

THE DERIVED DEFORMATION THEORY OF A POINT

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ABSTRACT. We provide a prorepresenting object for the noncommutative derived deformation problem of deforming a module X over a differential graded algebra. Roughly, we show that the corresponding deformation functor is homotopy prorepresented by the dual bar construction on the derived endomorphism algebra of X . We specialise to the case when X is one-dimensional over the base field, and introduce the notion of framed deformations, which rigidify the problem slightly and allow us to obtain derived analogues of the results of Ed Segal’s thesis. Our main technical tool is Koszul duality, following Pridham and Lurie’s interpretation of derived deformation theory. Along the way we prove that a large class of dgas are quasi-isomorphic to their Koszul double dual, which we interpret as a derived completion functor; this improves a theorem of Lu–Palmieri–Wu–Zhang. We also adapt our results to the setting of multi-pointed deformation theory, and furthermore give an analysis of universal prodeformations. As an application, we give a deformation-theoretic interpretation to Braun–Chuang–Lazarev’s derived quotient.

1. INTRODUCTION

Let k be a field of characteristic zero. We study the noncommutative derived deformation theory of modules over dg- k -algebras, with a focus on explicitly finding (pro)representing objects. Classically, commutative deformation theory studies certain functors $\mathbf{cArt}_k \rightarrow \mathbf{Set}$, where \mathbf{cArt}_k denotes the category of commutative Artinian local k -algebras. The value of a deformation functor on such an algebra Γ is supposed to behave like the set of deformations of some geometric object X over Γ : i.e. those objects \mathcal{X} , flat over $\mathrm{Spec}(\Gamma)$, whose base change along the point $\Gamma \rightarrow k$ is X . In this setting, deformation functors are rarely representable, but instead they are prorepresentable, meaning representable by an object of the procategory $\mathbf{pro}(\mathbf{cArt}_k)$. One typically embeds this procategory in the category of commutative complete local k -algebras, and often classical prorepresentability statements are presented in this form.

Classical noncommutative deformation theory is similar: one studies set-valued functors on the category \mathbf{Art}_k of noncommutative Artinian local k -algebras. To pass to the derived world, one makes two changes: firstly, one accepts connective noncommutative Artinian local dg- k -algebras as input to deformation functors. One thinks of such a dga Γ as a noncommutative derived affine scheme¹ given by some nilpotent thickening of k . Secondly, in

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¹The connective hypothesis is to avoid stacky pathologies: loosely, one thinks of the connective direction of some derived-geometric object as recording derived phenomena, and one thinks of the nonconnective direction as recording stacky phenomena. More explicitly, our results fail badly if one allows nonconnective dgas as input; see 5.3.4 for an example.

the derived setting one wants prorepresentable functors to be both left exact and quasi-isomorphism invariant. As a result, we are forced to use not just sets, but simplicial sets; one can obtain set-valued functors by truncating to π_0 . Consequently one can think of deformation functors as higher stacks. If A is a k -algebra and X is an A -module, we summarise the differences between the classical noncommutative and derived noncommutative deformation functors in the following table:

Functor	Test objects Γ	Output
classical $\mathcal{D}ef_A^{\text{cl}}(X)$	noncommutative Artinian local k -algebras	sets
derived $\underline{\mathcal{D}ef}_A(X)$	connective noncommutative Artinian local k -dgas	simplicial sets

Derived deformation functors are only really naturally defined up to weak equivalence of simplicial sets. Hence, one can only expect to obtain homotopy prorepresentability statements, as in e.g. [Jar11]. In other words, we expect to obtain statements of the form $\underline{\mathcal{D}ef}_A(X)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(P, \Gamma)$ for some pro-Artinian dga P , where $\mathbb{R}\text{Map}$ denotes the derived mapping space in a model category [Hov99, 5.4.9]. As for how we actually expect to get these statements, it should be through the machinery of Koszul duality. It is by now well known that commutative derived deformation theory, at least in characteristic zero, is a manifestation of the Koszul duality (a.k.a. bar-cobar duality) between the commutative and Lie operads [Pri10, Lur11]. Similarly, noncommutative derived deformation theory is effectively the Koszul self-duality of the associative operad [Lur11], and Koszul duality will be one of our major technical tools. Because $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ does not homotopy embed in a category of complete local algebras (unless one imposes extra finiteness conditions, as in 4.4.3), it is now necessary for us to work with pro-Artinian dgas as opposed to their limits.

