Global Koszul duality

Matt Booth
(joint work with Andrey Lazarev)

The theory of conilpotent Koszul duality has its roots in Quillen's comparison between the commutative and Lie approaches to rational homotopy theory [11]. The modern formulation of conilpotent Koszul duality, due to Positselski and Lefèvre-Hasegawa, is as a Quillen equivalence between the categories of augmented dg algebras and conilpotent dg coalgebras [7, 9]. The functors in question are the bar construction $B : \mathbf{dgAlg}^{\mathbf{aug}} \to \mathbf{dgCog}^{\mathbf{conil}}$, which roughly sends A to a twist of the tensor coalgebra on its augmentation ideal $\bar{A}$, and its left adjoint $\Omega : \mathbf{dgCog}^{\mathbf{conil}} \to \mathbf{dgAlg}^{\mathbf{aug}}$, defined analogously. The model structure on dg algebras here is the usual one - weak equivalences are the quasi-isomorphisms, fibrations are the degreewise surjections - but important here is both that the weak equivalences in $\mathbf{dgCog}^{\mathbf{conil}}$ are created by Ω, and that they are strictly stronger than the quasi-isomorphisms.[1]

One should think of this algebra-coalgebra Koszul duality as a noncommutative version of the derived-geometric Lurie–Pridham correspondence between formal moduli problems and dg Lie algebras [10, 8]. Indeed, in characteristic zero, dg Lie algebras are Koszul dual to cocommutative conilpotent dg coalgebras, which - following a philosophy going back to Hinich [5] - one should think of as formal moduli problems.[2] At a high level, one should think of this as calculus - a formal moduli problem has a 'linearisation' to its tangent complex, which is a dg Lie algebra, and working formally locally ensures that one can always go back via integration. From this perspective, the above Quillen equivalence shows that augmented dg algebras control noncommutative deformation problems via a similar sort of calculus.[3]

Two natural questions arise: firstly, is there a version of 'nonconilpotent Koszul duality', and secondly, what kind of deformation-theoretic interpretation should this have? Conilpotency in our coalgebras corresponds to the fact that our formal moduli problems accept Artinian local dg algebras as input. So if we want to drop conilpotency (and also the (co)augmentations), our resulting notion of deformation problems should accept all finite dimensional algebras as input. In the commutative world, every finite dimensional algebra splits as a product of local algebras, but this is false in noncommutative geometry (think of, for example, matrix algebras), so these moduli problems should contain interesting 'genuinely noncommutative' data that allows separate points to communicate.

Dropping the (co)augmentations corresponds to introducing **curvature** on the other side of the bar-cobar adjunction.[4] Essentially, a curved algebra is like a

---

[1] The cofibrations in $\mathbf{dgCog}^{\mathbf{conil}}$ are the degreewise injections.

[2] a.k.a. 'formal stacks' or 'derived deformation functors'.

[3] This works over any base field, essentially since dg algebras always model E_1-algebras. In positive characteristic, dg Lie algebras are not the correct objects to use, and one must instead use Brantner and Mathew's partition Lie algebras [3].

[4] A fact well known to Positselski, who also gives a Quillen equivalence between conilpotent curved coalgebras and all dg algebras [9].

dg algebra but instead of asking that the differential squares to zero we ask that $d^2(x) = [h, x]$ for some degree two 'curvature element' h (in particular, a curved algebra with zero curvature is the same thing as a dg algebra). Morphisms of curved algebras have two components: an algebra morphism and a change of curvature term.[5] This means, for example, that the natural inclusion $\mathbf{dgAlg} \to \mathbf{cuAlg}$ is not full. Curved coalgebras are defined similarly.

When removing the conilpotency assumption, one needs to replace the bar construction B by the extended bar construction $\check{B}$; loosely this is a completion of the usual bar construction.[6] For dg algebras the properties of the extended bar construction were first worked out in detail by Anel and Joyal [1] and in the curved setting, Guan and Lazarev [4] showed that there is an adjunction

$$\Omega : \mathbf{cuCog} \longleftrightarrow \mathbf{cuAlg} : \check{B}.$$

Our main theorem is that the categories **cuAlg** and **cuCog** admit model structures making the above adjunction into a Quillen equivalence.[7] As the notion of quasi-isomorphism does not make sense for curved (co)algebras, we need to formulate a new type of weak equivalence, the **Maurer–Cartan equivalences**. An MC element in a curved algebra is an element x of degree one with $dx + x^2 + h = 0$. We denote the set of MC elements in E by $\mathrm{MC}(E)$, and we caution that this set may be empty![8] Just as in the dg case, MC elements in the convolution algebra mediate the bar-cobar adjunction: if C is a curved coalgebra and A is a curved algebra, then the graded vector space $\hom(C, A)$ admits the structure of a curved algebra, and there are natural bijections $\mathbf{cuCog}(C, \check{B}A) \cong \mathrm{MC}\hom(C, A) \cong \mathbf{cuAlg}(\Omega C, A)$. If E is any curved algebra, we define a dg category $\mathrm{MC}_{\mathrm{dg}}(E) \subseteq \mathrm{Tw}(E)$ whose objects are the MC elements of E and whose hom-complexes are given by two-sided twists. Abbreviating $\mathrm{MC}_{\mathrm{dg}}(C, A) := \mathrm{MC}_{\mathrm{dg}}\hom(C, A)$, we can thus view $\mathrm{MC}_{\mathrm{dg}}(C, A)$ as a dg category of maps $C \to \check{B}A$ (equivalently, $\Omega C \to A$). We then say that a map f of curved algebras is an **MC equivalence** if for all[9] curved coalgebras C, the induced map $\mathrm{MC}_{\mathrm{dg}}(C, f)$ is a quasi-equivalence (a.k.a. Dwyer–Kan equivalence) of dg categories. MC equivalences for curved coalgebras are defined analogously.

