian) scheme then X can be recovered from $\mathbf{Coh}(X)$.

- The triangulated category $D^b(\mathbf{mod}\text{-}A)$ and variants; in particular the big unbounded derived category $D(\mathbf{Mod}\text{-}A)$. These are looser invariants that still know about the homological properties of A. Recovery theorems are fewer and far between here. One famous example in the geometric setting is the Bondal-Orlov reconstruction theorem, which says that if X is a smooth projective variety with (anti)ample canonical bundle then $D^b(X)$ recovers X. For a nice exposition of this see [Căl05].

- The triangulated category $\mathbf{per}(A)$ of perfect A-modules, i.e. those complexes of A-modules which are quasi-isomorphic to bounded complexes of finitely generated projectives.

- DG enhancements of the above triangulated categories. Triangulated categories have bad formal properties: for example, the category of triangulated categories doesn't have internal homs. Mapping cones are not functorial. Triangulated categories don't satisfy any reasonable form of geometric descent. Nobody knows how to recover invariants like the Hochschild cohomology $HH^*(A)$ from just the triangulated structure on $D^b(A)$. All of these problems are solved when passing to pretriangulated dg categories. For more on why you should like dg categories, see [Toë11].

- If A is reasonably commutative[2] , one can equip $D(A)$ or $\mathbf{per}(A)$ with the standard monoidal structure given by the (derived) tensor product. This is the starting point for the subject of tensor triangular geometry.

Here is one natural question to ask. As we will see, it leads to a rich theory. It is clear that $\mathbf{per}(A)$ is a subcategory of $D^b(\mathbf{mod}\text{-}A)$. What sort of difference is there between these two things?

[1] The point of this assumption is to make **mod**-A an abelian category. It is enough to assume that A is coherent and work with **coh**-A instead.

[2] E_2 is enough. A more down to earth example is when $A = B \otimes B^{\mathrm{op}}$ is the enveloping algebra of an algebra B, so that $D(B)$ is the derived category of B-bimodules.

Example 1.1. Let k be a field and consider the ring $A \coloneqq k[x]/x^2$. The module k has a projective resolution given by

$$\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A$$

from which it is clear that $\mathrm{Ext}^i(k,k) \cong k$ for all $i \geq 0$. Since perfect complexes must have bounded self-Exts, k cannot be perfect.

So we want to consider the difference between the triangulated category $D^b(\mathbf{mod}\text{-}A)$ and its triangulated subcategory $\mathbf{per}(A)$. In other words, we are interested in the (Verdier) quotient $D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$. We will soon see the following result, whose main ingredient is the Auslander–Buchsbaum–Serre theorem:

Theorem 1.2. *Let A be a commutative noetherian ring of finite Krull dimension. Then A is regular if and only if $D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$ vanishes.*

Recall that every smooth commutative ring is regular[3]. So for example, the ring $\mathbb{C}[x_1, \cdots, x_n]$ is regular, since it is the coordinate ring of complex n-space, which is smooth. The rings $\mathbb{C}[x,y]/xy$, $\mathbb{C}[x,y]/(x^2 - y^3)$, and $\mathbb{C}[x,y]/(x^3 + x^2 - y^2)$ are not smooth, since they are the coordinate rings of the coordinate axes $xy = 0$, the cuspidal cubic $x^2 = y^3$, and the nodal cubic $y^2 = x^2 + x^3$ respectively, all of which have singular points (to see this, either draw a picture or use calculus).

With this in mind we will call $D_{\mathrm{sg}}(A) \coloneqq D^b(\mathbf{mod}\text{-}A)/\mathbf{per}(A)$ the singularity category of A, and regard it as a homological invariant that detects the singularities of A (even when A is noncommutative!). Along the way we will see a purely homological characterisation of smoothness in terms of global dimension[4] - it should already be clear that the existence of finitely generated modules without a bounded projective resolution is an obstruction to the vanishing of $D_{\mathrm{sg}}(A)$.

Remark 1.3. The above motivates our choice of $D^b(\mathbf{mod}\text{-}A)$ as opposed to the sometimes more natural choice of $D(\mathbf{Mod}\text{-}A)$: the quotient $D(\mathbf{Mod}\text{-}A)/\mathbf{per}(A)$ fails to tell us much about the singularities of A, since $D(\mathbf{Mod}\text{-}A)$ is far too big of an object. From the perspective of homotopy theory, we may want A to be a differential graded algebra, in which case $D^b(A)$ is not necessarily well behaved (e.g. A need not be an object of $D^b(A)$). There are some fixes one can make here which we will discuss later.

After this we will do some computations. David outlined a nice computation of $D_{\mathrm{sg}}(k[x]/x^2)$ using AR theory.

In the next part of the seminar we will see two important alternate constructions of the singularity category.

[3]The converse is true if one works over a perfect field, and in particular a field of characteristic zero.

[4]Be warned that there is a terminology clash here. This homological characterisation of smoothness is not equivalent to 'homological smoothness'; i.e. asking that A be perfect as an A-bimodule.

The first description has a representation-theoretic flavour. Suppose that A is (Iwanaga)–Gorenstein; i.e. A has finite injective dimension over itself. Buchweitz [Buc86] noticed that the singularity category of A has a description as the stable category of maximal Cohen–Macaulay modules over A. Recall that a finitely generated A-module X is MCM if $\mathrm{Ext}^i_A(X, A)$ vanishes for $i > 0$ (there is a more general characterisation in terms of depth). Loosely, the stable category of MCM modules is what one gets by taking the category of MCM modules and quotienting out by projective modules. The shift of an MCM module X is its (inverse) syzygy.

The second description has a more geometric flavour. Suppose that $R = k[\![x_1, \ldots, x_n]\!]/f$ is a complete local hypersurface singularity. A matrix factorisation of f is a pair of free finite rank $k[\![x_1, \ldots, x_n]\!]$-modules M and N together with 'differentials' $d : M \to N$ and $d : N \to M$ such that $d^2 = f$. One can organise the collection of matrix factorisations into a category, and after modding out by a suitable notion of homotopy the category of matrix factorisations of f becomes equivalent to the singularity category of R. There is much literature in this direction, which we will mention later.

As a general reference for this part, see [Boo19, Chapter 6] or [Boo21, Sections 4 and 5] and the references contained therein. A good general reference is [Sym22].

1.1 Possible further directions

After this, there are lots of other directions we could go in, which will be dictated by the interests of participants. Here are some possible ones.

- Globalising the singularity category: one can define the singularity category of a scheme completely analogously to our definition- this was first done by Orlov [Orl04], who coined the name "singularity category". Orlov shows that the (idempotent completion of the) singularity category of a reasonable[5] scheme only depends on its formal completion at the singular locus [Orl11]. This allows one to do some nice geometric computations: e.g. the nodal cubic and the coordinate axes in the plane are singular equivalent, since they are (visibly!) analytically locally equivalent near their singular points. There is also an analogous description of the singularity category in terms of MCM sheaves.

- Knörrer periodicity: $f \in k[x_1, \ldots, x_n]$ has the same singularity category as $f + x_{n+1}x_{n+2} \in k[x_1, \ldots, x_{n+2}]$ [Knö87]. A nice proof is in [Dyc11].

[5] Orlov's term is ELF, which stands for 'separated, noetherian, of finite Krull dimension and with Enough Locally Free sheaves'. Recall that a scheme has enough locally free sheaves if for every coherent $\mathcal{F}$ there is an epi $\mathcal{E} \twoheadrightarrow \mathcal{F}$ with $\mathcal{E}$ locally free. This is also known as the resolution property, since this implies that every coherent sheaf has a resolution by vector bundles. Every divisorial scheme, and in particular every quasi-projective variety over a field, has the resolution property.

- Noncommutative computations, in particular those of [Cra20] who computes singularity categories for noncommutative deformations of Kleinian singularites (see also [KY18]).

- Homological mirror symmetry: Landau-Ginzburg models (or rather their categories of matrix factorisations) are mirror to Fano manifolds (or rather their Fukaya categories). There is a lot of literature on this, none of which I am an expert on.

- Relationship to various kinds of Koszul duality: see for example [CT13, Tu14]. For an algebraic approach, Keller and Wang notice that if A is a finite dimensional algebra, then there is a triangle equivalence

$$D_{\text{sg}}(A^{\text{op}})^{\text{op}} \cong \frac{\mathbf{per}(\Omega(A^*))}{\mathbf{thick}(R)}$$

  where R is the quotient of A by its Jacobson radical [CKWW21].

- DG singularity categories: one can enhance everything to the world of (pretriangulated) dg categories. In particular this requires some material on dg quotients, as in [Dri04]. For examples of the fully dg approach, see [BRTV18, Pip22]. See the next item for some applications.

- Hochschild theory: this first requires a brief discussion of dg singularity categories. One can then compute the Hochschild cohomology of D_{sg} of a hypersurface, following [Dyc11]: more or less it is the Milnor algebra of the singularity. Possible discussion of the Mather–Yau theorem [MY82, GP17] in the context of recovery theorems: the dg singularity catgeory is a complete invariant for quasi-homogeneous hypersurface singularities of fixed Krull dimension. As a follow-up, one can go into singular Hochschild cohomology [Kel18] and a stronger recovery theorem where one drops quasi-homogeneity [HK18, Gro92]. Application to the classification of singular threefold flops [Boo21].

- Krause's big singularity categories, and their relationship to Positselski's coderived categories [Bec14]. This links back to the Koszul duality story above. These big singularity categories can e.g. be used to give a definition of what the singularity category of an unbounded differential graded algebra is. See also [GS20] in this direction.

- Representation-theoretic aspects: Kalck and Yang's relative singularity categories [KY16, KY18, KY20]. Singularity categories as generalised cluster categories [HK18, Boo19, Boo21], especially in the smooth 3CY setting where they are controlled by a superpotential [VdB15]. This links back to the threefold flops story.

- Other types of relative singularity categories, in particular those associated to certain morphisms $X \to Y$ of schemes, like Efimov and Positselski's construction [EP15].

2 Triangulated categories

The notion of triangulated category is an axiomatisation of some of the properties that categories such as $D^b(A)$ enjoy. As intimated above, triangulated categories will not be completely sufficient for our uses, so we only give a sketch of the ideas. See [Nee01] for a comprehensive discussion.

Let k be a commutative ring. A k-linear triangulated category is a k-linear category $\mathcal{C}$ together with two extra pieces of data. The first piece of data is a linear autoequivalence Σ of $\mathcal{C}$, which we call the suspension or the shift functor. A triangle in $\mathcal{C}$ is a sequence of three morphisms

$$X \to Y \to Z \to \Sigma X$$

which we will frequently abbreviate by dropping the ΣX term and letting the rightmost arrow point to nowhere. The second piece of data is a class of 'exact' (or 'distinguished') triangles satisfying the following properties:

- TR0: Exact triangles are closed under isomorphisms.
- TR1: The triangle $X \xrightarrow{\text{id}} X \to 0 \to$ is exact.
- TR2: Every morphism $f : X \to Y$ has a cone Z, which fits into an exact triangle of the form $X \to Y \to Z \to$. It follows from the other axioms that cones are unique up to non-unique isomorphism. We caution that the cone is not functorial.
- TR3: One can rotate triangles: the triangle $X \to Y \to Z \to$ is exact if and only if $Y \to Z \to \Sigma X \xrightarrow{-}$ is, where one has to flip the sign on the indicated map.
- TR4: Given a morphism $f \to g$ in the arrow category (i.e. a commutative square from f to g!) then there is an induced morphism $\text{cone}(f) \to \text{cone}(g)$ fitting into a morphism of exact triangles. This morphism will not in general be unique.
- TR5: the famous octahedral axiom. Loosely this encodes the third isomorphism theorem, if one thinks of cones as homotopy cokernels.

These axioms are referred to by various different names in the literature. TR4 is a consequence of the others.

The idea is that an exact triangle behaves like a rolled-up long exact sequence. Indeed in our main example $D^b(A)$, the suspension functor Σ is precisely the shift [1], and the exact triangles are precisely those triangles isomorphic to triangles of the form $X \to Y \to C(f) \to$, where $C(f)$ denotes the usual mapping cone of chain complexes. It is well known that such triangles give long exact sequences of the form

$$\cdots \to H^i(X) \to H^i(Y) \to H^i(C(f)) \to H^{i+1}(X) \to \cdots$$

and indeed we see that the morphism $C(f) \to X[1]$ corresponds precisely to the connecting morphisms in this long exact sequence.

Remark 2.1. Say that a triangulated category has functorial cones if there is a functor $C : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ such that for each f, the object $C(f)$ is a cone of f. Then a triangulated category $\mathcal{C}$ has functorial cones if and only if $\mathcal{C}$ is semisimple abelian. This fact goes back to Verdier's thesis [Ver96, 1.2.13], but Greg Stevenson has given a modern proof [Ste]. The loose idea is that having functorial cones actually forces $\mathcal{C}$ to have kernels and cokernels. Then the claim follows because monos and epis split.

