# Stable Homotopy Theory

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These are notes for a pair of talks for the Hodge Club about stable homotopy theory. The notes are more expansive than what I'll actually say. I'll give a quick introduction to the very basics, and then the first examples of stable phenomena in topology. Later I'll talk about a more modern approach in terms of spectra, and explain how spectra relate to generalised cohomology theories.

## 1 The Basics

### 1.1 Homotopy

Let me first note that all spaces here will be pointed (i.e. equipped with a basepoint). Maps will be basepoint-preserving. The category of pointed spaces is denoted $\mathbf{Top}_*$. Note that a pointed space is necessarily nonempty! We can also think of $\mathbf{Top}_*$ as the undercategory $\{*\} \downarrow \mathbf{Top}$. Basepoints are really crucial for what we want to do[1], although I won't mention them much. Just remember that they're there.

It's worth mentioning what the coproduct in $\mathbf{Top}_*$ is. The **wedge product** $X \vee Y$ of two pointed spaces $X, Y$ is the analogue of the disjoint union: we simply take $X \sqcup Y$ but identify the basepoints in each copy. For example $S^1 \vee S^1$ is a figure eight. We can also define the **smash product** $X \wedge Y$ as $(X \times Y)/(X \vee Y)$. We have $S^n \wedge S^m = S^{n+m}$.

Given maps $f, g : X \to Y$, a **homotopy** from $f$ to $g$ is a map $H : X \times [0,1] \to Y$ with $H\big|_{X\times\{0\}} = f$ and $H\big|_{X\times\{1\}} = g$. We can think of $H$ as interpolating continuously between $f$ and $g$. If there is a homotopy from $f$ to $g$ then we say that they are **homotopic** and write $f \simeq g$. For example if $Y = \mathbb{R}^n$ then any two maps are homotopic via the **straight-line homotopy**

$$H(x,t) = (1-t)\cdot f(x) + t \cdot g(x)$$

[1] For example homotopy groups depend on the choice of basepoint. More importantly, some of the theorems we'll make use of (e.g. the Brown Representability Theorem) are false if we drop the 'pointed' assumption.

Homotopy is an equivalence relation on maps and we denote the corresponding quotient of $\mathrm{Hom}(X, Y)$ by $[X, Y]$. If we like, we can define a category $\mathbf{hTop}_*$, the **homotopy category of pointed spaces**[2], with the same objects as $\mathbf{Top}_*$ and morphism sets given by $\mathrm{Hom}_{\mathbf{hTop}_*}(X, Y) := [X, Y]$. Composition works as normal, since $f \simeq f'$ and $g \simeq g'$ implies that $fg \simeq f'g'$. The category $\mathbf{hTop}_*$ is not a concrete category; this is a theorem of Freyd, proved in [3].

## 1.2 The Fundamental Group

Consider the set $[S^1, Y]$, the set of homotopy classes of based loops in $Y$. This set admits a group structure: if $[\gamma]$ and $[\gamma']$ are homotopy classes of loops in $Y$ then their product is the homotopy class of the loop $\gamma \cdot \gamma'$ given by going around $\gamma$ and then going around $\gamma'$. Of course one needs to check that this operation is well-defined, admits inverses[3], et cetera.

The **fundamental group of** $Y$ is the group $\pi_1(Y) = [S^1, Y]$. If $A \subseteq X$ one can also define relative groups $\pi_1(X, A)$ consisting of those maps which are homotopic through homotopies that fix $A$. The relative groups will be important for the long exact sequences in homotopy that we'll use a little of later.

*Example* 1.2.1. We have $\pi_1(S^1) = \mathbb{Z}$ (homotopy classes of based loops in $S^1$ are completely described by their winding number), $\pi_1(S^2) = 0$ (any loop in $S^2$ can be shrunk to a point) and $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$ (so the fundamental group need not be abelian)[4].

The assignment $X \mapsto \pi_1(X)$ is functorial. Moreover it's homotopy invariant: if two maps are homotopic then they induce the same map between fundamental groups. Two spaces $X, Y$ are **homotopy equivalent** if there are maps $f : X \leftrightarrow Y : g$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. A **contractible** space is one that's homotopy equivalent to a point. Using functoriality of $\pi_1$ we can easily see that homotopy equivalent spaces have isomorphic fundamental groups. The converse is not true: for example $*$ and $S^2$ have isomorphic fundamental groups, but $S^2$ is not contractible[5].