One goal of this paper is to obtain a derived result of a representability theorem of Segal [Seg08] for deformations of one-dimensional modules. Typically, one is interested in deforming objects of derived or homotopy categories, which has been studied in detail by Efimov, Lunts, and Orlov [ELO09, ELO10, ELO11]. They give some prorepresentability theorems, although they do not identify the full simplicial set of deformations, and moreover they require some extra smoothness assumptions. Our first main theorem is the following:

Theorem A (5.4.9). Let A be a dga and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then there is a functorial weak equivalence

$$\underline{\mathcal{D}ef}_A(X)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\text{nu}}^{\sharp}E, \Gamma).$$

Here $B_{\text{nu}}E$ refers to the ‘nonunital bar construction’ on E (5.4.4), and $B_{\text{nu}}^{\sharp}E$ is the continuous nonunital Koszul dual of E : it is a pro-Artinian dga whose limit is B_{nu}^*E , the linear dual of $B_{\text{nu}}E$ (see 3.2.10 for the definition of the \sharp functor). Implicit in this theorem is the existence of a model structure on pro-Artinian algebras; setting up this model structure, and various related model structures, is the subject of §3. The model structure itself, at least for connective pro-Artinian algebras, is due to Pridham [Pri10]. We can deduce from our prorepresentability theorem a representability theorem for classical deformations:

Corollary A (5.5.8). Let A be a k -algebra and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then there is a functorial isomorphism

$$\mathcal{D}ef_A^{\text{cl}}(X)(\Gamma) \cong \text{Hom}_{\mathbf{aug. alg}_k}(H^0(B_{\text{nu}}^*E), \Gamma).$$

Essentially, one obtains Corollary A by taking π_0 of Theorem A and applying an inclusion-truncation adjunction.

Our representability theorems are not yet the end of the story: they involve the somewhat unsatisfying nonunital bar construction. When X is an A -module with $\dim_k(X) = 1$ (we call such X one-dimensional), its derived endomorphism algebra E is augmented, and one expects the continuous Koszul dual $B^\sharp E$ to prorepresent some naturally defined deformation problem. We rigidify slightly by introducing the notion of framed deformations; a framed deformation of S is essentially a deformation of S that respects a fixed choice of isomorphism $S \cong k$ (6.1.1, 6.1.2). After defining a functor of framed deformations $\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S)$, we obtain the following theorem:

Theorem B (6.2.2). Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the derived endomorphism dga of S . Then there is a functorial weak equivalence

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B^\sharp E, \Gamma).$$

Here, BE denotes the usual bar construction on E , so that the limit of $B^\sharp E$ is the usual Koszul dual $E^! := B^*E$. We regard Theorem B as an extension of the deformation-theoretic results of Segal's thesis [Seg08] to the derived setting: as before, taking π_0 of our prorepresentability theorem gives us a representability theorem for classical deformations, and in fact we obtain the following theorem of Segal [Seg08, 2.13]. For a vector space V , we denote its tensor algebra by $T(V)$ and its completed tensor algebra by $\hat{T}(V)$.

Corollary B (6.2.8). Let A be a k -algebra and let S be a one-dimensional A -module. Assume that $\text{Ext}_A^1(S, S)$ is finite dimensional. Let T be the k -algebra

$$T := \frac{\hat{T}(\text{Ext}_A^1(S, S)^*)}{m^*(\text{Ext}_A^2(S, S)^*)}$$

where $m : T(\text{Ext}_A^1(S, S)) \rightarrow \text{Ext}_A^2(S, S)$ is the homotopy Maurer–Cartan function (4.2.1). Then there is a functorial isomorphism

$$\mathcal{D}\text{ef}_A^{\text{cl}}(S)(\Gamma) \cong \frac{\text{Hom}_{\mathbf{aug. alg}_k}(T, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

Note that we are no longer using framed deformations. In fact, the functor of classical framed deformations is precisely $\text{Hom}_{\mathbf{aug. alg}_k}(H^0(E^!), -)$ (6.2.4), and the quotient by inner automorphisms is precisely the thing we need to do to forget the framing (6.2.5). The Ext condition is there to ensure that we have $H^0(E^!) \cong T$. Heuristically, the difference between the framed and unframed deformation functors is precisely the difference between the unital and nonunital bar construction, and on the set-valued level this presents itself as an extra action of the group of units of Γ^\times by inner automorphisms.