Remark 2.2. Suppose that $\mathcal{C}$ is a stable ∞-category, so that the homotopy category $h_0\mathcal{C}$ is canonically triangulated. In $\mathcal{C}$, one can make a functorial choice of cone, giving a morphism $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. One then obtains a functor $h_0\mathrm{Fun}(\Delta^1, \mathcal{C}) \to h_0\mathcal{C}$. There is a comparison map $h_0\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Ar}(h_0\mathcal{C})$, but it fails to be an equivalence, as spelled out in [Lur]. In particular, the 'functorial cone' does not factor through a morphism $\mathrm{Ar}(h_0\mathcal{C}) \to \mathcal{C}$, and so this does not prove that every enhanceable triangulated category is actually abelian.

A triangle functor (also called an exact functor or a triangulated functor) between triangulated categories is a linear functor which preserves shifts and cones - or equivalently, a functor which preserves exact triangles.

If $\mathcal{D}$ is a triangulated category, then a triangulated subcategory $\mathcal{C}$ is a full additive subcategory which is closed under shifts and cones[6]. This is equivalent to the inclusion functor $\mathcal{D} \to \mathcal{C}$ being a triangle functor. A triangulated subcategory is itself a triangulated category in an obvious way.

Let $\mathcal{D}$ be a triangulated category. A triangulated subcategory $\mathcal{C}$ of $\mathcal{D}$ is called thick (or épaisse) if it is closed under taking direct summands.

Example 2.3. When $\mathcal{D} = D^b(A)$, then $\mathbf{per}(A)$ is the thick closure of A, i.e. the smallest thick subcategory containing the object A.

Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion of a thick subcategory. There exists a triangulated category $\mathcal{D}/\mathcal{C}$, the Verdier quotient, which is initial among all triangulated categories $\mathcal{T}$ equipped with a triangle functor $\mathcal{D} \to \mathcal{T}$ whose kernel contains $\mathcal{D}$. In fact, the kernel of the projection map $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is precisely $\mathcal{C}$.

Loosely, to construct $\mathcal{D}/\mathcal{C}$, the objects are the same as the objects of $\mathcal{D}$, and the morphisms are equivalence classes of roofs $X \leftarrow Y \to Z$ where the cone of the left hand map is in $\mathcal{C}$.

More generally, one can form the quotient $\mathcal{D}/\mathcal{C}$ even when $\mathcal{C}$ is not thick in $\mathcal{D}$; the kernel of the quotient map is then the thick closure of $\mathcal{C}$ inside $\mathcal{D}$.

Example 2.4. Let $K^b(A)$ denote the chain homotopy category, where the morphisms are chain maps up to chain homotopy equivalence. Let $K^b_{\mathrm{ac}}(A)$ denote the subcategory of acyclic complexes; this is a thick subcategory. The Verdier quotient $K^b(A)/K^b_{\mathrm{ac}}(A)$ is precisely the bounded derived category $D^b(A)$.

We can finally define the singularity category.

[6]Note that a full subcategory that is closed under cones is automatically closed under finite direct sums and shifts, due to the existence of the exact triangles $X \to 0 \to \Sigma X \to$ and $X \to X \oplus Y \to Y \xrightarrow{0}$.

Definition 2.5. Let A be a ring. Then the singularity category is the Verdier quotient

$$D_{\mathrm{sg}}(A) \coloneqq \frac{D^b(\mathbf{mod}\text{-}A)}{\mathbf{per}\,(A)}$$

The singularity category comes equipped with a natural projection map $D^b(\mathbf{mod}\text{-}A) \to D_{\mathrm{sg}}(A)$. Note that since $\mathbf{per}\,(A)$ is a thick subcategory, the kernel of this projection map is precisely $\mathbf{per}\,(A)$.

3 Global dimension

Let A be a noetherian ring. If M is an A-module, the projective dimension of M is the minimal length of a projective resolution of M. So the modules of projective dimension 0 are precisely the projective modules. The global dimension of A is defined to be the supremum of the projective dimensions of all finitely generated[7] A-modules.

A priori, there is a left and a right notion of global dimension. However, for two-sided noetherian rings the two concepts agree, and we will use the two notions interchangeably.

Example 3.1. A ring A has global dimension zero if and only if every module is projective. These are precisely the semisimple rings. A commutative semisimple ring is a finite direct product of fields.

Example 3.2. If A has global dimension n, then if M, N are two A-modules, we must have $\mathrm{Ext}^i(M, N) \cong 0$ for $i > n$. In particular, our example earlier shows that when k is a field, the ring $k[x]/x^2$ must have infinite global dimension.

Lemma 3.3. *If A has finite global dimension then $D_{\mathrm{sg}}(A)$ vanishes.*

Proof. Take a bounded complex $M = M_p \to \cdots \to M_q$ of finitely generated modules. By hypothesis, each M_i has a bounded resolution P_i by finitely generated projectives, and moreover each differential $M_i \to M_{i+1}$ lifts to a morphism $P_i \to P_{i+1}$ of complexes. It follows that M is quasi-isomorphic to the totalisation of the double complex $P_p \to \cdots \to P_q$, which is clearly perfect. Hence M is quasi-isomorphic to a perfect complex. So $\mathbf{per}\,(A) = D^b(A)$ and hence $D_{\mathrm{sg}}(A)$ vanishes. □

We now pass to the setting of commutative rings. Recall that if $(R, \mathfrak{m}, k)$ is a commutative noetherian local ring, then it has finite Krull dimension, since there is an inequality $\dim(R) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, the dimension of the cotangent space (this latter number is also equal to the minimal number of generators of $\mathfrak{m}$). Say that R is regular if $\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$; or in other words, the (co)tangent space has the expected dimension. A commutative ring is defined

[7] One may drop this condition, if desired.

to be regular if all of its localisations at maximal (equivalently, prime) ideals are regular. Geometrically, a variety is regular if and only if it is smooth[8].

Theorem 3.4 ("Auslander–Buchsbaum–Serre[9]"). *Let R be a commutative local noetherian ring. The following are equivalent:*

1. *R is regular.*
2. $\mathrm{gldim}(R)$ *is finite.*
3. *$D_{\mathrm{sg}}(R)$ vanishes.*

Moreover, if any of the above hold, the global dimension of R is equal to its Krull dimension.

Proof sketch. The equivalence of the first two statements is [Lam99, 5.84]. The proof that (1) $\implies$ (2) is not terribly hard: one uses an induction on the Krull dimension to show the key equalities

$$\mathrm{gldim}(R) = \mathrm{pd}_R(k) = \dim(R)$$

where k is the residue field. This also proves the statement about Krull dimension. The proof of the converse is significantly more difficult and we avoid mentioning it here. (2) $\implies$ (3) is the previous Lemma and (3) $\implies$ (2) follows from the key equalities above. ☐

Corollary 3.5 ("Global Auslander–Buchsbaum–Serre"). *Let R be a commutative noetherian ring. The following are equivalent:*

1. *R is regular.*
2. *$D_{\mathrm{sg}}(R)$ vanishes.*

Moreover, if either of the above hold, the global dimension of R is equal to its Krull dimension.

Proof sketch. This follows from [Lam99, 5.94]. If R is regular then each $R_{\mathfrak{m}}$ is regular and hence of finite global dimension by ABS. For every finitely generated module M, there exists a maximal ideal $\mathfrak{m}$ of R such that $\mathrm{pd}_R(M) = \mathrm{pd}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$, by a compactness argument. In particular every finitely generated R-module has finite projective dimension and so $D_{\mathrm{sg}}(R)$ vanishes. For the converse, if $D_{\mathrm{sg}}(R)$ vanishes then so does each $D_{\mathrm{sg}}(R_{\mathfrak{m}})$, because resolutions localise. Hence each

[8] In general the two concepts differ when working with objects from arithmetic geometry: smoothness is a stronger condition. Note also that smoothness is a property of a morphism (for varieties, the structure morphism to the base field) whereas regularity is a property of a scheme.

[9] Historical remark: the ABS theorem is (1) $\iff$ (2), which long predates the invention of singularity categories. The implication (1) $\implies$ (2) was first noticed by Buchsbaum, and the implication (2) $\implies$ (1) was independently proved by both Serre and Auslander–Buchsbaum. The equivalence of both statements with (3) and the statement about the Krull dimension are easy corollaries of the proof.

$R_{\mathfrak{m}}$ is regular local by ABS and hence R is regular. The statement about Krull dimension follows from the equalities

$$\operatorname{gldim}(R) = \sup_{\mathfrak{m}} \operatorname{gldim}(R_{\mathfrak{m}}) = \sup_{\mathfrak{m}} \dim(R_{\mathfrak{m}}) = \dim(R)$$

where in the second equality we are using ABS. □

Note that we have not proved that a commutative noetherian regular ring must have finite global dimension. In fact this is false! Nagata gave an example of a commutative noetherian regular ring R with infinite Krull dimension (and hence, by global ABS, global dimension). Each localisation of R must have finite - but arbitrarily large - global dimension. Although $D_{\mathrm{sg}}(R)$ vanishes, it does not vanish in a 'uniform' way, in the sense that one cannot uniformly bound the projective dimension of all finitely generated modules.

Example 3.6 (Nagata [Nag62]). Let $I_n \subseteq \mathbb{N}$ denote the interval $[2^{n-1}, 2^n - 1]$, which has length 2^{n-1}. Let $A = \mathbb{C}[x_1, x_2, \cdots]$ be the infinite-dimensional polynomial ring and for each $n \in \mathbb{N}$ let $\mathfrak{p}_n$ denote the ideal generated by $\{x_i : i \in I_n\}$. Put $S \coloneqq A / \cup_n \mathfrak{p}_{\mathfrak{n}}$ and put $R \coloneqq A_S$ the localisation. Since $\mathfrak{p}_{\mathfrak{n}}$ has height 2^{n-1} in A it follows that R has infinite Krull dimension. To prove that it is regular, first use the Prime Avoidance Lemma to show that every maximal ideal of R is of the form $\mathfrak{p}_n R$, so that we need to check that each $A_{\mathfrak{p}_n}$ is regular; this holds since it is a localisation of a regular ring. To prove that it is noetherian boils down to checking that each $A_{\mathfrak{p}_n}$ is noetherian; again this holds since it can be written as a localisation of a noetherian ring.

4 Some explicit computations (David)

5 Buchweitz's stable category (Sofia)

Based on [Buc86].

A (noncommutative) ring A is **(Iwanaga)–Gorenstein** if A has finite injective dimension over itself; i.e. there exists a bounded complex I of injective A-modules with a quasi-isomorphism $I \simeq A$. A module M over a Gorenstein ring A is **maximal Cohen–Macaulay** (or just MCM for short) if there is a natural quasi-isomorphism $\mathbb{R}\mathrm{Hom}_A(M, A) \simeq \mathrm{Hom}_A(M, A)$. This is equivalent to the condition that $\mathrm{Ext}^i_A(M, A) \cong 0$ for $i > 0$.

Suppose that A is a commutative local ring with residue field k. The **depth** of a module M is the smallest number i for which $\mathrm{Ext}^i(k, M)$ is nonzero. When M is finitely generated, this is the length of a maximal regular sequence $(x_n) \in \mathfrak{m}$ for M. Then M is MCM if and only if $\mathrm{depth}(M) = \dim(A)$. The ring A is **Cohen–Macaulay** if and only if A is an MCM module over itself; a CM ring is necessarily Gorenstein. The **Auslander–Buchsbaum formula** says that if M has finite projective dimension then $\mathrm{pd}_A(M) + \mathrm{depth}(M) = \mathrm{depth}(A)$. In particular if A is CM then an MCM module has either 0 or infinite projective dimension.

Suppose from now on that A is Gorenstein. Let M be an MCM module. A **syzygy** of M is a module ΩM which fits into a short exact sequence of the form

$$0 \to \Omega M \to P \to M \to 0$$

where P is a finitely generated projective. One can check using the Ext characterisation that ΩM is again MCM. Note that one can stitch together the syzygy exact sequences for all $\Omega^i M$ into a projective resolution of M.

One can define the **stable category of MCM modules** $\underline{\mathbf{MCM}}(A)$ to have objects the MCM A-modules, and morphisms given by

$$\underline{\mathrm{Hom}}(M,N) := \frac{\mathrm{Hom}(M,N)}{\text{morphisms which factor through a projective module}}.$$

In particular, projective modules go to zero in $\underline{\mathbf{MCM}}(A)$. Hence taking syzygies produces a well defined endofunctor on $\underline{\mathbf{MCM}}(A)$.

The natural functor $\mathbf{MCM}(A) \to D_{\mathrm{sg}}(A)$ sends a morphism which factors through a projective to zero, and hence defines a functor $\Psi : \underline{\mathbf{MCM}}(A) \to D_{\mathrm{sg}}(A)$.

