Versal deformations

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1 Formal deformation theory

1.1 Setup

I'll work over an algebraically closed field k of characteristic zero. Algebraic closure is needed, but you can probably get away without the assumption on characteristic.

- **Set** is the category of sets.
- $\mathbf{Art}_k$ is the category of commutative Artinian local k-algebras with residue field k, and local algebra homomorphisms.
- $\hat{\mathbf{Art}}_k$ is the category of commutative Noetherian complete local k-algebras with residue field k, and local algebra homomorphisms.

Even in this favourable situation ($\bar{k} = k$) we need the condition that the residue field is k: if l is your favourite transcendental extension of k, then $l[t]/(t^2)$ satisfies all of the other conditions but is really too big. An Artinian local ring is complete[1] (and Noetherian), so $\mathbf{Art}_k$ is a (full) subcategory of $\hat{\mathbf{Art}}_k$. There are several quotient functors $\hat{\mathbf{Art}}_k \to \mathbf{Art}_k$ given on objects by $\Lambda \mapsto \Lambda/\mathfrak{m}^j$. We can reconstruct Λ from these quotients, in the sense that $\Lambda \cong \hat{\Lambda} \cong \varprojlim_n (\Lambda/\mathfrak{m}^{n+1})$. A weak version of the Cohen Structure Theorem tells us that every object of $\hat{\mathbf{Art}}_k$ is a quotient of a power series ring in finitely many variables.

Suppose we have a functor $F : \mathbf{Art}_k \to \mathbf{Set}$. If we have a diagram

$$R' \to R \leftarrow R'' \tag{*}$$

in $\mathbf{Art}_k$, then taking the pullback and applying F we obtain a natural map of sets

$$\eta : F(R' \times_R R'') \to F(R') \times_{F(R)} F(R'')$$

A **deformation functor** is a functor $F : \mathbf{Art}_k \to \mathbf{Set}$ such that:

i) $F(k)$ is a one-element set.

ii) In any diagram * as above, whenever $R' \to R$ is a surjection then η is a surjection.

iii) In *, if $R = k$ then η is a bijection.

I denote the dual numbers by $k[\epsilon] := k[t]/(t^2) \in \mathbf{Art}_k$. The **tangent space** to a deformation functor is the set $T^1F := F(k[\epsilon])$. It's a vector space, because $k[\epsilon]$ is a vector space object in $\mathbf{Art}_k$. It's not always finite-dimensional. A natural transformation $\phi : F \to G$ of deformation functors induces a linear map $T^1F \to T^1G$ of tangent spaces, the **differential** $\mathrm{d}\phi$.

[1] I think I got confused about this during the talk, but it's easy to see since the maximal ideal is nilpotent.

1.2 Formal elements

Let F be a deformation functor. A **formal element** of F is a ring $\Lambda \in \hat{\mathbf{Art}}_k$ together with a compatible[2] sequence of elements $\xi_n \in F(\Lambda/\mathfrak{m}^{n+1})$. We denote the set of formal elements of F over Λ by $\hat{F}(\Lambda) \cong \varprojlim_n F(\Lambda/\mathfrak{m}^{n+1})$. Note that $\hat{F}$ is a functor $\hat{\mathbf{Art}}_k \to \mathbf{Set}$.

A formal element $(\Lambda, \{\xi_n\})$ is **versal**, resp. **universal**, if the following holds:

Take a ring $R \in \mathbf{Art}_k$ with $\mathfrak{m}_R^{n+1} = 0$, and an element $\eta \in F(R)$. Suppose we have a map $\Lambda/\mathfrak{m}^{n+1} \to R$ sending ξ_n to η. Then, for all surjections $R' \to R$ and $\eta' \in F(R')$ mapping to η, there exists (resp. exists and is unique) a map $\Lambda/\mathfrak{m}^{n+1} \to R'$, lifting $\Lambda/\mathfrak{m}^{n+1} \to R$, that sends ξ_n to η'.