We apply our prorepresentability theorems to give a deformation-theoretic interpretation to Braun–Chuang–Lazarev's derived quotient [BCL18], which is a way to quotient an algebra by an idempotent in a homotopically well-behaved manner. If A is a k -algebra and S is a one-dimensional A -module, one obtains an idempotent $e_S \in S$ together with a factorisation of the action map $A \rightarrow \text{End}_k(S) \cong k$ through the quotient $A/Ae_S A$, which is necessarily augmented. We show that in this situation, under some mild finiteness conditions the derived quotient $A/{}^{\mathbb{L}}Ae_S A$ homotopy **represents** the functor of derived deformations of S :

Theorem C (6.3.9). Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Suppose that $A/\mathbb{L}Ae_SA$ is cohomologically locally finite and A/Ae_SA is Artinian local. Then there is a functorial weak equivalence

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{aug.dga}_k^{\leq 0}}(A/\mathbb{L}Ae_SA, \Gamma).$$

Surprising here is that neither Koszul duals nor pro-Artinian algebras seem to appear anywhere. The proof is based on a careful analysis of the Koszul double dual functor, which takes up the entirety of §4. The most important ingredient is the following theorem - valid in any characteristic - which may be of independent interest:

Theorem D (4.2.8). Let A be a connective augmented cohomologically locally finite dga with $H^0(A)$ Artinian local. Then there is a natural quasi-isomorphism $A^{\text{!!}} \simeq A$.

Theorem D is a generalisation of [LPWZ08, Theorem A], which, while true, originally appeared with an incorrect proof: see 4.1.15 for details. Note that Theorem D really is non-trivial, as there even exist finite-dimensional graded algebras which are not quasi-isomorphic to their own Koszul double duals (4.1.8). Our proof essentially identifies the Koszul double dual as the derived completion [GM92, DG02, Efi10, PSY14a, PSY14b, Sha19], and then argues that dgas satisfying the hypotheses are derived complete.

As usual, Theorem C specialises to the classical case and yields the following theorem, which appears in e.g. [DW19]:

Corollary C (6.3.10). Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Suppose that $A/\mathbb{L}Ae_SA$ is cohomologically locally finite and A/Ae_SA is Artinian local. Then there is a functorial isomorphism

$$\mathcal{D}\text{ef}_A^{\text{cl}}(S)(\Gamma) \cong \frac{\text{Hom}_{\mathbf{Art}_k}(A/Ae_SA, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

In §7, we extend our deformation functors via homotopy limits to accept pro-Artinian dgas as input, which allows us to associate universal prodeformations to modules. The use of simplicial sets is integral here. We make some progress towards concretely identifying universal prodeformations; there are unanswered questions here which may be a fruitful subject for further research.

In §8, we extend our results to the world of multi-pointed deformation theory [Lau02, Eri06]. Multi-pointed noncommutative deformations have recently found many applications within algebraic geometry, in particular the geometry of threefolds [Tod07, DW16, Kaw18] and we anticipate that our multi-pointed results will be of use in geometric settings.

Some of our results have already been obtained under various extra smoothness hypotheses by various authors. We have already mentioned Efimov–Lunts–Orlov, who obtain a representability statement for groupoid-valued functors under an Ext-vanishing hypothesis [ELO10]. Hua and Keller have also obtained a groupoid-valued multi-pointed representability theorem, under the additional assumption that S is perfect [HK18]. A weaker version of Theorem D appears in [KY20, §2.5], under the additional hypothesis that the A -module k given by the augmentation $A \rightarrow k$ is perfect over A , and a similar version for dg categories appears in [Efi10]. The advantage of our approach is that we can identify the full simplicial set of deformations, and moreover do not need to assume any sort of smoothness hypothesis.

Profinite algebras inherit the inverse limit topology, and instead of working with pro-Artinian algebras one can instead work with topological algebras, specifically the pseudo-compact ones. This is the context for [GLST20], who work with set-valued deformation functors, and also discuss dg Lie algebras as representing objects for commutative deformation problems. In this commutative setting, Yekutieli has set up the theory of Maurer–Cartan elements and Deligne groupoids for pronilpotent dglas in [Yek12]. Arguably, the simplest way to state our results would be as corepresentability theorems for coalgebras; this line of thought goes back to Hinich [Hin01].