Theorem 5.1. *Ψ is an equivalence. Moreover Ψ sends Ω to the inverse shift $[-1]$. Hence $\underline{\mathbf{MCM}}(A)$ is a triangulated category, with shift functor the 'inverse syzygy' Ω^{-1}.*

As a triangulated category, $\underline{\mathbf{MCM}}(A)$ admits a notion of Ext groups, which we denote by $\underline{\mathrm{Ext}}$ and refer to as the **stable Ext groups**. One pleasing fact is the following:

Proposition 5.2. *If A is a Gorenstein ring and M, N are MCM R-modules then for $j > 0$, there is an isomorphism $\underline{\mathrm{Ext}}^j_A(M,N) \cong \mathrm{Ext}^j_A(M,N)$. If $j < -1$ then there is an isomorphism $\underline{\mathrm{Ext}}^j_A(M,N) \cong \mathrm{Tor}^A_{-j-1}(N, M^\vee)$.*

Hence the stable Exts can be computed in positive degrees as the usual Exts. Buchweitz' proof uses the notion of a complete resolution. A fancier proof shows that (at least when $M = N$) there is an exact triangle

$$T \to \mathbb{R}\mathrm{Hom}_A(M,N) \to \mathbb{R}\underline{\mathrm{Hom}}_A(M,N) \to$$

where T is the standard bar complex computing $\mathrm{Tor}(N, M^\vee)$.

6 Matrix factorisations (Matt)

A nice reference here is [Dyc11]. See also [Sym22].

Let R be a complete local hypersurface singularity; i.e. R is of the form $k[\![\underline{x}]\!]/\sigma$ for some $\sigma \in \mathfrak{m}^2$. Observe that R is Gorenstein: one can prove this using the Koszul complex (more generally, the same proof shows that an lci ring is Gorenstein). Eisenbud was the first to notice the following surprising fact:

Theorem 6.1 ([Eis80])**.** *If R is a complete local hypersurface singularity then $\Omega^2 \cong \mathrm{id}$ as endofunctors of $\underline{\mathbf{MCM}}(R)$.*

In particular, Ω is the shift functor of $\underline{\mathbf{MCM}}(R)$. One can stitch the syzygy exact sequences for M together to build a 2-periodic resolution of M, and this was originally how Eisenbud's results were formulated. The converse is also true: in fact if R is a complete local Gorenstein ring with $\Omega^p \cong \mathrm{id}$ for some $p \geq 1$ then R is actually a hypersurface singularity.

So D_{sg} of a complete local hyperasurface singularity is actually a 2-periodic triangulated category. What sort of structure does this buy us? Can we write down a small 2-periodic model?

For brevity, write $A \coloneqq k[\![\underline{x}]\!]$, so that $R = A/\sigma$.

Definition 6.2. A **matrix factorisation** of σ is a free finitely generated $\mathbb{Z}/2$-graded $k[\![\underline{x}]\!]$-module $X = (X_0, X_1)$ together with an odd A-linear map $d : X \to X$ satisfying $d^2 = \mathrm{id}_X \cdot \sigma$.

We will draw matrix factorisations as diagrams $X_0 \underset{d_0}{\overset{d_1}{\rightleftarrows}} X_1$. If we choose bases for the X_i, we get a pair ϕ, ψ of matrices over A satisfying $\phi\psi = \sigma \cdot \mathrm{id}$ and $\psi\phi = \sigma \cdot \mathrm{id}$. In particular this forces ϕ and ψ to be square matrices of the same dimension.

Example 6.3. For the nodal cubic $\sigma = y^2 - x^2 - x^3$ in the plane, one matrix factorisation is

$$\phi = \psi = \begin{pmatrix} y & x + x^2 \\ -x & -y \end{pmatrix}$$

The category of matrix factorisations assembles into a $\mathbb{Z}/2$-graded dg category in the usual way: put

$$\mathrm{Hom}^i(X, Y) = \mathrm{Hom}_A(X_0, Y_i) \oplus \mathrm{Hom}_A(X_1, Y_{1+i})$$

with differential $\partial(f) = d_Y f - (-1)^{|f|} f d_x$. The homotopy category of this dg category is denoted $\mathbf{MF}(A, \sigma)$ and is referred to as the **homotopy category of matrix factorisations**. This is a triangulated category: the shift Σ sends $X_0 \underset{d_0}{\overset{d_1}{\rightleftarrows}} X_1$ to $X_1 \underset{d_1}{\overset{d_0}{\rightleftarrows}} X_0$ (i.e. it rotates the picture by 180 degrees).

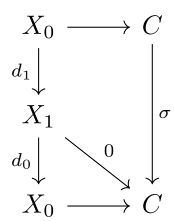
Theorem 6.4. *There is a triangle equivalence*

$$\mathrm{MF}(A, \sigma) \simeq \underline{\mathrm{MCM}}(R).$$

In fact, this enhances to an equivalence of dg categories.

We prove the theorem in several steps. We begin by constructing a functor.

If X is a matrix factorisation, put $C(X) \coloneqq \operatorname{coker}(d_0)$, which admits a projective resolution $X_1 \xrightarrow{d_0} X_0$. Note that $C(X)$ is naturally an R-module, since the diagram



commutes.

Lemma 6.5. *$C(X)$ is an MCM R-module.*

Proof. Since $\dim(A) = 1 + \dim(R)$, the Auslander–Buchsbaum formula tells us that an R-module M is MCM if and only if $\operatorname{pd}_A(M) = 1$. Since $C(X)$ is not projective, it must have projective dimension 1 as required. □

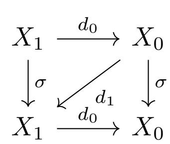
Lemma 6.6. *C extends to a triangle functor $C : \mathbf{MF}(A, \sigma) \to \underline{\mathrm{MCM}}(R)$.*

Proof. We first show that C is a functor; i.e. we need to show that if $f : X \to Y$ is a nullhomotopic morphism of matrix factorisations then $C(f)$ factors through a projective module. This uses the useful observation that the 'left unwinding mod σ'

$$\cdots \to X_1 \otimes_A R \to X_0 \otimes_A R \to X_1 \otimes_A R \to X_0 \otimes_A R \coloneqq \tilde{X}$$

resolves $C(X)$. In fact, using this one can show that if f is nullhomotopic then $C(f)$ will factor through both $X_1 \otimes_A R$ and $Y_0 \otimes_A R$. To show that C is a triangle functor, observe that $C(\Sigma X)$ is resolved by the brutal truncation $\tau_{\leq -1}(\tilde{X})[-1]$. But this complex resolves $\Omega(CX)$. □

Proof of the Theorem. We have a triangle functor $C : \mathbf{MF}(A, \sigma) \to \underline{\mathrm{MCM}}(R)$, so it is enough to check that C is an equivalence. To show that C is essentially surjective, let M be an MCM R-module. Since it has projective dimension 1 over A, it has a length two A-free resolution $X_1 \xrightarrow{d_0} X_0$. Multiplication by σ annihilates M, so is nullhomotopic on the resolution. This nullhomotopy is witnessed by a map $d_1 : X_0 \to X_1$ which fits into a commutative diagram



from which it is clear that $X \coloneqq (X_0 \oplus X_1, d_0 \oplus d_1)$ is a matrix factorisation of σ. Clearly $C(X) \cong M$, so that C is essentially surjective.

Next we need to show that C is full, for which we follow the proof of [Dyc11, Lemma 4.2]. Let $f : M \to N$ be a map in the stable category. Choose matrix factorisations X, Y with $C(X) \cong M$ and $C(Y) \cong N$. Lift f to a map

$$\begin{array}{ccc} X_1 & \xrightarrow{d_0} & X_0 \\ \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_0} \\ Y_1 & \xrightarrow{d_0} & Y_0 \end{array}$$

Since $d_0 d_1 = \sigma$, the pair $(\tilde{f}_0, \tilde{f}_1)$ defines a morphism $\tilde{f} : X \to Y$ of matrix factorisations which obviously lifts f. So C is full.

Finally we need to show that C is faithful, which is a bit tricky; see [Lan16, 5.2.2] for a hands-on proof. The loose idea is as follows. Take $f, g : X \to Y$ with $Cf = Cg$. Extending f periodically gives a map $\tilde{f} : \tilde{X} \to \tilde{Y}$. On the other hand, resolving Cf gives a different map $\tilde{f}' : \tilde{X} \to \tilde{Y}$. Necessarily we have a homotopy $\tilde{f} \simeq \tilde{f}'$, since they resolve the same thing. By assumption $\tilde{f}'$ is homotopic to $\tilde{g}'$ and so composing we get a homotopy $\tilde{f} \simeq \tilde{g}$. If we could choose this homotopy to be periodic we would be done, since then it would lift to a homotopy $f \simeq g$. So we have to modify the homotopy a bit to make it 2-periodic. □

7 Koszul duality (Björn)

The first part of the talk introduced dg coalgebras, the bar and cobar construction, and twisting cochains. A good elementary reference for this is [LV12].

The second part was about module-comodule Koszul duality. This is explained in [Pos11]. The key point is as follows. Let R be a commutative semisimple ring (i.e. a direct sum of fields). If C is a conilpotent dg R-coalgebra, then there is an equivalence

$$D(C) \to D(\Omega C)$$

which sends C to R and R to ΩC.

Finally, the goal was to prove the following theorem, which appears in the appendix of [CKWW21].

Theorem 7.1. *Let k be a field and let R be a finite dimensional commutative semisimple k-algebra (i.e. a direct sum of finite field extensions of k). Let A be an Artinian R-algebra. Then there is an equivalence*

$$D_{\mathrm{sg}}(A^{\mathrm{op}})^{\mathrm{op}} \simeq \frac{\mathbf{per}(\Omega(A^\vee))}{\mathrm{thick}(R)}.$$

Proof. Let $C = A^\vee$ be the linear dual coalgebra, which is a conilpotent R-coalgebra since k-linear duality coincides with R-bimodule duality. There is a functor $\mathbf{Mod}\text{-}A \to \mathbf{Com}\text{-}C$: an action map $M \otimes A \to M$ is adjoint to a coaction map $M \to \mathrm{Hom}(A, M) \cong M \otimes C$ making M into a C-comodule. In fact this extends to an equivalence $K(\mathrm{Inj}(A)) \to D(C)$. There is a fully faithful injective resolution functor $D^b(A) \to K(\mathrm{Inj}(A))$ and by composition we obtain a fully faithful triangle functor $\iota : D^b(A) \to D(C)$.

Since A is finite dimensional, both $R \cong R^\vee$ and $A^\vee = C$ are injective A-modules. In particular, ι induces a triangle equivalence

$$\frac{\mathbf{thick}_{D^b(A)}(R)}{\mathbf{thick}_{D^b(A)}(A^\vee)} \simeq \frac{\mathbf{thick}_{D(C)}(R)}{\mathbf{thick}_{D(C)}(C)}.$$

We note that Keller and Wang write $D^b(\mathrm{com}C)$ for what we call $\mathbf{thick}_{D(C)}(R)$. Since A is Artinian, R is a thick generator of $D^b(A)$ and so the left hand side is nothing more than $D^b(R)/\mathbf{thick}(A^\vee)$. But linear duality is an equivalence $D^b(A^{\mathrm{op}})^{\mathrm{op}} \to D^b(A)$, and it follows that the left hand side is exactly $D_{\mathrm{sg}}(A^{\mathrm{op}})^{\mathrm{op}}$. Turning to the right hand side, Koszul duality gives us an equivalence

$$\frac{\mathbf{thick}_{D(C)}(R)}{\mathbf{thick}_{D(C)}(C)} \simeq \frac{\mathbf{per}(\Omega C)}{\mathbf{thick}(R)}$$

as required. □

One can check that $\Omega(C)$ is isomorphic to the Koszul dual $A^!$ of A, defined to be the linear dual of the coalgebra $B(A)$. If A is the path algebra of a quiver Q with admissible relations, then one can take $R \cong \oplus_{i \in Q_0} ke_i$ to be the sum of the vertex idempotents; this is the motivating example. For a possible extension of this theorem, see [GS20]. We also remark that the category $K(\mathrm{Inj}(A))$ appearing in the proof is closely related to Krause's **big singularity category** of A; cf. [Bec14, Kra05].

8 Relative singularity categories (Nick)

Let R be a commutative noetherian k-algebra. Take a finitely generated R-module M and look at the ring $A \coloneqq \mathrm{End}_R(R \oplus M)$. The idea is that A should behave like a 'noncommutative partial resolution' of R. There's a base change functor $\mathbf{Mod}\text{-}R \to \mathbf{Mod}\text{-}A$ given by $X \mapsto X \otimes_R (R \oplus M)$, and its derived functor is a fully faithful embedding $D(R) \to D(A)$. Since this sends R to the bounded complex $R \oplus M$, this gives an embedding $\mathbf{per}(R) \hookrightarrow D^b(A)$. The **relative singularity category**, whose definition is due to Burban and Kalck [BK12], is the Verdier quotient

$$\Delta_R(A) \coloneqq \frac{D^b(A)}{\mathbf{per}(R)}.$$

Note that if $M \cong 0$ then $\Delta_R(A) \simeq D_{\mathrm{sg}}(R)$, so this is a generalisation of the notion of singularity category.