So having a formal (uni)versal element is essentially a (unique) lifting condition against square-zero extensions[3]. This looks a bit like formal smoothness (formal étaleness): we can draw loose analogies

formal versal element	formally smooth map	surjective differential
formal universal element	formally étale map	bijective differential

We'll see soon that formal versal (resp. universal) elements induce surjections (resp. bijections) on appropriate tangent spaces.

1.3 Prorepresentability and weakenings

If $\Lambda \in \hat{\mathbf{Art}}_k$ then it defines a functor $h_\Lambda : \mathbf{Art}_k \to \mathbf{Set}$ by sending R to $\mathrm{Hom}_{\hat{\mathbf{Art}}_k}(\Lambda, R)$. In particular, if $\Lambda \in \mathbf{Art}_k$ then such a functor is just a representable functor. Say that a deformation functor F is **prorepresentable** if it's isomorphic to h_Λ for some Λ. Such a Λ is unique up to unique isomorphism. Intuitively, a functor is prorepresentable if it's the functor of deformations of some point of a scheme.

Theorem 1.3.1 ([4], 2.3.1)**.** *F is prorepresentable if and only if it is left exact (i.e. preserves pullbacks) and has a finite-dimensional tangent space.*

The 'only if' part is easy to prove, but the 'if' part is considerably harder. The following theorem gives us a link between prorepresentability and existence of formal (uni)versal elements:

Theorem 1.3.2. *Let F be a deformation functor. Then F has a formal versal (resp. universal) element if and only if there's a ring $\Lambda \in \hat{\mathbf{Art}}_k$ and a smooth surjection[4] (resp. an isomorphism) $\phi : h_\Lambda \to F$.*

So F has a formal universal element if and only if it's prorepresentable. The proof of 1.3.2 proceeds via the following intermediate lemma:

Lemma 1.3.3 ([4], 2.2.2)**.** *Let F be a deformation functor. Then for any $\Lambda \in \hat{\mathbf{Art}}_k$ there's a bijection*

$$\hat{F}(\Lambda) \xrightarrow{\cong} \mathrm{Hom}(h_\Lambda, F)$$

Proof. Given a pair $(\Lambda, \hat{\xi})$, we must construct a map $h_\Lambda \to F$. We have elements $\xi_n \in F(\Lambda/\mathfrak{m}^{n+1})$ that induce maps $h_{\Lambda/\mathfrak{m}^{n+1}} \to F$ by the Yoneda lemma. Fix an $R \in \mathbf{Art}_k$. Since the maximal ideal of R is nilpotent, every map $\Lambda \to R$ factors through some $\Lambda/\mathfrak{m}^{N+1}$. So we have $h_\Lambda(R) \cong h_{\Lambda/\mathfrak{m}^{N+1}}(R)$. Compose with the map induced by ξ_N to get a map $h_\Lambda(R) \to F(R)$. One checks that these fit together into a natural transformation. Conversely, given $\phi : h_\Lambda \to F$, we define $\xi_n := \phi(\pi_n)$ where $\pi_n \in h_\Lambda(\Lambda/\mathfrak{m}^{n+1})$ is the quotient map. These two constructions are inverse. □

[2]There's an inverse system $\Lambda/\mathfrak{m} \leftarrow \Lambda/\mathfrak{m}^2 \leftarrow \Lambda/\mathfrak{m}^3 \leftarrow \cdots$ whose limit is Λ, and 'compatible' means that the induced map $F(\Lambda/\mathfrak{m}^{n+1}) \to F(\Lambda/\mathfrak{m}^n)$ sends ξ_n to ξ_{n-1}.

[3]Every surjection in $\mathbf{Art}_k$ factors as a composition of square-zero extensions - cf. [2].

[4]A natural transformation $G \to H$ is a **surjection** if every $G(X) \to H(X)$ is surjective, and **smooth** if for all surjections $X \to Y$, the map $G(X) \to G(Y) \times_{H(Y)} H(X)$ is a surjection.