The functor $\pi_1$ commutes with both products and coproducts for reasonable[6] spaces. Moreover it's independent of choice of basepoint for path-connected spaces (it's clearly not for all spaces - e.g. $S^1 \sqcup S^2$).

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[2]This category is the same as the localisation of $\mathbf{Top}_*$ at the homotopy equivalences.

[3]The identity element is the constant loop $S^1 \to *$. The inverse of $\gamma$ is '$\gamma$ but in the opposite direction'.

[4]In fact it's true that for nice spaces $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$; this is a weak form of the **Van Kampen Theorem**.

[5]We can see this either by computing $\pi_2$ or using (co)homology.

[6]path-connected, locally contractible

Finally, one can phrase the fact that $[S^1, Y]$ is a group in a more sophisticated manner: Suppose $[f], [g]$ are homotopy classes of loops in $Y$. Then the product $[f][g]$ is the homotopy class of the map given by

$$S^1 \to S^1 \vee S^1 \to Y$$

where the map $S^1 \to S^1 \vee S^1$ is the folding map given by contracting an equator, and the map $S^1 \vee S^1 \to Y$ does $f$ on the first copy and $g$ on the second.

### 1.3 Higher homotopy groups

One can repeat the above constructions for all mapping sets $[S^n, Y]$ for $n > 0$ and hence obtain functors $\pi_n$ for all positive $n$. One can define $\pi_0(Y)$ as the set of connected components of $Y$, but this does not necessarily get a group structure - we'll see later that we can get around this in the stable world. It's also possible to define relative homotopy groups, and there's an associated long exact sequence for pairs.

An important property of the higher groups is that they are all abelian: the best way to see this is to regard $S^n = I^n/\partial^n$, realise that composition is equivalent to just placing each map in one half of $I^n$, and then simply switching the maps around by a homotopy (see [4], pg. 340). This doesn't work for $S^1$ because there isn't enough space to switch the maps around.

A map $X \to Y$ is a **weak homotopy equivalence** if it induces isomorphisms on all homotopy groups. Clearly any homotopy equivalence is a weak homotopy equivalence. **Whitehead's Theorem** gives a partial converse and says that a weak homotopy equivalence between CW complexes is a homotopy equivalence. Weak homotopy equivalence is a stronger condition than simply having all homotopy groups isomorphic: $S^2 \times \mathbb{RP}^3$ and $S^3 \times \mathbb{RP}^2$ have the same homotopy groups, but are not weakly homotopy equivalent[7] since there's no map of spaces witnessing this isomorphism.

The homotopy groups of spheres are very badly behaved, which is at first surprising since spheres are 'simple' spaces[8]. For example, we have an isomorphism $\pi_{14}(S^3) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{84}$. Historically, the first nontrivial example is $\pi_3 S^2 = \mathbb{Z}$, generated by the Hopf fibration. We'll come back to spheres several times.

[7] They're both 5-dimensional manifolds, and one is orientable and the other is not, so their fifth homology groups must differ. So they're not (weakly) homotopy equivalent.

[8] Of course spheres are only 'simple' in a homological sense. The corresponding notion for homotopy is that of an Eilenberg-Mac Lane space; see 3.2.2, iv).

## 1.4 Operations on spaces

The **cone**[9] on $X$ is the space $CX := (X \times [0,1])/\{(x,1) : x \in X\}$ a.k.a. a cylinder on $X$ with one end collapsed down. It's contractible - we can contract it to the vertex. Coning is functorial: given $f : X \to Y$ we can define $Cf : CX \to CY$ by just using $f$ levelwise (one needs to check continuity at the vertex).

The **suspension** $\Sigma X$ of $X$ is the space $(CX \sqcup CX)/X$, or two cones on $X$ joined at the base (we can think of 'suspending' $X$ between two points). A key observation is that $\Sigma X = S^1 \wedge X$, and as a consequence we have $\Sigma S^n = S^{n+1}$. Suspending spaces is homotopically much more interesting since, unlike the cone, the suspension is not usually contractible.

Suspension is also functorial, and the functoriality gives us the **suspension map** $\sigma : \pi_n X \to \pi_{n+1}\Sigma X$ between homotopy groups, defined by the following procedure:

Take $[f] \in \pi_n(X)$. Then $\Sigma f : S^{n+1} \to \Sigma X$. Taking homotopy classes we get $\sigma[f] := [\Sigma f] \in \pi_{n+1}(\Sigma X)$. This map is well-defined since a nullhomotopy of $f$ lifts to a nullhomotopy of $\Sigma f$. It's a homomorphism because $\Sigma$ commutes with $\vee$ - recall the definition of the group operation in terms of the folding map. More generally, $\wedge$ distributes over $\vee$.