Remark 8.1. If A is any k-algebra with a idempotent e, then one can define the **relative singularity category** to be the Verdier quotient

$$\Delta_e(A) \coloneqq \frac{D^b(A)}{\mathbf{thick}(eA)}.$$

There is a fully faithful triangle functor $j_! : D(eAe) \to D(A)$ given on objects by $j_!X \coloneqq X \otimes^{\mathbb{L}}_{eAe} eA$, and clearly we have $\mathbf{thick}(eA) = j_!\mathbf{per}(eAe)$. In fact $j_!$ is part of a recollement [KY16, 2.10]. If R, M, and A are now as above, note that A comes with an idempotent $e = \mathrm{id}_R$. We have isomorphisms $eA \cong R \oplus M$ and $eAe \cong R$, and hence get a triangle equivalence

$$\Delta_R(A) \simeq \Delta_e(A).$$

If R is Gorenstein and M is an MCM module then we have an isomorphism $A/AeA \cong \underline{\mathrm{End}}_R(M)$, where the target is the endomorphism ring of M in the stable category.

Where does the definition of a relative singularity category come from? The motivation is from algebraic geometry. Assume that k has characteristic zero. For readability, all functors in this section will be derived. Let X be a k-variety and $\pi : \tilde{X} \to X$ a resolution of singularities. There is a derived pushforward functor $\pi_* : D^b(\tilde{X}) \to D^b(X)$, and a derived pullback functor $\pi^* : D^b(X) \to D(\tilde{X})$. They are adjoints[10]. The projection formula tells us that the unit of this adjunction is $\mathcal{F} \to \pi_*\pi^*\mathcal{F} \simeq \mathcal{F} \otimes \pi_*\mathcal{O}_{\tilde{X}}$. Say that X **has rational singularities** if the natural map $\mathcal{O}_X \to \pi_*\mathcal{O}_{\tilde{X}}$ is a quasi-isomorphism. If this is the case, then by the projection formula π^* is fully faithful, and in particular we get an embedding $\mathbf{per}(X) \hookrightarrow D^b(\tilde{X}) \simeq \mathbf{per}(\tilde{X})$.

The ring R is to be thought of as X, and the ring A is supposed to be like its resolution $\tilde{X}$. The pullback π^* gets replaced by $j_!$, and hence the relative singularity category is supposed to behave like the Verdier quotient $\mathbf{per}(\tilde{X})/\mathbf{per}(X)$, which measures the failure of $\pi^*\mathbf{per}(X)$ to be all of $\mathbf{per}(\tilde{X})$.

With this in mind, let's try to globalise the construction of our noncommutative partial resolution A. Let X be a variety with isolated Gorenstein singularities (i.e. X has isolated singularities, and at each singular point $p \in X$ the formal completion $\hat{X}_p$ is a complete local Gorenstein ring). If $\mathcal{F}'$ is a coherent sheaf on X, put $\mathcal{F} \coloneqq \mathcal{F}' \oplus \mathcal{O}_X$. Let $\mathcal{A} \coloneqq \mathcal{E}\mathrm{nd}(\mathcal{F})$ be the endomorphism sheaf, which is a coherent sheaf of rings on X. This defines a ringed space $\mathbb{X} \coloneqq (X, \mathcal{A})$ which is supposed to behave like a partial resolution of X. Note that $\mathbb{X}$ is a sheaf of noetherain rings so it makes sense to consider the abelian category $\mathrm{Coh}(\mathbb{X})$ and its bounded derived category $D^b(\mathbb{X})$. Obnserve that there are natural functors $\psi \coloneqq \mathcal{F} \otimes^{\mathbb{L}} - : \mathbf{per}(X) \to D^b(\mathbb{X})$ and $\psi \coloneqq \mathbb{R}\mathrm{Hom}(\mathcal{F}, -) : D^b(\mathbb{X}) \to D^b(X)$.

[10] Note that this does not quite make sense since the codomain of π^* is not the domain of π_*. Really they are defined on larger derived categories where they are actually adjoints. So "$\pi^* \dashv \pi_*$" means that if $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(\tilde{X})$, then there is a natural isomorphism $\mathrm{Hom}(\pi^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \pi_*\mathcal{G})$.

Theorem 8.2 (Burban–Drozd). *$\phi \dashv \psi$ and the unit $\mathrm{id} \to \psi\phi$ is an isomorphism; in particular ϕ is fully faithful.*

Hence we may define the **relative singularity category**

$$\Delta_X(\mathbb{X}) \coloneqq \frac{D^b(\mathbb{X})}{\mathbf{per}(X)}.$$

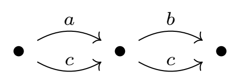
In [BK12], the relative singularity category is defined to be $\Delta_X(\mathbb{X})^\omega$, the idempotent completion of our relative singularity category. Burban and Kalck give a description of this category in a special case.

Let X be a nodal curve with n singular points and let $\mathcal{F}'$ be the ideal sheaf of the singular locus[11]. With this setup, Burban and Kalck give the following theorem.

Theorem 8.3 ([BK12]). *The ringed space $\mathbb{X}$ is a noncommutative resolution of X (in fact it has global dimension 2). There is an equivalence of triangulated categories*

$$\Delta_X(\mathbb{X})^\omega \simeq \bigoplus_{i=1}^{n} \left(\frac{D^b(\Lambda)}{\mathrm{Band}(\Lambda)} \right)^\omega$$

where Λ is the path algebra of the quiver



with relations $ab = 0 = cd$, and the category of band modules is defined to be

$$\mathrm{Band}(\Lambda) \coloneqq \{C \in D^b(\Lambda) : \quad \tau(C) \simeq C\}$$

where τ, the Auslander–Reiten translate, is given by $\tau(C) \coloneqq \Lambda^* \otimes^{\mathbb{L}}_{\Lambda} C[-1]$.

Remark 8.4. In particular, the two triangulated categories $\Delta_X(\mathbb{X})$ and $\bigoplus_{i=1}^n \frac{D^b(\Lambda)}{\mathrm{Band}(\Lambda)}$ are equivalent up to direct summands, and hence have the same K-theory.

Remark 8.5. The above theorem can be thought of as a relative version of the following general theorem, due to Orlov. Let X be an ELF[12] scheme with isolated singular locus $\{p_1, \dots, p_n\}$. Then there are triangle equivalences

$$D_{\mathrm{sg}}(X)^\omega \quad \simeq \quad \bigoplus_{i=1}^{n} D_{\mathrm{sg}}(X_{p_i})^\omega \quad \simeq \quad \bigoplus_{i=1}^{n} D_{\mathrm{sg}}(\hat{X}_{p_i})^\omega$$

One cannot in general drop the idempotent completions. We may return to a proof of this later. In particular one expects a relative singularity category to have an analogous decomposition into blocks.

[11]So Proj of the Rees algebra of $\mathcal{F}'$ is the blowup of X at the singular locus, which is a resolution since X has nodal singularities.

[12]Enough Locally Free sheaves; which geometrically is a very mild condition. Concretely ELF means separated, noetherian, of finite Krull dimension, has closed singular locus (e.g. locally of finite type over a perfect field), and has the resolution property. Over a perfect field, quasiprojective varieties satisfy ELF. A good discussion is in [Sym22].

9 DG categories (Callum G.)

Let k be a commutative ring.

Definition 9.1. A k-linear **dg category** is a category $\mathcal{C}$ enriched over the monoidal category ($\mathbf{Mod}$-$k, \otimes$) of dg k-modules with the usual tensor product. In other words, to every pair of elements $(x, y) \in \mathcal{C}^2$ we assign a chain complex $\mathcal{C}(x, y)$, to every triple (x, y, z) we assign a chain map $\mu_{xyz} : \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ satisfying associativity, and for every $x \in \mathcal{C}$ we assign a map $\eta_x : k \to \mathcal{C}(x, x)$ which is a unit with respect to composition.

Note in particular that for any object $x \in \mathcal{C}$, the complex $\mathcal{C}(x, x)$ naturally has the structure of a (unital) dga.

Definition 9.2. A **dg functor** $F : \mathcal{C} \to \mathcal{D}$ between two dg categories is an enriched functor; i.e. a map of objects $\mathcal{C} \to \mathcal{D}$ together with, for every pair $(x, y) \in \mathcal{C}^2$, a map of complexes $F_{xy} : \mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$. We require that F satisfies the associativity condition $\mu_{Fx\ Fy\ Fz} \circ (F_{xy} \otimes F_{yz}) = F_{xz} \circ \mu_{xyz}$ and the unitality condition $F_{xx} \circ \eta_x = \eta_{Fx}$.

In particular, a dg functor $F : \mathcal{C} \to \mathcal{D}$ induces dga morphisms $F_{xx} : \mathcal{C}(x, x) \to \mathcal{D}(Fx, Fx)$ for every $x \in \mathcal{C}$.

Example 9.3. Examples of dg categories include:

- $\mathbf{Mod}$-k itself. More generally, if A is a k-dga then $\mathbf{Mod}$-A is a k-dg category.
- If $\mathcal{C}$ is a dg category then so is $\mathcal{C}^{\mathrm{op}}$.
- If X is a topological space with a sheaf of rings $\mathcal{O}$, then the category $\mathbf{Mod}$-$\mathcal{O}$ of complexes of sheaves of $\mathcal{O}$-modules is a dg category.
- A dg category with one object is the same thing as a dg algebra. The inclusion of dg algebras into dg categories is fully faithful.

Definition 9.4. Let $\mathcal{C}$ be a dg category. The **homotopy category** of $\mathcal{C}$ is the k-linear category $H^0\mathcal{C}$ whose objects are the same as $\mathcal{C}$ and whose hom-spaces are given by $(H^0\mathcal{C})(x, y) \coloneqq H^0(\mathcal{C}(x, y))$. Composition is inherited from $\mathcal{C}$.

Example 9.5. $H^0(\mathbf{dgMod}_k)$ is the chain homotopy category.

Definition 9.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a dg functor.

- F is **quasi-fully faithful** if all of its components F_{xy} are quasi-isomorphisms.
- F is **quasi-essentially surjective** if the induced functor H^0F is essentially surjective.
- F is a **quasi-equivalence** if it is quasi-fully faithful and quasi-essentially surjective.

Two dg algebras are quasi-equivalent as dg categories if and only if they are quasi-isomorphic as dg algebras. Note that a quasi-equivalence induces an equivalence on H^0, but the converse is not true. For a simple example, the morphism of dgas $k \hookrightarrow k[x]$, where x has degree 1, is an equivalence on H^0 but clearly not a quasi-equivalence.

If $\mathcal{C}$ is a dg category, it has a category of **right modules**, the dg category $\mathbf{Mod}\text{-}\mathcal{C} \coloneqq \mathbf{dgFun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Mod}\text{-}k)$. In other words, to an object x of $\mathcal{C}$ we assign a complex $M(x)$ together with action maps $\mathcal{C}(x,y) \otimes M(y) \to M(x)$. If x is an object of $\mathcal{C}$, then we obtain a module $h_x \coloneqq \mathcal{C}(-,x)$. The assignment $x \mapsto h_x$ is a (quasi-)fully faithful dg functor $\mathcal{C} \to \mathbf{Mod}\text{-}\mathcal{C}$ which we call the **Yoneda embedding**.

The category $\mathbf{Mod}\text{-}\mathcal{C}$ has shifts and cones, defined pointwise: if $f : M \to N$ is a $\mathcal{C}$-linear map then we may define its cone by $\mathrm{cone}(f) : x \mapsto \mathrm{cone}(f(x))$ and similarly for morphisms. This makes the homotopy category $H^0(\mathbf{Mod}\text{-}\mathcal{C})$ into a triangulated category.

Definition 9.7. Say that a dg category is **strongly pretriangulated** if, for all morphisms $f : x \to y$, the following modules are representable:

- The zero module 0.
- The shifts $h_x[n]$ for all n.
- The cone $\mathrm{cone}(h_f)$.

When these are representable, we define $x[n]$ to be the object representing $h_x[n]$ and $\mathrm{cone}(f)$ to be the object representing $\mathrm{cone}(h_f)$.

If $\mathcal{C}$ is strongly pretriangulated then the homotopy category $H^0\mathcal{C}$ is canonically triangulated, with translation functor given by the shift.

Example 9.8. If $\mathcal{C}$ is a dg category then $\mathbf{Mod}\text{-}\mathcal{C}$ is strongly pretriangulated.