Remark 1.3.4. One can abstractly see the existence of the above bijection if one uses the language of Kan extensions; I claim that $\hat{F}$ is the right Kan extension of F along the inclusion $\iota : \mathbf{Art}_k \to \hat{\mathbf{Art}}_k$. To prove this, use the characterisation $(\mathrm{Ran}_\iota F)(\Lambda) \cong \lim J$ where J is the diagram defined by the composition $(\Lambda \downarrow \iota) \to \mathbf{Art}_k \xrightarrow{F} \mathbf{Set}$. A piece of J looks like $F(A) \xrightarrow{F(f)} F(A')$, where A and A' are in $\mathbf{Art}_k$, and there exists a commutative triangle

$$\begin{array}{ccc} & \Lambda & \\ \swarrow & & \searrow \\ A & \xrightarrow{\quad f \quad} & A' \end{array}$$

in $\hat{\mathbf{Art}}_k$. Since the maximal ideals of A and A' are nilpotent, the maps from Λ must factor through some $\Lambda/\mathfrak{m}_\Lambda^N$ for some $N > 0$, and we hence obtain a commutative triangle

$$\begin{array}{ccc} & \Lambda/\mathfrak{m}_\Lambda^N & \\ \swarrow & & \searrow \\ A & \xrightarrow{\quad f \quad} & A' \end{array}$$

in $\mathbf{Art}_k$. Hence the subdiagram

$$F(\Lambda/\mathfrak{m}_\Lambda) \leftarrow F(\Lambda/\mathfrak{m}_\Lambda^2) \leftarrow F(\Lambda/\mathfrak{m}_\Lambda^3) \leftarrow \cdots$$

is final in J, and so we see that

$$\lim J \cong \varprojlim_n F(\Lambda/\mathfrak{m}^{n+1}) = \hat{F}(\Lambda) \quad .$$

Finally, one uses the weighted limit definition of right Kan extension to compute

$$(\mathrm{Ran}_\iota F)(\Lambda) \cong \lim^{h_\Lambda} F \cong \mathrm{Hom}(h_\Lambda, F) \quad .$$

If $(\Lambda, \hat{\xi})$ is a formal element of F, the induced map $\mathrm{d}\phi : T^1 h_\Lambda \to T^1 F$ is called the **Kodaira-Spencer map** of $(\Lambda, \hat{\xi})$. The formal versal elements of Λ whose Kodaira-Spencer map is a bijection are called **miniversal** or **semiuniversal**. We have obvious inclusions

$$\{\text{universal elements}\} \subseteq \{\text{miniversal elements}\} \subseteq \{\text{versal elements}\}$$

2 Computing (uni)versal deformations

2.1 Schlessinger's algorithm

Schlessinger's theorem not only tells us when a deformation functor has a formal (uni)versal element, but can be adapted to provide an algorithm to find one. Let F be a deformation functor with a formal versal element. We can compute it inductively - the algorithm follows Schlessinger's original proof from [3]:

Base case Say the dimension of $T^1 F$ is d. We know that $d < \infty$ since F has a versal element. Set $P := k[\![z_1, \dots, z_d]\!]$. Let J_1 be the square of the maximal ideal of P, and set

$$U_1 := P/J_1 \cong \underbrace{k[\epsilon] \times_k \cdots \times_k k[\epsilon]}_{d \text{ copies}}$$

where $X \times_k Y$ is the pullback[5] along the canonical quotient maps to k. Observe that by construction $T^1 h_{U_1} \cong T^1 F$.

[5] Around this point I made a bad mistake and said that by $X \times_k Y$ I meant the tensor product $X \otimes_k Y$ - observe that $\dim(k[\epsilon] \times_k k[\epsilon]) = 3$ but $\dim(k[\epsilon] \otimes_k k[\epsilon]) = 4$.