The very first question we ask in stable homotopy theory is:

**When is the map $\sigma$ an isomorphism?**

The answer cannot always be 'yes' - for example, if $n = 1$ and we pick a space $X$ with nonabelian fundamental group, then clearly $\sigma$ cannot be an isomorphism since $\pi_2(\Sigma X)$ is abelian. This is not just restricted to $n = 1$ either: $\pi_n(S^1) = 0$ for all $n > 1$, but there are infinitely many $n$ with $\pi_n(S^2) \neq 0$. This surprising result is in fact true for all spheres of dimension greater than 1: this is the consequence of a theorem of Serre on torsion in homotopy groups.

If we consider (reduced) homology, then this question is uninteresting since one can show that $\tilde{H}_n X \cong \tilde{H}_{n+1}\Sigma X$. This is because we have excision for homology. The lack of excision for homotopy groups is what makes stable homotopy questions interesting.

[9] Technically we need to use the reduced cone - I'll be lax about the difference between reduced and unreduced things. The resulting things we get are homotopy equivalent anyway.

Before we move on, we need to look at one more operation. The **loopspace** $\Omega X$ of a space $X$ is the space of based loops in $X$. It's an **h-group** (i.e. a group object up to homotopy - equivalently a group object in $\mathbf{hTop}_*$) since we can concatenate loops. The important property of loop spaces is that we have an adjunction $[\Sigma X, Y] = [X, \Omega Y]$. The adjunction map sends $f : \Sigma X \to Y$ to the map $X \to \Omega Y$ that sends a point $x \in X$ to the loop $f\big|_{\{x\}\times[0,1]}$. This adjunction is actually an instance of a more general adjunction: if $Y$ is a locally compact Hausdorff space, then for any $Z$ we can define a space $\underline{\mathrm{Hom}}(Y, Z)$ by equipping the set $\mathrm{Hom}(X, Z)$ with the compact-open topology. Then we have an adjunction $\mathrm{Hom}(X \wedge Y, Z) = \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$. Putting $Y = S^1$ gives us back the adjunction above.

Since $\Omega Y$ is an h-group, the sets $[X, \Omega Y]$ all get the structure of groups. Using the adjunction, this means that suspensions are **h-cogroups**, or cogroup objects up to homotopy[10]. The cogroup structure is given by the fold map we saw earlier. By the Eckmann-Hilton argument, a double loop space is an abelian group object, and so $[X, \Omega^2 Y]$ is always an abelian group. Abstractly, this is why $[S^n, Y]$ is a group for $n \geq 1$, and why it's abelian for $n \geq 2$.

# 2 Stabilisation

## 2.1 The Freudenthal Suspension Theorem

**Definition 2.1.1.** A space $X$ is said to be $n$**-connected** if $\pi_i X = 0$ for $i \leq n$. For example '0-connected' means just 'path-connected' and '1-connected' means 'simply-connected'. Some sources use the terminology '$n$-simply-connected' to emphasise this.

**Definition 2.1.2.** A map $f : X \to Y$ is $n$**-connected** if its homotopy fibre is $(n-1)$-connected. Concretely, this means that the induced map on homotopy groups $f_* : \pi_i X \to \pi_i Y$ is an isomorphism for $i < n$ and a surjection for $i = n$.

Here's a partial answer to the suspension isomorphism question for CW complexes:

**Theorem 2.1.3** (Freudenthal, 1937)**.** *Suppose that $X$ is an $n$-connected CW complex. Then the unit $X \to \Omega\Sigma X$ of the adjunction $\Sigma \dashv \Omega$ is $(2n+1)$-connected.*

Noting that $\pi_i(\Omega Y) \cong \pi_{i+1}(Y)$, we can see that this is equivalent to the statement that the suspension maps $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ are isomorphisms for $i \leq 2n$ and a surjection for $i = 2n+1$. The most elementary proof (the one you'll find in [4]) uses:

[10] Not all h-cogroups are suspensions.