This paper is an extension and improvement of Part I of the author’s PhD thesis [Boo19]. The author would like to express his gratitude to his supervisor, Jon Pridham, who patiently explained much of the modern approach to deformation theory to him. The author would also like to thank David Edwards, Bernhard Keller, Andrey Lazarev, Michael Wemyss and Amnon Yekutieli for useful discussions and suggestions which improved both the presentation and the content of the paper.

2. NOTATION AND CONVENTIONS

Throughout this paper, k will denote a field. In sections 3 and 4, k is arbitrary, and in sections 5 onwards, k will be assumed to be characteristic zero. Occasionally we may in addition require k to be algebraically closed; we will indicate where this is necessary. Modules are right modules, unless stated otherwise. Consequently, all notions are ‘right’: e.g. ‘noetherian’ means right noetherian. Unadorned tensor products are by default over k . We denote isomorphisms (of modules, functors, ...) with \cong and weak equivalences with \simeq .

We use cohomological grading conventions, so that the differential of a complex has degree 1. If we refer to an object as just **graded**, then by convention we mean that it is \mathbb{Z} -graded. If X is a complex, we will denote its cohomology complex by $H(X)$ or just HX . If X is a complex, let $X[i]$ denote ‘ X shifted left i times’: the complex with $X[i]^j = X^{i+j}$ and the same differential as X , but twisted by a sign of $(-1)^i$. This sign flip can be worked out using the **Koszul sign rule**: when an object of degree p moves past an object of degree q , one should introduce a factor of $(-1)^{pq}$. If x is a homogeneous element of a complex of modules, we denote its degree by $|x|$.

A **differential graded algebra** (**dga** for short) over k is a monoid in the category of chain complexes of vector spaces. A k -algebra is equivalently a dga concentrated in degree zero, and a graded k -algebra is equivalently a dga with zero differential. We will sometimes refer to k -algebras as **ungraded algebras** to emphasise that they should be considered as dgas concentrated in degree zero.

A **dg module** (or just a **module**) over a dga A is a complex of vector spaces M together with an action map $M \otimes A \rightarrow M$ satisfying the obvious identities, or equivalently a dga map $A \rightarrow \text{End}_k(M)$. Note that a dg module over an ungraded ring is exactly a complex of modules. If A is an algebra, write **Mod- A** for its category of right modules; it is a closed monoidal abelian category.

Let X be a complex of k -vector spaces. The **total dimension** or just **dimension** of X is $\sum_{n \in \mathbb{Z}} \dim_k X^n$. Say that X is **finite-dimensional** or just **finite** if its total dimension is finite. Say that X is **locally finite** if each $\dim_k X^n$ is finite. Say that X is **cohomologically locally finite** if the cohomology dg vector space HX is locally finite. Say that X is **bounded** if X^n vanishes for all but finitely many n , **bounded above** if X^n vanishes for

all $n \gg 0$, and **bounded below** if X^n vanishes for all $n \ll 0$. We use the same terminology in the case that X admits extra structure.

Say that a complex X is **connective** if one has $X^i = 0$ for $i > 0$. Up to quasi-isomorphism, by taking the good truncation to nonpositive degrees it is enough to assume that $H^i(X) \cong 0$ for $i > 0$. Say that X is **coconnective** if one has $X^i = 0$ for $i < 0$ (or equivalently $H^i(X) \cong 0$ for $i < 0$ up to quasi-isomorphism). We use the same terminology in case X admits extra structure (e.g. that of a dga). Note that if A is a connective dga in the weak sense then the good truncation map $\tau_{\leq 0}A \hookrightarrow A$ is a dga quasi-isomorphism. The reason we use ‘connective’ instead of ‘concentrated in nonpositive degrees’ is that the former notion is not dependent on grading conventions. See e.g. [Yek20] for further facts about dgas.

We freely use terminology and results from the theory of model categories; see for example [Qui67, Hov99, DS95, Rie14] for references. We will in particular assume that the reader knows the basics of the homotopy theory of simplicial sets, and that a model category admits derived mapping spaces which are (weak equivalence classes of) simplicial sets [Hov99, 5.4.9]. See [GJ09] for a comprehensive textbook account of simplicial homotopy theory.