Every dg category $\mathcal{C}$ admits a **strongly pretriangulated envelope** $\mathrm{tri}(\mathcal{C})$, defined to be the closure of the image of the Yoneda embedding under the zero module, cones and shifts. The assignment $\mathcal{C} \mapsto \mathrm{tri}(\mathcal{C})$ is left adjoint to the inclusion of strongly pretriangulated dg categories in all dg categories. The unit $\mathcal{C} \to \mathrm{tri}(\mathcal{C})$ is the Yoneda embedding.

Definition 9.9. A dg category $\mathcal{C}$ is **pretriangulated** if the natural map $C \hookrightarrow \mathrm{tri}(\mathcal{C})$ is a quasi-equivalence.

Remark 9.10. The following are equivalent for a dg category $\mathcal{C}$:

- $\mathcal{C}$ is pretriangulated.
- The natural map $H^0C \to H^0\mathrm{tri}(\mathcal{C})$ is an equivalence.
- The natural map $H^0C \to H^0\mathrm{tri}(\mathcal{C})$ is essentially surjective.

- $\mathcal{C}$ is quasi-equivalent to a pretriangulated dg category.
- $\mathcal{C}$ is quasi-equivalent to a strongly pretriangulated dg category.
- Zero, shifts, and cones are all homotopy representable (i.e. representable in $H^0(\mathbf{Mod}\text{-}\mathcal{C})$).

In particular if $\mathcal{C}$ is pretriangulated then $H^0\mathcal{C}$ is canonically triangulated. If $\mathcal{T}$ is a triangulated category then a pretriangulated dg category $\mathcal{C}$ is an **enhancement** of $\mathcal{T}$ if there is a triangle equivalence $H^0(\mathcal{C}) \simeq \mathcal{T}$.

Remark 9.11. A functor $F : \mathcal{C} \to \mathcal{D}$ between pretriangulated dg categories is a quasi-equivalence if and only if H^0F is a triangle equivalence. To see that the latter implies the former, use the isomorphism $H^i(\mathcal{C}(x, y)) \simeq (H^0\mathcal{C})(x, y[i])$.

Now assume that k is a field[13]. Let $\mathcal{A}$ be a dg category and $\mathcal{B} \hookrightarrow \mathcal{A}$ be a full dg subcategory. Then there exists a dg category $\mathcal{A}/\mathcal{B}$, the **dg quotient** of $\mathcal{A}$ by $\mathcal{B}$, which is universal (up to quasi-equivalence) among those dg functors $\mathcal{A} \to \mathcal{C}$ which homotopy annihilate $\mathcal{B}$ (i.e. for every $b \in \mathcal{B}$, the morphism id_b becomes 0 in $H^0\mathcal{C}$). The idea of the dg quotient is to kill all of the objects in $\mathcal{B}$.

The first construction of a dg quotient was given by Keller [Kel99]. Drinfeld [Dri04] came up with a simple construction: the idea is to adjoin to $\mathcal{A}$, for every object in b, a contracting homotopy h with $dh = \mathrm{id}_b$. Drinfeld also gives another description using ind-categories which can be useful. Tabuada showed that $\mathcal{A}/\mathcal{B}$ is the homotopy cofibre of $\mathcal{B} \hookrightarrow \mathcal{A}$, taken in the homotopy category of dg categories [Tab10].

Example 9.12. Let A be a dg algebra, regarded as a one-object dg category. This does not have many full subcategories: only the empty subcategory $\emptyset$ along with A itself. The quotient $A/\emptyset$ is simply A again. The quotient A/A is the dga $A\langle h\rangle$ with $dh = 1$; in particular the ring $H^0(A)$ is the zero ring and hence A/A is acyclic.

Theorem 9.13 (Drinfeld)**.** *If both $\mathcal{B}$ and $\mathcal{A}$ are pretriangulated then so is $\mathcal{A}/\mathcal{B}$. Moreover, there is a triangle equivalence*

$$H^0(\mathcal{A}/\mathcal{B}) \simeq H^0(\mathcal{A})/H^0(\mathcal{B})$$

where the right hand side is the Verdier quotient.

In other words, the above theorem of Drinfeld says that if $\mathcal{T}$ is a triangulated category, $\mathcal{T}'$ a full triangulated subcategory, and the inclusion $\mathcal{T}' \hookrightarrow \mathcal{T}$ is enhanceable, then the Verdier quotient $\mathcal{T}/\mathcal{T}'$ is also enhanceable.

[13] More generally, we need our dg categories to be homotopically flat; if not we need to first replace them by suitable resolutions, as in [Dri04].

In particular, let A be a ring. The pretriangulated dg category $\mathbf{Mod}\text{-}A$ has a pretriangulated full dg subcategory $\mathrm{Acyc}(A)$ of acyclic complexes of modules. The dg quotient $\mathbf{Mod}\text{-}A/\mathrm{Acyc}(A)$ is the **dg derived category** of A, denoted by $D_{\mathrm{dg}}(A)$. The homotopy category of $D_{\mathrm{dg}}(A)$ is the usual triangulated derived category $D(A)$ of A. One can then define the dg bounded derived category $D^b_{\mathrm{dg}}(A)$ to be the full subcategory on those objects with bounded cohomology, and the dg perfect derived category $\mathbf{per}_{\mathrm{dg}}(A)$ to be the full subcategory on those objects which represent perfect complexes.

Remark 9.14. There are other ways of constructing the dg derived category, but all of them yield quasi-equivalent results. Usually the dg derived category is constructed by taking the dg localisation of $\mathbf{Mod}\text{-}A$ at the quasi-isomorphisms. A simple alternate construction is given by taking the full dg subcategory of $\mathbf{Mod}\text{-}A$ on the bifibrant objects in either the projective or injective model structures.To construct $D^b_{\mathrm{dg}}(A)$ one can simply take the dg category of right bounded complexes of projectives with bounded cohomology; with this construction $\mathbf{per}_{\mathrm{dg}}(A)$ is then the category of strictly perfect complexes.

Definition 9.15. If A is a noetherian ring, its **dg singularity category** is the dg category

$$D^{\mathrm{dg}}_{\mathrm{sg}}(A) := \frac{D^b_{\mathrm{dg}}(A)}{\mathbf{per}_{\mathrm{dg}}(A)}.$$

As before, $H^0 D^{\mathrm{dg}}_{\mathrm{sg}}(A)$ is the usual triangulated singularity category of A.

10 Generators in matrix factorisation categories

In this section we will follow [Dyc11]. We'll work over a field of characteristic zero. All dg algebras and categories will be $\mathbb{Z}/2$-graded.

Let $R = k[\![x_1, \dots, x_n]\!]$ be a commutative power series ring in n variables[14] and $w \in \mathfrak{m}$ a nonzero element defining an isolated hypersurface singularity. Recall that the singularity category of the hypersurface singularity R/w has a description as the category of matrix factorisations of w. This is a $\mathbb{Z}/2$-graded dg category $\mathrm{MF}(R, w)$ whose objects are matrix factorisations of w and whose morphism complexes are defined in the usual way.

Remark 10.1. In the non-complete setting, one has to make various technical modifications since the category $\mathrm{MF}(R, w)$ may not be idempotent complete. In fact $\mathrm{MF}(\hat{R}, w)$ can be identified with the idempotent completion of $\mathrm{MF}(R, w)$. This is loosely because the category $\mathrm{MF}^\infty(R, w)$ of infinite rank matrix factorisations can be identified with the dg derived category of $\mathrm{MF}(R, w)$, and the compact objects are then identified with $\mathrm{MF}(\hat{R}, w)$.

[14]The Cohen structure theorem says that any commutative complete local noetherian regular ring containing a field is of this form.

Recall that there are triangle equivalences

$$D_{\mathrm{sg}}(R/w) \simeq \underline{\mathrm{MCM}}(R/w) \simeq \mathrm{MF}(R,w).$$

In particular an R/w-module M, thought of as an object in the singularity category, functorially corresponds to a matrix factorisation M^{stab}, the **stabilisation** of M. A construction of this stabilisation functor was given by Eisenbud; we give a description in the case when $M := R/I$ is cut out by a regular sequence $f_1, \ldots, f_r$ and $w \in I$. Note that (w) need not equal I (in fact it won't, unless R/w is smooth). Consider the Koszul complex K of I, which in degree n is given by $\Lambda^n(R^{\oplus r})$ and whose differential s_0 is given by contraction:

$$s_0(e_{i_1} \wedge \cdots \wedge e_{i_n}) = \sum_j (-1)^j f_{i_j} e_{i_1} \wedge \cdots \wedge \tilde{e}_{i_j} \wedge \cdots \wedge e_{i_n}$$

where the $e_1, \ldots, e_r$ are the obvious basis vectors for R^r and the tilde denotes that we leave out the corresponding factor from the wedge product. View K as an R-free resolution of M. Write $w = \sum w_i f_i$ and consider the element $(w_1, \cdots, w_r) \in R^r$. Exterior multiplication with this element defines a contracting homotopy s_1 for the multiplication of w on M (which must exist since w annihilates M). Placing R^r in odd degree, the $\mathbb{Z}/2$-graded object

$$(\Lambda^*(R^r), s_0 + s_1)$$

is then a matrix factorisation representing M^{stab}.

The R-module $\Lambda^*(R^r)$ can be identified with the free supercommutative algebra $R[\theta_1, \ldots, \theta_r]$ with all θ_i in odd degree. Note that this really is a finite R-module, since $\theta_i\theta_j = -\theta_j\theta_i$. The differential can then be identified as the left action of the differential operator

$$\sum_{i=1}^{r} f_i \frac{\partial}{\partial\theta_i} + w_i\theta_i.$$

Every R-linear endomorphism of $R[\theta_1, \ldots, \theta_r]$ can be represented as a polynomial differential operator (via a dimension count), and so the underlying $\mathbb{Z}/2$-graded algebra of $\mathrm{End}(M^{\mathrm{stab}})$ can be itentified with the ring of polynomial differential operators on $R[\theta_1, \ldots, \theta_r]$. Concretely, this has generators $\theta_1, \ldots, \theta_n$ and $T_1, \ldots, T_n$ of odd degree, where we think of $T_i = \frac{\partial}{\partial\theta_i}$. The relations are

- $\theta_i\theta_j = -\theta_j\theta_i$.
- the graded Weyl relations $T_i\theta_j + \theta_j T_i = \delta_{ij}$. This is the graded Leibniz rule: if $u \in R[\theta_1, \ldots, \theta_r]$ then $(T_i\theta_j)(u) = T_i(\theta_j u) = \delta_{ij}u - \theta_j T_i(u)$.
- $T_iT_j = -T_jT_i$. This follows from the graded Leibniz rule again: for a polynomial $p \in R[\theta_1, \ldots, \theta_r]$, write $p = \theta_i\theta_j q + p'$ where $\theta_i\theta_j$ does not divide p'. Since both sides vanish on p' we may take $p' = 0$. Clearly the

claim also holds for $p = 0$ so we can assume that neither θ_i nor θ_j divide q, so both $T_i(q)$ and $T_j(q)$ are zero. The graded Leibniz rule then tells us that $T_iT_j(p) = -q$ and hence by symmetry $T_jT_i(-p) = -q$ too.

With this identification the differential d on the endomorphism algebra satisfies $d\theta_i = f_i$ and $dT_i = w_i$.

Example 10.2. Let's do a concrete example. Take $R = k[\![x]\!]$, $w = x^2$, and $I = (x)$. So $n = 1$ and $f_1 = w_1 = x$. In this case, the endomorphism algebra A is of the form

$$\frac{k[\![x]\!]\langle\theta, T\rangle}{(\theta^2, T^2, T\theta + \theta T = 1)}$$

with $k[\![x]\!]$-linear differential defined on generators by $d\theta = x = dT$. The element $t \coloneqq T - \theta$ is a cocycle and the Weyl relations give $t^2 = -1$. We have $d(\theta T) = -d(T\theta) = xt$, which implies that the cohomology algebra of A is $k[t]/(t^2 + 1)$. In fact, the k-subalgebra of A generated by t is also $k[t]/(t^2 + 1)$, which implies that the inclusion $k[t]/(t^2 + 1) \hookrightarrow A$ is a quasi-isomorphism. In particular, the singularity category of $\mathbb{R}[x]/x^2$ is $D^b(\mathbb{C})$.

Example 10.3. The previous example was a special case of a theorem which says that if w is a quadratic form then A is quasi-isomorphic to the associated Clifford algebra [Dyc11, §5.5]. More precisely, there is a quasi-isomorphism $Cl(w) \hookrightarrow A$. Suppose that we are interested in the singularity $R = k[\![x, y]\!]/xy$ defined by the coordinate axes in the plane. After a change of coordinates $x = u-v, y = u+v$, we may take $R \cong k[\![u, v]\!]/(u^2 - v^2)$ to be given by a quadratic form. Then A is quasi-isomorphic to the algebra $Cl_{1,1}(k) \coloneqq \frac{k[U,V]}{(U^2=-1,V^2=1,UV=-VU)}$ with U, V in odd degree and zero differential. The assignment

$$U \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism $Cl_{1,1}(k) \cong M_2(k)$. Since $M_2(k)$ is Morita equivalent to k, we obtain a quasi-equivalence $\mathrm{MF}(k[\![x, y]\!], xy) \simeq D^b(k)$. This is an instance of Knörrer periodicity.