**Theorem 2.1.4** (homotopy excision[11])**.** *Let $X$ be a CW complex that's the union of subcomplexes $A$ and $B$ with $A \cap B \neq \emptyset$. If the pairs $(A, A \cap B)$ and $(B, A \cap B)$ are $m$-connected and $n$-connected respectively, then the map of pairs $(A, A \cap B) \to (X, B)$ induced by the inclusion map is $(m+n)$-connected.*

The proof of the homotopy excision theorem is elementary but nasty - it relies on some involved CW complex computations. Now we can prove the suspension theorem:

Write $\Sigma X = C^+X \cup C^-X$ with $C^+X \cap C^-X = X$. The suspension map $\sigma : \pi_i X \to \pi_{i+1}\Sigma X$ is the same as the map $\pi_i(C^+X, X) \to \pi_{i+1}(\Sigma X, C^-X)$ via the long exact sequence for pairs. Applying homotopy excision gets us the result.

## 2.2 Consequences

**Proposition 2.2.1.** *Let $X$ be an $n$-connected pointed CW complex. Then $\Sigma X$ is $(n+1)$-connected.*

**Proposition 2.2.2.** *Let $X$ be a pointed CW complex. Fix a positive integer $n$. Then for all $i \geq n+2$, the groups $\pi_{n+i}(\Sigma^i X)$ are all isomorphic.*

*Proof.* $\Sigma X$ is at least path-connected, so $\Sigma^i X$ is at least $(i-1)$-connected. So we have isomorphisms $\pi_{n+i}(\Sigma^i X) \cong \pi_{n+i+1}(\Sigma^{i+1} X)$ as long as $n+i \leq 2(i-1)$. □

We call this stable limit the the $n^{\text{th}}$ **stable homotopy group** of $X$, denoted $\pi_n^S(X)$. We see that $\pi_n^S(X) = \pi_{n+i}(\Sigma^i X)$ whenever $i \geq n+2$. If we consider the sequence

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$

then we see that the colimit $\varinjlim_i \pi_{n+i}(\Sigma^i X)$ is exactly $\pi_n^S(X)$.

More generally, if $X$ and $Y$ are finite pointed CW complexes then the sequence

$$[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to \cdots$$

always stabilises. After the first two terms, this is a sequence of abelian groups. We define $[\Sigma^\infty X, \Sigma^\infty Y]$ to be the colimit $\varinjlim_i [\Sigma^i X, \Sigma^i Y]$. This limit is eventually attained, so for suitably large $N$ we have $[\Sigma^\infty X, \Sigma^\infty Y] = [\Sigma^N X, \Sigma^N Y]$.

**Proposition 2.2.3.** $\pi_n(S^n) = \mathbb{Z}$ *for all* $n > 0$.

*Proof.* We have a suspension sequence

$$\pi_1 S^1 \to \pi_2 S^2 \to \pi_3 S^3 \to \cdots$$

and Freudenthal tells us that the first map is onto and all subsequent maps are isomorphisms. There are at least two ways of seeing that the map $\mathbb{Z} \twoheadrightarrow \pi_2 S^2$ is an isomorphism:

[11] This theorem is a weak form of the Blakers-Massey theorem, and is sometimes referred to by that name.

i) homotopy-theoretic: one can use the long exact sequence of the Hopf bundle $S^1 \to S^3 \to S^2$ to conclude that $\pi_2(S^2) \cong \mathbb{Z}$ and the result follows.

ii) homological: the degree of a self-map of $S^2$ is a $\mathbb{Z}$-valued homotopy invariant. Since there exist maps of arbitrary degree, $\pi_2(S^2)$ must surject onto $\mathbb{Z}$. In fact the degree map must be an isomorphism.

□

### 2.3 Stable homotopy groups of spheres

Of paticular interest are the groups $\pi_i^S(S^0)$; the $i^{\text{th}}$ group is sometimes abbreviated to just $\pi_i^S$ and called the **stable *i*-stem**. Note that we have $\pi_i^S(S^0) = \pi_{i+N}(S^N)$ for $N > i+1$. As an example, we have $\pi_0^S = \mathbb{Z}$. A theorem of Serre says that this is the only infinite stable stem:

**Theorem 2.3.1** (Serre, 1953, [5]). *$\pi_i^S$ is finite for all $i > 0$.*

In fact Serre proved something stronger: the **only** infinite homotopy groups of spheres are the groups $\pi_n(S^n)$ and $\pi_{4m-1}(S^{2m})$, which are all of the form $\mathbb{Z} \oplus F$, where $F$ is finite. The idea of the proof is to first prove a generalised Hurewicz theorem, and use a spectral sequence argument to show that up to a finite summand the groups $H_i S^n$ and $\pi_i S^n$ are isomorphic (unless $i = 4m-1$ and $n = 2m$).