Since F is a deformation functor we see that $F(U_1) \cong T^1F \times \cdots \times T^1F$. Fix a basis $e_1, \ldots, e_d$ of T^1F and set $\xi_1 := (e_1, \ldots, e_d) \in F(U_1)$. One can check that ξ_1 induces an isomorphism $T^1h_{U_1} \to T^1F$.

Induction step Suppose we've already found $J_n \subseteq P$, $U_n = P/J_n$ and $\xi_n \in F(U_n)$. Consider the set S of ideals $J \subseteq P$ such that

i) $\mathfrak{m}_P J_n \subseteq J \subseteq J_n$

ii) ξ_n lifts to $F(P/J)$ along the map defined by the surjection $P/J \twoheadrightarrow U_n$

Note that S is nonempty because $J_n \in S$. Let J_{n+1} be a minimal element of S. Set $U_{n+1} := P/J_{n+1}$, and let ξ_{n+1} be any lift of ξ_n to $F(U_{n+1})$. Such a lift exists by condition ii).

Theorem 2.1.1 (Schlessinger)**.** *The previous procedure works. More specifically, the U_n fit together into an inverse system with limit $U := \varprojlim_n U_n$, and the pair $(U, \hat{\xi})$ is a formal versal element for F. Moreover, if F has a formal universal element, then $(U, \hat{\xi})$ will also be universal.*

2.2 An improved version for algebras

Let's assume we're trying to deform some finite-type algebra $A := k[x_1, \ldots, x_p]/(f_1, \ldots, f_q)$. Schlessinger's algorithm works, but it isn't very computational. The following modifications for this setup seem to be due to Artin - they're certainly in [1]. He leaves the base case alone, but changes the induction step considerably - instead of picking a maximal algebra first, and then a flat deformation, he writes down a 'suitably generic' flat deformation and finds the maximal algebra that it lifts to. I'm emphasising 'flat' because we first write down a general 'non-flat' deformation, and flatness will impose some constraints - i.e. some relations in P. The ideal generated by these relations will be exactly J_{n+1}.

For $n \geq 1$, let $\mathcal{R}_n$ denote the subcategory of $\mathbf{Art}_k$ of those rings R with $\mathfrak{m}_R^{n+1} = 0$. Note that the $\mathcal{R}_n$ are nested, so that we have a sort of filtration $\mathcal{R}_1 \hookrightarrow \mathcal{R}_2 \hookrightarrow \cdots$ of $\mathbf{Art}_k$. If we define $P_n := P/\mathfrak{m}_P^{n+1}$, then P_n is a free object in $\mathcal{R}_n$.

We see that a formal element $(\Lambda, \hat{\xi})$ of F is (uni)versal if and only if each ξ_n has a (unique) lifting property with respect to extensions in $\mathcal{R}_n$. In this case, we can also say that each ξ_n is (uni)versal for the category $\mathcal{R}_n$, so that the algorithm starts with a universal[6] deformation for $\mathcal{R}_1$ and inductively lifts it to (uni)versal deformations for the $\mathcal{R}_n$.

Lemma 2.2.1. *The algebra U_n is a quotient of P_n, and hence in $\mathcal{R}_n$.*

Proof. Certainly true for $n = 1$. For $n \geq 2$, by construction of the J_n we have $\mathfrak{m}_P J_{n-1} \subseteq J_n$, and hence inductively a chain

$$\mathfrak{m}_P^{n-1} J_1 \subseteq \mathfrak{m}_P^{n-2} J_2 \subseteq \cdots \subseteq \mathfrak{m}_P^j J_{n-j} \subseteq \cdots \subseteq \mathfrak{m}_P J_{n-1} \subseteq J_n$$

Now using that $J_1 = \mathfrak{m}_p^2$, we get an inclusion $\mathfrak{m}_p^{n+1} \subseteq J_n$ and hence a surjection
$P_n = P/\mathfrak{m}_p^{n+1} \twoheadrightarrow P/J_n = U_n$. □

As a corollary, at the n^{th} step of the induction we may restrict our attention to the rings in $\mathcal{R}_n$ - in fact those rings that are quotients of P_n.