Theorem 10.4 (Dyckerhoff)**.** *The object k^{stab} is a compact generator for* $\mathrm{MF}(R, w)$*.*

Proof. In more detail, what is actually proved is that k^{stab} is a classical generator for the category $\mathrm{MF}^\infty(R, w)$ of infinite rank matrix factorisations. It then follows that k^{stab} is a thick generator for the category $\mathrm{MF}(R, w)$. The proof is a difficult piece of homological algebra and makes use of a homological version of the Nakayama lemma. □

Corollary 10.5 (Dyckerhoff)**.** *There is a quasi-equivalence of $\mathbb{Z}/2$-graded dg categories* $\mathrm{MF}(R, w) \simeq \mathbf{per}\,\mathrm{End}(k^{\mathrm{stab}})$*.*

Proof. This is an application of Keller's theorem to the compact generator k^{stab} of $\mathrm{MF}^\infty(R, w)$. Putting $A \coloneqq \mathrm{End}(k^{\mathrm{stab}})$, we get a quasi-equivalence $\mathrm{MF}^\infty(R, w) \simeq D_{\mathrm{dg}}(A)$. This restricts to a quasi-equivalence $\mathbf{per}\,\mathrm{MF}^\infty(R, w) \simeq \mathbf{per}\,(A)$. Since $\mathrm{MF}(R, w)$ is idempotent complete, there is a quasi-equivalence $\mathbf{per}\,\mathrm{MF}(R, w) \simeq \mathrm{MF}(R, w)$. □

In particular we know how to compute $\mathrm{End}_R(k^{\mathrm{stab}})$! Our earlier description tells us that it is the $\mathbb{Z}/2$-graded dga of polynomial differential operators on $R[\theta_1, \ldots, \theta_n]$, with differential given on a basis by $d\theta_i = x_i$ and $d(\frac{\partial}{\partial\theta_i}) = w_i$, where we write $w = \sum_i w_i x_i$.

Remark 10.6. The ring R is the linear dual of the cosymmetric coalgebra C on the variables $x_1^*, \ldots, x_n^*$, and w is dual to a functional $h : C \to k$. View C as a curved coalgebra with zero differential and curvature given by k. Then matrix factorisations are precisely finite rank twisted complexes over C. Tu proves that Dyckerhoff's generator 'arises from Koszul duality'. More precisely, the algebra $\mathrm{End}_R(k^{\mathrm{stab}})$ is quasi-isomorphic to ΩC, and Dyckerhoff's equivalence is in fact a Koszul duality equivalence [Tu14].

The dga $\mathrm{End}_R(k^{\mathrm{stab}})$ is a very computable invariant, and the identification of $\mathrm{MF}(R, w)$ as perfect modules over it is a powerful theorem. Dyckerhoff uses it to give a quick conceptual proof of Knörrer periodicity, and a proof that if w is a quadratic form then its category of matrix factorisations is the category of perfect modules over the associated Clifford algebra. For the rest of the talk, we will focus on Dyckerhoff's main application to Hochschild theory.

One can define the Hochschild (co)homology of a ring or more generally a dg algebra. Thinking of a dg category as a many-object dg algebra, one can make analogous definitions. If $\mathcal{C}$ is a dg category, then it has an enveloping dg category $\mathcal{C}^e \coloneqq \mathcal{C} \otimes \mathcal{C}^{\mathrm{op}}$. The category of right modules over $\mathcal{C}^e$ is the same as the category of $\mathcal{C}$-bimodules. This is a perfectly good abelian category and one can define Tor and Ext functors via the usual machinery of projective resolutions. One can then define the Hochschild homology and cohomology of $\mathcal{C}$ as

$$HH^*(\mathcal{C}) \coloneqq \mathrm{Ext}^*_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C})$$
$$HH_*(\mathcal{C}) \coloneqq \mathrm{Tor}^{\mathcal{C}^e}_*(\mathcal{C}, \mathcal{C})$$

Just like for usual Hochschild (co)homology, one can compute these using bar complexes.

Back to the situation at hand, let A be the endomorphism dga of k^{stab}. Dyckerhoff proves that there is a quasi-equivalence

$$\mathrm{MF}(R \hat{\otimes} R, \tilde{w}) \simeq \mathbf{per}(A^e)$$

where $\tilde{w}$ is the matrix factorisation $1 \otimes w - w \otimes 1$. Note that the completed tensor product $R \hat{\otimes} R$ is isomorphic to $k[\![x_1, \ldots, x_{2n}]\!]$. Across this equivalence, the module A corresponds to the matrix factorisation Δ^{stab}, where Δ is the $(R \otimes R)/\tilde{w}$-bimodule R.

Remark 10.7. The dg category $\mathrm{MF}(R \hat{\otimes} R, \tilde{w})$ can be identified with the derived internal endomorphism dg category of the dg category $\mathrm{MF}(R, w)$. This fits with Toën's interpretation of Hochschild cohomology as derived endomorphisms of the identity bimodule.

One can explicitly compute the endomorphism dga of Δ^{stab}, which leads us to the following theorem:

Theorem 10.8 (Dyckerhoff)**.** *The Hochschild cohomology of the $\mathbb{Z}/2$-graded dg category* $\mathrm{MF}(R,w)$ *is concentrated in even degree, where it is the Milnor algebra*

$$R/\left(\frac{\partial w}{\partial x_1},\dots,\frac{\partial w}{\partial x_n}\right).$$

Similarly, one can compute the Hochschild homology of A as a morphism complex in $\mathrm{MF}(R\hat{\otimes}R,-\tilde{w})$, and doing this one obtains the following:

Theorem 10.9 (Dyckerhoff)**.** *The Hochschild homology of* $\mathrm{MF}(R,w)$ *is the Milnor algebra of w, concentrated in degree n, the Krull dimension of R.*

Remark 10.10. This should be a manifestation of the fact that $\mathrm{MF}(R,w)$ is a n-Calabi-Yau dg category.

{hhderinv} *Remark* 10.11. If $\mathcal{C}$ is a dg category, then there are isomorphisms

$$HH_*(\mathcal{C}) \simeq HH_*(\mathbf{per}\mathcal{C})$$
$$HH^*(\mathcal{C}) \simeq HH^*(\mathbf{per}\mathcal{C})$$

The first is due to Keller and the second is due to Lowen and Van den Bergh[15]. In particular, there are isomorphisms

$$HH_*(A) \simeq HH_*(\mathrm{MF}(R,w))$$
$$HH^*(A) \simeq HH^*(\mathrm{MF}(R,w))$$

Dyckerhoff makes use of the first.

Remark 10.12. Let $k[u,u^{-1}]$ be the graded Laurent polynomial ring, where u has degree 2. Then a $\mathbb{Z}/2$-graded k-linear dg category is the same thing as a $\mathbb{Z}$-graded $k[u,u^{-1}]$-linear dg category: note that a $\mathbb{Z}/2$-graded complex $X^0 \rightleftarrows X^1$ is the same thing as a 2-periodic $\mathbb{Z}$-graded complex $\cdots \to X^0 \to X^1 \to X^0 \to X^1 \to X^0 \to \cdots$. Then a 2-periodic complex is the same thing as a complex over $k[u,u^{-1}]$, with the action of u giving the periodicity isomorphisms. There is an obvious algebra map $k \to k[u,u^{-1}]$, and by restriction along this map we can view a $\mathbb{Z}/2$-graded dg category as a $\mathbb{Z}$-graded dg category. In particular, if $\mathcal{C}$ is a $\mathbb{Z}/2$-graded dg category, it has two different kinds of Hochschild cohomology: firstly $HH^*_{k[u,u^{-1}]}(\mathcal{C})$, where we work over the base ring of Laurent polynomials, and $HH^*_k(\mathcal{C})$, where we work over the base ring k. **There is no need for these to agree!** Next week we will see that $HH^*_k(\mathrm{MF}(R,w))$ is a (generically nontrivial) quotient of $HH^*_{k[u,u^{-1}]}(\mathrm{MF}(R,w))$.

[15] The chief difficulty in proving this for cohomology is that HH^* is not a functor, since $\mathrm{Ext}^*(\mathcal{C},\mathcal{C})$ is contravariant in the first variable and covariant in the other.

11 Loose ends

Let k be an algebraically closed field of characteristic zero, $R = k[\![x_1, \ldots, x_n]\!]$ and $w \in \mathfrak{m}_R$ defines an isolated singularity. Regard $\mathrm{MF}(R, w)$ as a $\mathbb{Z}$-graded dg category, by cobase change along the forgetful functor induced by the map $k \to k[u, u^{-1}]$.

Theorem 11.1 (Keller)**.** *When $\mathrm{MF}(R, w)$ is viewed as a $\mathbb{Z}$-graded dg category, $HH^0(\mathrm{MF}(R, w))$ is the algebra M_w/w.*

The algebra $T_w \coloneqq M_w/w$ is the **Tjurina algebra** of the singularity R/w. When w is a quasi-homogenous singularity, one has $T_w \cong M_w$, since w is contained in the ideal $\left(\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\right)$ by Euler's theorem on homogenous functions.

Proof sketch. One first develops a theory of **singular Hochschild cohomology** for rings, and in the situation of interest one has isomorphisms

$$HH^0(D^{\mathrm{dg}}_{\mathrm{sg}})(R/w) \cong HH^0_{\mathrm{sg}}(R/w) \cong HH^0(R/w).$$

One can compute the zeroth Hochschild cohomology of R/w to be the Tjurina algebra by a direct computation. □

Note that due to the existence of Knörrer periodicity, singularities of different dimensions may have isomorphic Tjurina algebras. However, when the dimension is fixed, the Tjurina algebra is a complete invariant of the singularity:

Theorem 11.2 (Mather–Yau; cf. [GP17])**.** *Let $w_1, w_2 \in \mathfrak{m}_R$ define isolated singularities. Then $R/w_1 \cong R/w_2$ if and only if $T_{w_1} \cong T_{w_2}$.*

As stated the theorem is false in characteristic p - consider $f \coloneqq x^{p+1} + y^{p+1}$ and $f + x^p$. However a modified version is true. The Milnor algebra also classifies complete local isolated singularities, up to a different notion of equivalence.

Corollary 11.3. *Let $w_1, w_2 \in \mathfrak{m}_R$ define isolated singularities. Then $R/w_1 \cong R/w_2$ if and only if the $\mathbb{Z}$-graded dg categories $\mathrm{MF}(R, w_1)$ and $\mathrm{MF}(R, w_2)$ are quasi-equivalent.*

Similarly, if the w_i are quasi-homogenous, then the quasi-equivalence type of the $\mathbb{Z}/2$-graded dg category $\mathrm{MF}(R, w_i)$ is a complete invariant.

In preparation for the next talk we will need some preliminaries on relative singularity categories, which we take from [KY18]. Let R be any commutative Gorenstein ring and M any MCM R-module. Recall that we regard the endomorphism algebra $A \coloneqq \mathrm{End}_R(R \oplus M)$ as a noncommutative partial resolution of R. Note that A comes with an idempotent $e = \mathrm{id}_R$ and the quotient A/AeA is isomorphic to $\underline{\mathrm{End}}_R(M)$, the stable endomorphism algebra of M (i.e. the endomorphism algebra of M in the stable category $\underline{\mathrm{MCM}}(R)$). Recall that if $\mathcal{T}$ is a triangulated category then $\mathcal{T}^\omega$ denotes its idempotent completion.

Proposition 11.4. *There exists a connective differential graded algebra B and a triangle functor* $\mathbf{per}(B) \to \Delta_R(A)^\omega$ *which sends B to M. Moreover if A has finite global dimension (i.e. if it is a noncommutative resolution of R) then this functor is a triangle equivalence.*

In fact, one can compute B as a sort of derived quotient of A by e: if $\tilde{A} \to A$ denotes a dga resolution of A then there is a quasi-isomorphism $\tilde{A}/\tilde{A}e\tilde{A} \simeq B$. In particular this implies that $H^0(B) \cong A/AeA$.

Remark 11.5. In fact, B literally is the derived quotient of A by e, in the sense of [BCL18]. This description shows that $D(A)$ fits into a recollement

$$D(B) \to D(A) \to D(R)$$

and one can use this recollement to prove the theorem.

Suppose now that R is a complete local isolated hypersurface singularity. This is not strictly necessary, but it simplifies things: in particular it ensures that $D_{\mathrm{sg}}(R)$ is idempotent complete and hom-finite, and in particular A/AeA is a finite dimensional algebra.