Stable homotopy groups of spheres are notoriously badly behaved and hard to compute: the first few are

$$\mathbb{Z},\ \mathbb{Z}/2,\ \mathbb{Z}/2,\ \mathbb{Z}/24,\ 0,\ 0,\ \mathbb{Z}/2,\ \mathbb{Z}/240,\ \mathbb{Z}/2 \times \mathbb{Z}/2,\ \ldots$$

Composition of maps turns the group $\pi_*^S := \bigoplus_i \pi_i^S$ into a graded-commutative ring. In fact, this ring is the coefficient ring of a homology theory! We'll see why in 4.2.2.

**Theorem 2.3.2** (Nishida, 1973). *All elements of positive grading in the ring $\pi_*^S$ are nilpotent.*

# 3 Spectra

## 3.1 Finite spectra

Can we 'stabilise' our spaces? That is, for every space $X$ can we meaningfully define a space $\Sigma^\infty X$ such that

i) $\pi_i(\Sigma^\infty X) = \pi_i^S(X)$

ii) $\Sigma$ is an autoequivalence of the category of such spaces?

Our answer will be **yes**, as long as we weaken the concept of 'space'.

**Definition 3.1.1.** The category $\mathcal{F}$ of **finite spectra**[12] has objects $\Sigma^{\infty+n}X$ for every finite CW complex $X$ and integer $n$. The homsets are defined to be

$$\mathrm{Hom}_{\mathcal{F}}(\Sigma^{\infty+n}X, \Sigma^{\infty+m}Y) := \varinjlim_i [\Sigma^{i+n}X, \Sigma^{i+m}Y]$$

and by Freudenthal the limit is attained at some finite stage.

The functors $\Sigma$, $\vee$ and $\wedge$ all have extensions to $\mathcal{F}$, and $\Sigma$ becomes an autoequivalence. The wedge product makes $\mathcal{F}$ into an additive category - it's enriched over abelian groups since we can write $\mathrm{Hom}_{\mathcal{F}}(A, B) = \mathrm{Hom}_{\mathcal{F}}(\Sigma^2 A, \Sigma^2 B)$, which is a finitely generated abelian group. However $\mathcal{F}$ is not abelian, but it is triangulated, with shift functor $\Sigma$. Since $\Sigma$ is an autoequivalence, we can desuspend objects arbitrarily many times - so we can think of the objects $\Sigma^{\infty-n}S^0$ for $n > 0$ as negative-dimensional spheres.

It's easy to recover stable homotopy groups as certain morphism sets of $\mathcal{F}$: we see that $\mathrm{Hom}_{\mathcal{F}}(\Sigma^{\infty+n}S^0, \Sigma^\infty X) \cong \pi_n^S X$.

## 3.2 Infinite spectra

We'd like to extend this definition to all CW complexes, not just finite ones. We could simply define a spectrum to be a directed system of finite spectra, analogous to how a CW complex is the direct limit of its finite subcomplexes (more formally, this is the ind-completion of $\mathcal{F}$). Or we could repeat the definition of $\mathcal{F}$, but with not-necessarily-finite CW complexes. Unfortunately these constructions don't work very well; the categories we get have bad technical properties. Our definition of spectra will follow Adams in [1], III.[13]

**Definition 3.2.1.** A **CW-spectrum** (we'll ususally just say **spectrum**) is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of CW complexes together with inclusions of subcomplexes $\Sigma E_n \to E_{n+1}$. We call these maps the **structure maps**.

Before we go any further, let's see some examples. We'll see later (in 3.2.9) that without any loss of generality we can take $n \in \mathbb{N}$ in the above definition. Hence I'll just write down what happens in nonnegative degrees.

*Example* 3.2.2.

i) The **suspension spectrum** of a CW complex $X$ is the spectrum $\Sigma^\infty X$ with $(\Sigma^\infty X)_n = \Sigma^n X$ and the obvious structure maps.

ii) The **sphere spectrum** $\mathbb{S}$ is the suspension spectrum of $S^0$.

iii) An **$\Omega$-spectrum** is a sequence of CW complexes $E_n$ with weak homotopy equivalences $E_n \to \Omega E_{n+1}$. Using adjointness, we see that every $\Omega$-spectrum defines a spectrum.

[12] Also known as the **Spanier-Whitehead category**.