We show that **cuCog** is a model category, where the cofibrations are the injections and the weak equivalences are the MC equivalences. Moreover, we show that **cuAlg** is a model category, where the fibrations are the maps p inducing fibrations $\mathrm{MC}_{\mathrm{dg}}(C, p)$ for all curved coalgebras C, and the weak equivalences are

[5] A curved algebra is a curved A_∞-algebra with only three nonzero operations m_0, m_1, m_2, and a morphism is then the same as an A_∞ morphism with only two components f_1, f_2.

[6] Heuristically, $\check{B}$ is like B but where one replaces the 'cofree conilpotent coalgebra' functor - which is the tensor coalgebra functor - with the 'cofree coalgebra' functor, which is much wilder. For example, the cofree coalgebra on a one-dimensional vector space has dimension at least as large as the number of closed points in $\mathbb{A}^1_k$ - a sharp contrast to the tensor coalgebra, which always has dimension $\aleph_0$.

[7] Strictly, **cuAlg** is not cocomplete as it lacks an initial object, so we formally add one; dually we must also finalise **cuCog**.

[8] We have $\mathrm{MC}(E) \cong \mathbf{cuAlg}(k, E)$, and this set is nonempty precisely when E is curved isomorphic to a dg algebra; in fact, this gives an equivalence $\mathbf{cuAlg}_{k/} \simeq \mathbf{dgAlg}$.

[9] It is actually enough to test against all finite dimensional curved coalgebras.

the MC equivalences.[10] Finally, we show that the extended bar-cobar adjunction is a Quillen equivalence.[11]

Following [1], we also show that **cuCog** is a closed symmetric monoidal model category under $\otimes$, and that **cuAlg** is model enriched over **cuCog**. The external homs are given by setting $\underline{\hom}(\Omega C, A) = \check{B}\hom(C, A)$ and then Kan extending in the first variable. We moreover show that our Koszul duality equivalence is compatible with both the curved and uncurved versions of conilpotent Koszul duality, as well as Holstein and Lazarev's categorical Koszul duality [6]; in particular we show that the left adjoint of the $\mathrm{MC}_{\mathrm{dg}}$ functor gives a Quillen coreflection of **dgCat** into **cuAlg**, and hence that the homotopy theory of dg categories fully faithfully embeds into that of dg algebras.

Finally, we study the global analogue of noncommutative formal moduli problems, which we call **Maurer-Cartan stacks**, defined as the left exact ∞-functors from $\mathbf{cuAlg_{fd}}$ to any finitely complete ∞-category. These are geometric objects modelled on (curved) profinite completions, rather than pro-Artinian completions. We give (pro)representability results for MC stacks valued in simplicial sets and in dg categories, and moreover show that these are compatible with Pridham and Lurie's (pro)representability results for formal moduli problems.

References

[1] M. Anel and A. Joyal, *Sweedler Theory for (co)algebras and the bar-cobar constructions*, arXiv:1309.6952 (2013).

[2] M. Booth and A. Lazarev, *Global Koszul duality*, arXiv:2304.08409 (2023).

[3] L. Brantner and A. Mathew, *Deformation Theory and Partition Lie Algebras*, arXiv:1904.07352 (2019).

[4] A. Guan and A. Lazarev, *Koszul duality for compactly generated derived categories of second kind*, J. Noncommut. Geom., **15**(4):1355–1371 (2021).

[5] V. Hinich, *DG coalgebras as formal stacks*, J. Pure Appl. Algebra **162** (2001).

[6] J. Holstein and A. Lazarev, *Categorical Koszul duality*, Adv. Math. **409** (2022).

[7] K. Lefèvre-Hasegawa, *Sur les A_∞-catégories*, PhD thesis, Université Paris 7 (2003).

[8] J. Lurie, *Derived Algebraic Geometry X: Formal Moduli Problems* (2011).

[9] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Mem. Amer. Math. Soc. **212** (2011).

[10] J.P. Pridham, *Unifying derived deformation theories*, Adv. Math. **224** (2010).

[11] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295.

[10] To partly alleviate this apparent asymmetry, a key intermediate step is to show that a morphism i of curved coalgebras is an injection if and only if, for all curved algebras A, the map $\mathrm{MC}_{\mathrm{dg}}(i, A)$ is a fibration. The rough idea of the proof is to reduce to cosquare-zero extensions and finite dimensional cosemisimple coalgebras. Whilst every fibration of algebras is a surjection, the converse is not true, and so some asymmetry remains.

[11] Using the results of [4] it is relatively straightforward to show that the corresponding ∞-categories are equivalent; the difficult part of [2] consists of actually constructing the model structures.