Proposition 11.6. *If M is nonzero in $D_{\mathrm{sg}}(R)$, then there is a surjective triangle functor* $\Sigma : \mathbf{per}(B) \to D_{\mathrm{sg}}(R)$*. If A has finite global dimension then the kernel of Σ is the category $D_{\mathrm{fd}}(B)$, the category of dg B-modules N whose cohomology has finite total dimension (i.e. only finitely many H^iN are nonzero, and they are all finite dimensional) and we hence obtain a triangle equivalence*

$$\frac{\mathbf{per}(B)}{D_{\mathrm{fd}}(B)} \xrightarrow{\simeq} D_{\mathrm{sg}}(R).$$

Proof idea. Existence of the functor comes from composing the map from $\mathbf{per}(B)$ with the natural projection $\Delta_R(A)^\omega \to D_{\mathrm{sg}}(R)^\omega \simeq D_{\mathrm{sg}}(R)$. Surjectivity follows from a theorem of Takahashi saying that singularity categories of isolated singularities have no nontrivial thick subcategories. □

Remark 11.7. Implicit in the statement of the theorem is that, when A has finite global dimension, every object in $D_{\mathrm{fd}}(B)$ is perfect. This is not necessarily true when A has infinite global dimension.

{kycor}

Corollary 11.8. *If R admits a noncommutative resolution A, then there exists a differential graded algebra B and a triangle equivalence*

$$\frac{\mathbf{per}(B)}{D_{\mathrm{fd}}(B)} \simeq D_{\mathrm{sg}}(R).$$

One can compute B as the derived quotient of A by an idempotent.

Remark 11.9. In fact, one can show that, if R_i are two complete local isolated hypersurface singularities of the same dimension, with associated dgas B_i coming from an irreducible module M_i, that $R_1 \cong R_2$ if and only if there is a quasi-isomorphism $B_1 \simeq B_2$. To do this it is enough to show that the quasi-isomorphism class of B recovers the quasi-equivalence class of $D_{\mathrm{sg}}(R)$. In fact $D_{\mathrm{sg}}(R)$ can be described as the category of perfect modules over a certain localisation of B, as in [Boo20].

12 Deformations of Kleinian singularities (Callum B.)

This section mostly follows [KY18, §9]; the main result is originally due to Crawford [Cra20]. In this section we'll work over $\mathbb{C}$. If Q is a quiver, we denote by $\bar{Q}$ its double, which has for every arrow $a : i \to j$ of Q an opposite arrow $a^* : j \to i$. If Q is a finite quiver, the **preprojective algebra** of Q is the algebra $\Pi(Q)$ obtained as the path algebra $\mathbb{C}\bar{Q}$ of the doubled quiver modulo the relation $\sum_a [a, a^*] = 0$. Label the vertices of Q by $1, \dots, n$. A **weight** is a vector $\lambda \in \mathbb{C}^n$. The **deformed preprojective algebra** is the quotient of $\mathbb{C}\bar{Q}$ by the deformed preprojective relations $\sum_a e_i[a, a^*]e_i = \lambda_i e_i$, one for each i. Clearly $\Pi^0(Q) \cong \Pi(Q)$.

A **Kleinian singularity** is a quotient of $\mathbb{C}^2$ by a finite subgroup of $SL_2(\mathbb{C})$. They are isolated hypersurface singularities, and they have a minimal resolution whose exceptional locus is a tree of rational curves, linked in an ADE configuration. All ADE Dynkin diagrams can be obtained in this way, and each such diagram corresponds to a unique Kleinian singularity.

Crawley-Boevey and Holland [CBH98] constructed noncommutative deformations of Kleinian singularities using deformed preprojective algebras, according to the following recipe. Fix a Kleinian singularity R. Let Q be an arbitrary quiver whose underlying graph is the affine Dynkin diagram of the Dynkin type of R. Choose an extending vertex e_0 of Q and label the other vertices $1, \dots, n$. Fix a weight λ on Q. The ring $e_0\Pi^\lambda(Q)e_0$ is then a noncommutative deformation of R: indeed if $\lambda = 0$ then it is isomorphic to R.

We will study the singularity category of $e_0\Pi^\lambda(Q)e_0$ via the method of 11.8. Crawley-Boevey and Holland prove that $\Pi^\lambda(Q)$ has finite global dimension, so we need to find a dga resolution $\tilde{A} \to \Pi^\lambda(Q)$, take the quotient $B \coloneqq \tilde{A}/\tilde{A}e_0\tilde{A}$, and then the singularity category of $e_0\Pi^\lambda(Q)e_0$ will be the triangle quotient $\mathbf{per}(B)/D_{\mathrm{fd}}(B)$. Our resolution will be given by a derived version of the preprojective algebra.

Let Q be any finite quiver. Let $\tilde{Q}$ be the graded quiver consisting of $\bar{Q}$ placed in degree zero, along with a loop t_i in degree -1 at each vertex i. The **derived preprojective algebra** is the dg algebra $\underline{\Pi}(Q)$ whose underlying graded algebra is $\mathbb{C}\tilde{Q}$, and whose differential satisfies $d(a) = d(a^*) = 0$ and $d(t_i) = \sum_a e_i[a, a^*]e_i$. The **deformed derived preprojective algebra** $\underline{\Pi}^\lambda(Q)$ is the same, but the differential on t_i is modified to be the sum $d(t_i) = \sum_a e_i[a, a^*]e_i - \lambda_i e_i$. Clearly we have an isomorphism $H^0(\underline{\Pi}^\lambda(Q)) \cong \Pi^\lambda(Q)$. Observe that $\underline{\Pi}^\lambda(Q)$ is a cofibrant dga (i.e. its underlying graded algebra is free and the differential is upper-triangular).

Remark 12.1. The dga $\underline{\Pi}^\lambda(Q)$ is a deformed 2-Calabi–Yau completion of the path algebra $\mathbb{C}Q$, in the sense of Keller [Kel11]. More generally, to construct the n-CY completion one puts a^* in degree $2-n$ and t_i in degree $1-n$. The differential remains the same. Deformed n-CY completions are more subtle; we remark that Ginzburg dgas of quivers with potential are examples of 3-CY completions (the potential should be thought of as the deformation parameter).

Proposition 12.2 (Hermes [Her16]). *If Q is a finite quiver without oriented cycles and not a Dynkin quiver then the projection map $\underline{\Pi}(Q) \to \Pi(Q)$ is a quasi-isomorphism.*

In other words, when $\lambda = 0$ the derived preprojective algebra is a resolution of the usual preprojective algebra.

Remark 12.3. If Q is a finite quiver without cycles then $\Pi(Q)$ is finite dimensional precisely when Q is of Dynkin type; since finite dimensional algebras cannot be 2CY, the conclusion of the proposition cannot hold when Q is a Dynkin quiver. In this case, Hermes constructs a minimal model of $\underline{\Pi}(Q)$. If Q is a finite quiver without cycles then $\Pi(Q)$ is Koszul precisely when Q is not of Dynkin type; one can prove the proposition using this.

By filtering away the λ_i, one can prove the above proposition for the deformed derived preprojective algebra:

Proposition 12.4 (Kalck–Yang). *If Q is a finite quiver without oriented cycles and not a Dynkin quiver and λ is a weight on Q then the projection map $\underline{\Pi}^\lambda(Q) \to \Pi^\lambda(Q)$ is a quasi-isomorphism.*

Proof. Introduce a secondary Adams grading on $\underline{\Pi}^\lambda(Q)$ by putting a and a^* in degree -1, e_i in degree 0, and t_i in degree -2. This secondary grading induces a filtration on $\underline{\Pi}^\lambda(Q)$ and $\underline{\Pi}(Q)$ by Adams path length. One compares the induced spectral sequences for $\underline{\Pi}^\lambda(Q)$ and for $\underline{\Pi}(Q)$ and finds that the cohomology of $\underline{\Pi}^\lambda(Q)$ must be concentrated in degree zero, and hence the result holds. □

Let's return to the Kleinian situation. The above gives us an equivalence

$$D_{\mathrm{sg}}(e_0\Pi^\lambda(Q)e_0) \simeq \frac{\mathbf{per}(B)}{D_{\mathrm{fd}}(B)}$$

where B is the dga $\underline{\Pi}^\lambda(Q)/(e_0)$. Recall that Q was an affine Dynkin quiver, with extending vertex 0; let Q' be the full subquiver on the vertices $1,\ldots,n$ and λ' the corresponding weight on Q'. It is not too hard to see that B is isomorphic to $\underline{\Pi}^{\lambda'}(Q')$. This dga is still rather mysterious, so we would like to pass to a simpler model to remove the noncommutative data.

Let $Q'_{\lambda'}$ be the full subquiver of Q' on those vertices i with $\lambda'_i = 0$. There is a natural surjection $\underline{\Pi}^{\lambda'}(Q') \to \underline{\Pi}(Q'_{\lambda'})$ which Crawford proves to be a quasi-isomorphism as long as λ is **quasi-dominant**, meaning that each of the λ'_i either has positive real part, or zero real part and nonnegative imaginary part. Since one may choose λ to be quasi-dominant without affecting the isomorphism type of the noncommutative deformation, we may assume that this is the case.

Observe that $Q'_{\lambda'}$ must be a disjoint union of Dynkin quivers $Q^1, \dots, Q^s$; let $R^1, \dots, R^s$ denote the corresponding Kleinian singularities. The above arguments tell us that $\mathbf{per}\,(\underline{\Pi}(Q^j))/D_{\mathrm{fd}}(\underline{\Pi}(Q^j))$ is triangle equivalent to the singularity category of R^j. We hence have triangle equivalences

$$D_{\mathrm{sg}}(e_0\Pi^\lambda(Q)e_0) \simeq \frac{\mathbf{per}\,(\underline{\Pi}(Q'_{\lambda'}))}{D_{\mathrm{fd}}(\underline{\Pi}(Q'_{\lambda'}))} \simeq \bigoplus_{j=1}^{s} \frac{\mathbf{per}\,(\underline{\Pi}(Q^j))}{D_{\mathrm{fd}}(\underline{\Pi}(Q^j))} \simeq \bigoplus_{j=1}^{s} D_{\mathrm{sg}}(R^j)$$

exhibiting the singularity category of the noncommutative deformation as a sum of singularity categories of Kleinian singularities.

13 Hochschild cohomology via Koszul duality

Recall that I mentioned, as an aside, that matrix factorisation categories were categories of twisted modules over certain curved dg (co)algebras. I'll expand on this before describing how to recover Dyckerhoff's results on Hochschild cohomology from this perspective. I'll follow [CT13, Tu14].

Definition 13.1. A **curved dg algebra** is the data of

- A graded algebra A
- A degree 1 derivation $d : A \to A$
- An element $h \in A^2$

such that

- $d(h) = 0$
- $d^2(x) = [h, x] = hx - xh$ for all $x \in A$

We call d the **differential** and h the **curvature element**.

Example 13.2. If A is a dg algebra, then it is naturally a curved dg algebra with zero curvature.

Example 13.3. If A is a graded algebra and $h \in Z(A)$ is of degree 2 then the pair (A, h) is a curved dg algebra with zero differential (a.k.a. a **curved graded algebra**).

Example 13.4. Curved dg coalgebras are defined similarly: they are graded coalgebras C with a coderivation d and a curvature functional $h : C \to k$. If C is a curved dg coalgebra then one can define its cobar construction ΩC, which is a curved dg algebra. If C was coaugmented then ΩC is a dg algebra. Similarly if A is a dg algebra it has a bar construction BA which is a conilpotent curved dg coalgebra[16].

[16] Defining a bar construction for curved dg algebras is possible, but a little more subtle, since one can no longer expect it to be a conilpotent curved coalgebra.

A morphism of curved algebras $A \to B$ is a pair (f, b) where $f : A \to B$ is a map of graded algebras, and $b \in B$ is a degree 1 element satisfying the formulas

- $f(da) = d(fa) + [b, fa]$
- $f(h_A) = h_B + db + b^2$.

Morphisms compose by putting $(g, b)(f, a) = (gf, b + g(a))$. Observe that the inclusion of dg algebras into curved dg algebras is faithful but not full; there are more curved maps than uncurved maps between dg algebras.

A (left) **dg module** over a curved dg algebra is a graded A-module M with a differential d of degree 1 satisfying the Leibniz rule with respect to the A-action, such that $d^2(m) = hm$. Note that A need not be a module over itself! These assemble into a dg category of dg-A-modules - the hom-complexes are defined exactly as in the uncurved case. This has an important full subcategory $\mathrm{Tw}(A)$ of finitely generated twisted modules, whose objects are those A-modules whose underlying graded A-modules are free of finite rank over the underlying graded algebra of A.

Remark 13.5. If A is a dga, then $\mathrm{Tw}(A)$ is a model for the pretriangulated hull of the one-object dg category A. A twisted module can be described as a twisting of the free module $A^{\oplus n}$.