[13] There are many other constructions; another method using **orthogonal spectra** is given in in [6]. We're using **sequential spectra**.

iv) Fix an integer $n > 0$ and a group $G$ (abelian if $n > 1$). An **Eilenberg-Mac Lane space** is a space $K(G, n)$ with $\pi_n K(G, n) \cong G$ and all other homotopy groups trivial. They exist, are unique up to weak homotopy equivalence, and can be constructed as CW complexes. The most important property of Eilenberg-Mac Lane spaces is that they represent cohomology: we have $H^n(X; A) \cong [X, K(A, n)]$. Since $\pi_i(\Omega X) = \pi_{i+1} X$, we see that $\Omega K(G, n+1)$ is a $K(G, n)$. Hence we must have a weak homotopy equivalence $K(G, n) \to \Omega K(G, n+1)$. This turns the collection of Eilenberg-Mac Lane spaces for a given group $G$ into an $\Omega$-spectrum $HG$ with $(HG)_n = K(G, n)$.

Morphisms of spectra take a little bit of work to define. We'll need to distinguish between functions, maps, and morphisms. A **function $f$ of degree $r$** between spectra $E$ and $F$ is a collection of maps $f_n : E_n \to F_{n-r}$ commuting with the structure maps. There are some issues with this definition: let $\eta : S^3 \to S^2$ be the Hopf fibration. We want to lift this to a degree 1 self-map of $\mathbb{S}$. However, $\eta$ doesn't desuspend to a map $S^2 \to S^1$ or a map $S^1 \to S^0$. We want to define a stable version of functions that allow us to lift $\eta$ to a map of spectra.

Start by saying that a subspectrum $E' \subseteq E$ is **cofinal** (or **dense**) when every cell in $E_m$ is eventually mapped to a cell in some $E'_{m+N}$. The point is that we only care about what happens stably, so to define a map between spectra we may as well just define a map on a cofinal subspectrum, since all cells in $E$ eventually end up in $E'$. Adams says "cells now – maps later". With this in mind, we define:

**Definition 3.2.3.** Let $E, F$ be spectra and $U, V$ be two cofinal subspectra of $E$. Let $f : U \to F$ and $g : V \to F$ be functions of spectra. Say that $f$ and $g$ are **equivalent** if they agree on the cofinal subspectrum $U \cap V$. A **map** from $E$ to $F$ is an equivalence class of such functions.[14]

*Example* 3.2.4 (The Kan-Priddy map). For each $n \geq 1$, let $X_n$ be $\mathbb{RP}^{n-1}$ with a disjoint basepoint. There's a map $X_n \to O(n)$ that sends a line to $L$ to reflection in the hyperplane orthogonal to $L$ containing the origin. Noting that $\underline{\mathrm{Hom}}(S^n, S^n) = \underline{\mathrm{Hom}}(S^0, \Omega^n S^n) = \Omega^n S^n$ we see that there's also a map $O(n) \to \Omega^n S^n$, sending $x \in O(n)$ to the corresponding transformation of $S^n$. Composing and using the adjunction gives a family of maps $\Sigma^n X_n \to S^n$, which we can collect together into a map of spectra $\Sigma^\infty X_\infty \to \mathbb{S}$. This map is homotopy surjective, but the restrictions $\Sigma^n X_n \to S^n$ are nullhomotopic for all $n$, so this map can only exist stably.

[14] More abstractly, we may partially order the collection of cofinal subspectra of a given spectrum $E$ by saying that $U \leq V$ if and only if $V \subseteq U$. If $F^U$ is the set of functions from $U$ to $F$, then the set of maps from $E$ to $F$ is the direct limit $\varinjlim_U F^U$.

Finally, we want to define morphisms so as to make the collection of spectra into a category. A morphism will be a homotopy class of maps, and homotopy will be defined as usual: in terms of maps out of a cylinder.[15]

**Definition 3.2.5.** Let $I^+$ be the unit interval $I$ with a disjoint basepoint added. Let $E$ be a spectrum. The **cylinder spectrum** $\mathrm{Cyl}(E)$ of $E$ has terms $(\mathrm{Cyl}(E))_n = I^+ \wedge E_n$ and structure maps induced from those of $E$, using $\Sigma(X \wedge Y) = X \wedge \Sigma Y$.

**Definition 3.2.6.** Say that two maps $f, g : E \to F$ are **homotopic** if there's a map from $\mathrm{Cyl}(E)$ to $F$ restricting to $f, g$ at the ends of the cylinder. Homotopy is an equivalence relation, and a **morphism** of spectra is a homotopy class of maps.