[6] It's a general fact that any first-order versal deformation will be universal, essentially because the category $\mathcal{R}_1$ is so small: it's equivalent to the category of finite-dimensional vector spaces.

As a note on terminology, when I say that an algebra B is 'given by' relations S, I mean that B is the quotient of a polynomial ring by the ideal generated by S. Since polynomial rings are Noetherian, we may take S to be finite, and I usually just write down a generic element s_i.

Recall that we're trying to deform A, which is an algebra given by relations f_i. Let's suppose we've found J_n and ξ_n, and are looking to find J_{n+1} and ξ_{n+1}. Say that ξ_n is the deformation of A over U_n given by relations f_i+r_i, where the r_i are elements of $\mathfrak{m}_{U_n}\otimes_k A \subseteq U_n \otimes_k A$. Since this tensor product is simply a quotient of $P_n[x_1,\dots,x_p]$, we may as well regard the r_i as being elements of $\mathfrak{m}_{P_n}[x_1,\dots,x_p]$. Note that $\mathfrak{m}_{P_n}$ can be identified with the subset of $k[z_1,\dots,z_d]$ consisting of the polynomials of degree n with no constant term, so we can think of the r_i as belonging to $k[z_1,\dots,z_d][x_1,\dots,x_p]$.

Now we try to naïvely lift ξ_n to a deformation over P_{n+1}. Consider a new P_{n+1}-algebra $\tilde{A}$ given by relations $f_i + r_i + r_i'$, where now the r_i' are polynomials in $x_1,\dots,x_p$ with coefficients from

$$\{\text{polynomials in } z_1,\dots,z_d \text{ homogeneous of degree } n+1\}$$

Certainly we already have $\tilde{A}\otimes_{P_{n+1}} k \cong A$. We now find the minimal set of relations between the coefficients of the $f_i+r_i+r_i'$ such that $\tilde{A}$ is a flat deformation of A. We can often find these relations by Gröbner basis methods, as we'll see later. These relations will generate some ideal of P_{n+1}, and J_{n+1} will be exactly this ideal. The required deformation ξ_{n+1} over U_{n+1} will be $\tilde{A}$. Essentially, $\tilde{A}$ is (uni)versal because it's the most general choice of deformation we could have made - because we chose a minimal set of relations, every deformation of A over a quotient of P_{n+1} is induced by ξ_{n+1} over U_{n+1}.

2.3 A sample computation

Let A be the first-order neighbourhood of $0 \in k^2$; i.e. the k-algebra $k[x,y]/(x^2,xy,y^2)$. Note that $A \cong k[\epsilon]\times_k k[\epsilon]$. We're going to compute the formal versal deformation of A. The first step is to compute the first-order universal deformation.

I'll omit this computation - we did it a few weeks ago, following Artin in [1]. Every first-order deformation is given by relations

$$\begin{aligned} x^2 &= z_1\epsilon x + z_2\epsilon y \\ xy &= z_3\epsilon x + z_4\epsilon y \\ y^2 &= z_5\epsilon x + z_6\epsilon y \end{aligned}$$

where the z_i are in k. The tangent space manifestly has dimension 6, so we set $P := k[\![z_1,\dots,z_6]\!]$ and $U_1 := P/\mathfrak{m}_p^2$ as usual. The universal deformation ξ_1 is just the U_1-algebra given by relations

$$\begin{aligned} x^2 &= z_1 x + z_2 y \\ xy &= z_3 x + z_4 y \\ y^2 &= z_5 x + z_6 y \end{aligned}$$

To be more precise, I mean that the deformation is the U_1-algebra

$$\frac{k[x,y]}{(x^2 - z_1x - z_2y, \quad xy - z_3x - z_4y, \quad y^2 - z_5x - z_6y)}$$

It's easy to see that maps $U_1 \to k[\epsilon]$ are in bijection with assignments of the z_i to elements of $k\epsilon \subseteq k[\epsilon]$, and that such an assignment determines a deformation of A over $k[\epsilon]$.