Example 13.6. Let R be any commutative ring and let $w \in R$ be any element. Define a curved $\mathbb{Z}/2$-graded algebra R_w by:

- The underlying graded algebra of R_w is R, concentrated in even degree.
- R_w has zero differential and the curvature element is given by w.

Then $\mathrm{Tw}(R_w)$ is precisely the $\mathbb{Z}/2$-graded dg category $\mathrm{MF}(R, w)$.

The main question of this talk: is there a good theory of Hochschild (co)homology HH' for curved dg algebras, such that $HH(\mathrm{Tw}(A)) \simeq HH'(A)$? This would reduce the computation of Hochschild (co)homology of matrix factorisation categories to that of HH' of certain curved graded algebras.

Fix an abelian grading group G; note that there's a unique map $\mathbb{Z} \to G$. From now on, our curved dg algebras will be G-graded. If A is a curved dg algebra, it has a Hochschild complex $HH^*(A)$. As a graded vector space, it is the product $\prod_{i=0}^{\infty} \mathrm{Hom}(A^{\otimes i}, A)$. Note that this is the direct product totalisation of a $\mathbb{Z} \times G$-graded vector space and hence has a G-grading. The Hochschild differential sends a degree k cochain f to the sum

$$(a_1 \dots, a_l) \mapsto \sum_{\substack{j \\ k \leq l}} (-1)^{j+|a_1|+\dots+|a_j|} m_{l-k+1}(a_1, \dots, f(a_{j+1}, \dots, a_{j+k}), \dots, a_l)$$

$$+ \sum_i (-1)^{|f|+|a_1|+\dots+|a_i|} f(a_1, \dots, m_{l-k+1}(a_{i+1}, \dots, a_{i+l-k+1}), \dots, a_l)$$

where $m_0() = h$, $m_1(a) = da$, and $m_2(a, b) = ab$.

There is a similar definition for $HH_*(A)$ as the totalisation of the bigraded vector space $A \otimes A^i$ equipped wih a similar differential.

Proposition 13.7. *Let A be a curved graded algebra. Then both $HH^*(A)$ and $HH_*(A)$ vanish.*

Proof sketch. A spectral sequence argument reduces to the case where A is a curved algebra with zero multiplication, and one can compute $HH(A)$ explicitly in this situation. □

So usual HH is not a good invariant for our purposes. We introduce a modified version.

Definition 13.8 ([PP12])**.** Let A be a curved dg algebra. The **compactly supported Hochschild cohomology complex** is the G-graded vector space $HH_c^*(A) \coloneqq \bigoplus_i \mathrm{Hom}(A^{\otimes i}, A)$ equipped with the same Hochschild differential. Similarly, the **Borel–Moore Hochschild homology complex** is the G-graded vector space $HH_*^{\mathrm{BM}}(A) \coloneqq \prod_i (A \otimes A^i)$ equipped with the Hochschild differential.

Proposition 13.9. *If A is a connective $\mathbb{Z}$-graded finite dimensional curved dg algebra then the natural maps*

$$HH_c^*(A) \to HH^*(A)$$

$$HH_*(A) \to HH_*^{\mathrm{BM}}(A)$$

are quasi-isomorphisms.

Proof idea. Under the hypotheses, $HH^*(A)$ is the completion of $HH_c^*(A)$ with respect to the $\mathbb{Z}$-grading. □

Proposition 13.10. *Let $R = k[\![x_1, \ldots, x_n]\!]$ and $w \in \mathfrak{m}_R$ defining an isolated singularity. Let R_w be the associated $\mathbb{Z}/2$-graded curved algebra whose twisted modules are the matrix factorisations of w. Then there are isomorphisms*

$$HH^*(R_w) \cong M_w$$

$$HH_*(R_w) \cong M_w[\dim R].$$

The proof is a computation. At least for homology, one can deduce this from some general theory, as we now describe. The curved graded algebra R_w is the linear dual of the curved graded coalgebra $C_w \coloneqq k[y_1, \ldots, y_n]$ with curvature functional w^*. We have a quasi-equivalence of $\mathbb{Z}/2$-graded dg categories

$$\mathrm{Tw}(R_w) \simeq \mathrm{Tw}(C_w)$$

where $\mathrm{Tw}(C_w)$ denotes the category of finitely generated twisted comodules. Since R was a local ring, it is augmented, and hence C_w is coaugmented. Hence it has a Koszul dual dg algebra $\Omega(C_w)$, and general results in Koszul duality give us a quasi-equivalence of $\mathbb{Z}/2$-graded dg categories

$$\mathrm{Tw}(C_w) \simeq \mathbf{per}\,(\Omega C_w).$$

Putting these quasi-equivalences together, we get a quasi-equivalence of $\mathbb{Z}/2$-graded dg categories

$$\mathrm{MF}(R, w) \simeq \mathbf{per}\,(\Omega C_w).$$

Remark 13.11. ΩC_w is quasi-isomorphic to the dga constructed by Dyckerhoff defined in terms of polynomial differential operators.

Hochschild (co)homology is invariant under quasi-equivalences, and derived Morita invariant by 10.11, and so we get quasi-isomorphisms[17]

$$HH^*(\mathrm{MF}(R, w)) \simeq HH^*(\Omega C_w)$$

$$HH_*(\mathrm{MF}(R, w)) \simeq HH_*(\Omega C_w).$$

Tu then proves that $HH_*(\Omega C_w)$ is naturally isomorphic to the linear dual of $HH_*^{\mathrm{BM}}(R_w)$, using the intermediate notion of the Hochschild complex of a (curved) coalgebra. Heuristically, this is because the linear dual of $HH_*(C_w)$ is $HH_*^{\mathrm{BM}}(R_w)$, since to get from left to right we pass the linear dual inside the coproduct (turning it into a product), through the tensor products, and finally $C_w^* \cong R_w$.

Since $HH_*^{\mathrm{BM}}(R_w)$ is finite dimensional, it is isomorphic to its own dual, and it follows that $HH_*(\mathrm{MF}(R_w)) \simeq HH_*^{\mathrm{BM}}(R_w)$, without doing any computation. I'm not aware if the same strategy works for HH^*.

References

[BCL18] C. Braun, J. Chuang, and A. Lazarev. Derived localisation of algebras and modules. *Adv. Math.*, 328:555–622, 2018.

[Bec14] Hanno Becker. Models for singularity categories. *Adv. Math.*, 254:187–232, 2014.

[BK12] Igor Burban and Martin Kalck. The relative singularity category of a non-commutative resolution of singularities. *Adv. Math.*, 231(1):414–435, 2012.

[Boo19] Matt Booth. *The derived contraction algebra.* PhD thesis, The University of Edinburgh, 2019. ArXiv:1911.09626.

[Boo20] Matt Booth. Singularity categories via the derived quotient, 2020. arXiv:2003.05439.

[Boo21] Matt Booth. Singularity categories via the derived quotient. *Adv. Math.*, 381:Paper No. 107631, 56, 2021.

[17]For HH_* this is a quasi-isomorphism of dg vector spaces and so witnessed by an actual map. For HH^* is it an isomorphism in the homotopy category of B_∞-algebras; there may not be a direct map between the two, rather just a zigzag.

[BRTV18] Anthony Blanc, Marco Robalo, Bertrand Toën, and Gabriele Vezzosi. Motivic realizations of singularity categories and vanishing cycles. *J. Éc. polytech. Math.*, 5:651–747, 2018.

[Buc86] Ragnar-Olaf Buchweitz. Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings, 1986. Available at `https://tspace.library.utoronto.ca/handle/1807/16682`.

[Căl05] Andrei Căldăraru. Derived categories of sheaves: a skimming. In *Snowbird lectures in algebraic geometry*, volume 388 of *Contemp. Math.*, pages 43–75. Amer. Math. Soc., Providence, RI, 2005.

[CBH98] William Crawley-Boevey and Martin P. Holland. Noncommutative deformations of Kleinian singularities. *Duke Math. J.*, 92(3):605–635, 1998.

[CKWW21] Xiao-Wu Chen, Bernhard Keller, Yu Wang, and Zhengfang Wang. The dg leavitt algebra, singular yoneda category and singularity category, 2021. arXiv:2109.11278.

[Cra20] Simon Crawford. Singularity categories of deformations of Kleinian singularities. *Algebra Number Theory*, 14(2):349–382, 2020.

[CT13] Andrei Căldăraru and Junwu Tu. Curved A_∞ algebras and Landau-Ginzburg models. *New York J. Math.*, 19:305–342, 2013.

[Dri04] Vladimir Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.

[Dyc11] Tobias Dyckerhoff. Compact generators in categories of matrix factorizations. *Duke Math. J.*, 159(2):223–274, 2011.

[Eis80] David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.

[EP15] Alexander I. Efimov and Leonid Positselski. Coherent analogues of matrix factorizations and relative singularity categories. *Algebra Number Theory*, 9(5):1159–1292, 2015.

[GP17] G.-M. Greuel and T. H. Pham. Mather-Yau theorem in positive characteristic. *J. Algebraic Geom.*, 26(2):347–355, 2017.

[Gro92] Buenos Aires Cyclic Homology Group. Hochschild and cyclic homology of hypersurfaces. *Adv. Math.*, 95(1):18–60, 1992.

[GS20] J. P. C. Greenlees and Greg Stevenson. Morita theory and singularity categories. *Adv. Math.*, 365:107055, 51, 2020.

[Her16] Stephen Hermes. Minimal model of Ginzburg algebras. *J. Algebra*, 459:389–436, 2016.

[HK18] Zheng Hua and Bernhard Keller. Cluster categories and rational curves, 2018. arXiv:1810.00749.

[Kel99] Bernhard Keller. On the cyclic homology of exact categories. *J. Pure Appl. Algebra*, 136(1):1–56, 1999.

[Kel11] Bernhard Keller. Deformed Calabi-Yau completions. *J. Reine Angew. Math.*, 654:125–180, 2011. With an appendix by Michel Van den Bergh.

[Kel18] Bernhard Keller. Singular Hochschild cohomology via the singularity category. *C. R. Math. Acad. Sci. Paris*, 356(11-12):1106–1111, 2018.

[Knö87] Horst Knörrer. Cohen-Macaulay modules on hypersurface singularities. I. *Invent. Math.*, 88(1):153–164, 1987.

[Kra05] Henning Krause. The stable derived category of a Noetherian scheme. *Compos. Math.*, 141(5):1128–1162, 2005.

[KY16] Martin Kalck and Dong Yang. Relative singularity categories I: Auslander resolutions. *Adv. Math.*, 301:973–1021, 2016.

[KY18] Martin Kalck and Dong Yang. Relative singularity categories II: DG models, 2018. arXiv:1803.08192.

[KY20] Martin Kalck and Dong Yang. Relative singularity categories III: cluster resolutions, 2020. arXiv:2006.09733.

[Lam99] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[Lan16] Marit Buset Langfeldt. Triangulated categories and matrix factorizations, 2016.

[Lur] Jacob Lurie. Functorial kernel in derived category. MathOverflow. URL:https://mathoverflow.net/q/386369 (version: 2021-03-13).

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 2012.

[MY82] John N. Mather and Stephen S. T. Yau. Classification of isolated hypersurface singularities by their moduli algebras. *Invent. Math.*, 69(2):243–251, 1982.

[Nag62] Masayoshi Nagata. *Local rings*. Interscience Tracts in Pure and Applied Mathematics, No. 13. Interscience Publishers a division of John Wiley & Sons New York-London, 1962.

[Nee01] Amnon Neeman. *Triangulated Categories*. Princeton University Press, 2001.

[Orl04] D. O. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004.

[Orl11] Dmitri Orlov. Formal completions and idempotent completions of triangulated categories of singularities. *Adv. Math.*, 226(1):206–217, 2011.

[Pip22] Massimo Pippi. On the structure of dg categories of relative singularities. *High. Struct.*, 6(1):375–402, 2022.

[Pos11] Leonid Positselski. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Mem. Amer. Math. Soc.*, 212(996):vi+133, 2011.

[PP12] Alexander Polishchuk and Leonid Positselski. Hochschild (co)homology of the second kind I. *Trans. Amer. Math. Soc.*, 364(10):5311–5368, 2012.

[Ste] Greg Stevenson. On the failure of functorial cones in triangulated categories.

[Sym22] Julie Symons. On singularity categories, 2022.

[Tab10] Gonçalo Tabuada. On Drinfeld's dg quotient. *J. Algebra*, 323(5):1226–1240, 2010.

[Toë11] Bertrand Toën. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, volume 2008 of *Lecture Notes in Math.*, pages 243–302. Springer, Berlin, 2011.

[Tu14] Junwu Tu. Matrix factorizations via Koszul duality. *Compos. Math.*, 150(9):1549–1578, 2014.

[VdB15] Michel Van den Bergh. Calabi-Yau algebras and superpotentials. *Selecta Math. (N.S.)*, 21(2):555–603, 2015.

[Ver96] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.