Write $[E, F]_r$ for the set of morphisms of degree $r$ from $E$ to $F$. With this definition the collection of CW spectra becomes a graded category, the **stable homotopy category**, which I'll denote **Spe**. As before, the functors $\Sigma$, $\vee$ and $\wedge$ all extend to **Spe** (proving this for the smash products is not so easy[16]). We see that $\Sigma E$ is just the spectrum with $(\Sigma E)_n = E_{n+1}$, and hence it's obvious that $\Sigma$ admits an inverse, the **desuspension** $\Sigma^{-1}$. It's clear that $[E, F]_r = [\Sigma^r E, F]_0 = [E, \Sigma^{-r} F]_0$.

We define the **homotopy groups of a spectrum** to be $\pi_n(E) := [\Sigma^n \mathbb{S}, E]_0 = [\mathbb{S}, E]_n$.

**Proposition 3.2.7** ([1], III.2.8)**.** *If $E$ is a spectrum, then*

$$\pi_n(E) = \varinjlim_k \pi_{n+k}(E_{n+k}) = \varinjlim_k \pi^S_{n+k}(E_{n+k})$$

As a corollary, we see that for a suspension spectrum, $\pi_n(\Sigma^\infty X) = \pi^S_n(X)$. In particular we have $\pi^S_n = [\mathbb{S}, \mathbb{S}]_n$. So the sphere spectrum already contains all of the information about the stable homotopy groups of spheres!

We finish by proving a couple of easy but important lemmas:

**Proposition 3.2.8.** *If $E'$ is cofinal in $E$ then the inclusion $E' \hookrightarrow E$ is an isomorphism.*

*Proof.* The morphism $E \to E'$ represented by the identity function $E' \to E'$ is an inverse. □

[15] The collection of spectra together with the maps of spectra is already a category, but it's very badly behaved. We're about to definine the **homotopy category of spectra**. A true 'category of spectra' needs $\infty$-categorical tools; see Remark 3.2.10.

[16] We'd like the smash product to be symmetric monoidal, but we can only make this true up to homotopy. One can fix this with more complicated theories of spectra, e.g. **symmetric spectra**. Once we have a smash product functor we can define the internal hom of spectra to be its right adjoint.

**Proposition 3.2.9.** *If $E$ is a spectrum, then define the spectrum $E'$ by*

$$E'_n = \begin{cases} E_n & n \geq 0 \\ \{*\} & \text{otherwise} \end{cases}$$

*Then $E \cong E'$. Hence we may as well index our spectra with nonnegative natural numbers.*

*Proof.* $E'$ is cofinal in $E$. □

*Remark* 3.2.10. Let **CW** be the (1-)category of CW complexes. Glossing over some technicalities, the category **CW** has an $(\infty, 1)$-categorical enhancement, i.e. an $(\infty, 1)$-category $\infty$**Grpd** whose homotopy category **Ho**($\infty$**Grpd**) is isomorphic to **hCW**, the category of CW complexes and homotopy classes of maps. Every $(\infty, 1)$-category $\mathcal{C}$ has a **stabilisation**, which essentially consists of forming 'spectrum objects' of $\mathcal{C}$. The stabilisation is usually denoted **Sp**($\mathcal{C}$). Then **Spe** is equivalent to the category **Ho**(**Sp**($\infty$**Grpd**)). So we can view the formation of **Spe** from **CW** as a 1-categorical shadow of the procedure of stabilisation of an $(\infty, 1)$-category. This shows 'why' **Spe** is triangulated: it's the homotopy category of a stable $(\infty, 1)$-category.

# 4 Homology and cohomology

## 4.1 Representability

If $E$ is a spectrum and $X$ a CW complex, then $X \wedge E$ is the spectrum with $(X \wedge E)_n = X \wedge E_n$ and obvious structure maps (compare the definition of a homotopy in 3.2.5). This agrees with the smash product of spectra $\Sigma^\infty X \wedge E$.

Spectra are very closely related to generalised homology and cohomology theories:

**Definition 4.1.1.** Let $E$ be a spectrum. Define the $E$**-homology** of a pointed CW complex $X$ to be $E_n X := \pi_n(X \wedge E)$. Define the $E$**-cohomology** to be $E^n X := [\Sigma^\infty X, E]_{-n}$.

**Proposition 4.1.2.** *If $E$ is a spectrum then the sequence of functors $E_n$ (resp. $E^n$) is a reduced homology (resp. cohomology) theory for pointed CW complexes.*

Remarkably, we get every cohomology theory this way:

**Theorem 4.1.3.** *Every reduced cohomology theory on connected pointed CW complexes is $E$-cohomology for some spectrum $E$.*

The proof is an application of the Brown Representability Theorem. We may in fact take $E$ to be an $\Omega$-spectrum.