Now for the lifting step. Consider the algebra $\tilde{A}$ over P_{n+1} given by relations

$$x^2 = z_1x + z_2y + \alpha_1$$
$$xy = z_3x + z_4y + \alpha_2$$
$$y^2 = z_5x + z_6y + \alpha_3$$

where the α_i are polynomials in x, y with coefficients from the maximal ideal of P_{n+1}. We're looking to find a minimal set S of relations between the z_j such that $\tilde{A}$ is flat over $P_{n+1}/(S)$. Clearly we may reduce to the case where the α_i are linear in x and y. Write $\alpha_i = \beta_i^1 x + \beta_i^2 y + \beta_i^3$. By a change of coordinates $z_1 \mapsto z_1 + \beta_1^1$, $z_2 \mapsto z_2 + \beta_1^2$, . . . ,$z_6 \mapsto z_6 + \beta_3^2$ in P, we may eliminate these linear terms and assume without loss of generality that the α_i are constant in x, y.

One can now use results on Gröbner bases to find the minimal set of relations: Artin proves that $\tilde{A}$ is flat if and only if 'the overlaps are consistent'. For example if we have relations $xy^3 = p_1$ and $x^3y = p_2$, then we want $x^2p_1 = y^2p_2$ to hold. Of course this equation will hold in the quotient, but there is no reason for it to hold amongst the relations: for a stupid example, in $k[x]/(x-1, x-2)$ we have $x - 1 = x - 2$ but of course $1 \neq 2$. So all that we need to do is check the finite number of overlaps, and the relations we get will be the minimal ones needed.

Getting back to the example, there are two overlaps we need to check:
$(x^2)y = x(xy)$ and $x(y^2) = (xy)y$. Writing these equations out, reducing them to linear equations in x, y, and equating coefficients, we obtain the three relations

$$\alpha_1 = z_2z_3 - z_1z_4 + z_3^2 - z_2z_6$$
$$\alpha_2 = z_2z_5 - z_3z_4$$
$$\alpha_3 = z_4z_5 - z_3z_6 + z_3^2 - z_1z_5$$

Note that we didn't obtain any relations between the z_i (in other words, the deformation is unobstructed). So $J_{n+1} = 0$ and hence $U_{n+1} = P_{n+1}$. The formal versal algebra is hence $U = P$, and the formal versal deformation is given by

$$x^2 = z_1x + z_2y + z_2z_3 - z_1z_4 + z_3^2 - z_2z_6$$
$$xy = z_3x + z_4y + z_2z_5 - z_3z_4$$
$$y^2 = z_5x + z_6y + z_4z_5 - z_3z_6 + z_3^2 - z_1z_5$$

Or to be more precise, the formal versal deformation is the P-algebra obtained by quotienting $k[x, y]$ by these relations. If $R \in \mathcal{R}_n$, then a surjection $P_n \twoheadrightarrow R$ induces a deformation of A over R, in the same manner as before.

This deformation is in fact universal - Brent pointed out that one can see this by thinking geometrically. Namely, A is a point of $X = \mathrm{Hilb}_3(k^2)$, which is a smooth variety over k of dimension $3 \cdot 2 = 6$. So the formal neighbourhood of $A \in X$ is isomorphic to P, and hence P prorepresents the functor $\mathrm{Def}_{A\in X} \cong \mathrm{Def}_A$.

References

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[3] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130, 1968, pp. 208-222.

[4] E. Sernesi, *Deformations of Algebraic Schemes*. Grundlehren der mathematischen Wissenschaften **334**, Springer, 2006.