In addition, every morphism of spectra defines a cohomology operation between cohomology theories. However, the notions of reduced cohomology theory and $\Omega$-spectra are not quite the same! There are maps of spectra (called **phantom maps**) which induce the zero map between cohomology theories.

One can also define $E$-homology and cohomology for spectra, and the same results hold once we translate the Eilenberg-Steenrod axioms into the world of spectra.

## 4.2 Examples

Which spectrum gives us ordinary (reduced) cohomology? Since Eilenberg-Mac Lane spaces represent the cohomology functors $H^*$, it's perhaps not so surprising that $HA$ represents ordinary cohomology: we have

$$HA^n X = [\Sigma^\infty X, HA]_n = [X, K(A,n)] = H^n(X; A)$$

Perhaps more surprising is that Eilenberg-Mac Lane spectra also corepresent homology; this is harder to see.

*Example* 4.2.1. Let $O$ (resp. $U$) be the infinite-dimensional orthogonal (resp. unitary) group. Then a version of **Bott periodicity** says that we have weak homotopy equivalences $O \to \Omega^8 O$ and $U \to \Omega^2 U$, giving us periodic $\Omega$-spectra $KO$ and $KU$. The associated (periodic) cohomology theories are **real** and **complex $K$-theory**, respectively.

*Example* 4.2.2. What does the sphere spectrum $\mathbb{S}$ give us? The associated homology theory is simply $X \mapsto \pi_n^S X$. The coefficient ring of this theory is the ring $\pi_*^S$ we saw earlier. The associated cohomology theory is known as **stable cohomotopy**, and the stable cohomotopy groups of $X$ are denoted by $\pi_S^n X$.

From now on, we'll focus on the extended example of **Thom spectra**. A nice reference is [2].

**Definition 4.2.3.** Let $E \to B$ be a vector bundle. We can define a sphere bundle $\mathrm{Sph}(E) \to B$ by taking the one-point compactification of each of the fibres. The **Thom space** of the vector bundle is the space $\mathrm{Th}(E) := \mathrm{Sph}(E)/B$. So we obtain $\mathrm{Th}(E)$ from $\mathrm{Sph}(E)$ by identifying all of the new points.

*Remark* 4.2.4. If $B$ is compact, then $\mathrm{Th}(E)$ is the one-point compactification of $E$.

*Example* 4.2.5. If $\mathbb{R}^n$ is the trivial bundle over $B$, then $\mathrm{Th}(\mathbb{R}^n) = \Sigma^n(B_+)$, where $B_+$ is $B$ with a disjoint basepoint.

*Example* 4.2.6. Let $V, W$ be any vector bundles over $B$. Then there is a homeomorphism $\mathrm{Th}(V \oplus W) \cong \mathrm{Th}(V) \wedge \mathrm{Th}(W)$. In particular, if $W = \mathbb{R}^n$ is the trivial bundle over $B$, we have $\mathrm{Th}(V \oplus \mathbb{R}^n) \cong \Sigma^n \mathrm{Th}(V)$.

**Definition 4.2.7.** Let $EO(n) \to BO(n)$ be the universal vector bundle of rank $n$. Define $MO(n) := \mathrm{Th}(EO(n))$. Note that if we pull back $EO(n+1)$ along the inclusion $BO(n) \hookrightarrow BO(n+1)$, we get the vector bundle $EO(n) \oplus \mathbb{R}$. Since pullback along a function $f$ induces a map of Thom spaces $\mathrm{Th}(f^*V) \to \mathrm{Th}(V)$, we get maps $\Sigma MO(n) \to MO(n+1)$. Hence the spaces $MO(n)$ form a spectrum, the **Thom Spectrum** $MO$.

**Definition 4.2.8.** The $n^{\text{th}}$ **cobordism group** $\Omega_n$ is the set of cobordism classes of compact smooth $n$-manifolds with group operation the disjoint union.

**Theorem 4.2.9** (Thom)**.** $\pi_n(MO) \cong \Omega_n$.

The associated cohomology theory of $MO$ is **cobordism**, and the associated homology theory is **bordism**. These theories are 2-torsion, since $M \sqcup M$ is the boundary of $M \times I$.

There are variants where we require our smooth manifolds to have more vector bundle structure (e.g. framed, oriented, almost complex). In particular, if we require our manifolds to be stably framed, a result of Pontryagin tells us that we recover the stable homotopy groups of spheres!

# References


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