We will moreover assume that the reader has a good familiarity with the theory of triangulated and derived categories; see [Nee92] and [Wei94, Huy06] respectively for references. In particular we will make use of the fact that the derived category of a dga is the homotopy category of a model category. By convention we use the projective model structure on dg-modules where every object is fibrant, and over a ring the cofibrant complexes are precisely the perfect complexes (this is the ‘q-model structure’ of [BMR14]). We also assume familiarity with the theory of dg categories and simplicially enriched categories, although we provide a quick overview to set notation.

We will make use of the following categories and model structures. We will not use multiple model structures on the same category.

Category	Objects	Model structure
Mod-A	dg modules over a dga A	projective, e.g. [BMR14]
dgvect$_k$	dg k -vector spaces	projective
dga$_k$	dg algebras over k	Hinich [Hin97]
aug.dga$_k$	augmented k -dgas	Hinich
Art$_k$	ungraded Artinian algebras	trivial
pro(Art$_k$)	pro-ungraded Artinian algebras	trivial
dgArt$_k$	Artinian local k -dgas	
pro(dgArt$_k$)	pro-Artinian dgas	See 3.2.12
pro(dgArt$_k^{\leq 0}$)	connective pro-Artinian dgas	Pridham [Pri10]
dgc_k	dg coalgebras over k	
con.dgc_k	conilpotent dgcs over k	As in [Pos11] or [LH03]
con.dg$c_k^{\geq 0}$	coconnective conilpotent dgcs over k	See 3.3.10
Set	sets	trivial
sSet	simplicial sets	Quillen, e.g. [GJ09]
Grp	groups	trivial
Grpd	groupoids	Canonical, e.g. [Str00]
dgCat	dg categories	Tabuada [Tab05]
sSetCat	simplicially enriched categories	Bergner [Ber07]
fd-\mathcal{C}	finite-dimensional objects in \mathcal{C}	
pro(\mathcal{C})	pro-objects in \mathcal{C}	
ind(\mathcal{C})	ind-objects in \mathcal{C}	
$\mathcal{C}^{\leq 0}$	connective objects in \mathcal{C}	
$\mathcal{C}^{\geq 0}$	coconnective objects in \mathcal{C}	

We summarise some of our functors of interest in the following commutative diagram. For readability, we denote functors $X \mapsto X^\square$ by simply \square . The $*$ functor is the k -linear dual, and the $!$ functor is the Koszul dual. The \sharp and \circ functors, which are equivalences as well as Quillen equivalences, are defined in §3.2. Functors running to the right are right Quillen, and functors running to the left are left Quillen; the bar construction B is a left adjoint as we are using opposite categories. The Koszul dual is not Quillen, although it does preserve weak equivalences.

$$\begin{array}{ccccc}
(\text{con.dg}c_k^{\geq 0})^{\text{op}} & \xleftarrow{\quad} & \text{con.dg}c_k^{\text{op}} & \xleftarrow{B} & \text{aug.dga}_k^{\text{op}} \\
\circ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \sharp & & \circ \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) \sharp & \searrow^* & \downarrow ! \\
\text{pro(dgArt}_k^{\leq 0}) & \xleftarrow{\quad} & \text{pro(dgArt}_k) & \xrightarrow{\varprojlim} & \text{aug.dga}_k
\end{array}$$

3. MODEL STRUCTURES ON ALGEBRAS AND COALGEBRAS

In this section, k is any field. We review the standard model structures on algebras and coalgebras, for which the bar-cobar adjunction is a Quillen equivalence. We also consider pro-Artinian algebras, which we think of as formal duals of coalgebras. New here is a model structure on unbounded pro-Artinian algebras which extends Pridham's [Pri10] and is distinct from the level model structures [EH76, EH80, Isa04]. We remark that if one wants to work with (co)commutative (co)algebras, then k should be characteristic zero, since

Pridham’s model structure depends on the identification of connective dgas with simplicial algebras.

3.1. Bar and cobar constructions. We follow Positselski [Pos11]; for other references see Loday–Vallette [LV12] or Lefèvre–Hasegawa’s thesis [LH03].

Definition 3.1.1. Let \mathbf{dga}_k be the category of dgas over k . Let $\mathbf{aug.dga}_k$ be the category of **augmented** dgas over k ; these are the dgas A for which the unit map $k \rightarrow A$ admits a k -linear retraction. A morphism of augmented dgas must respect the augmentation.

The **augmentation ideal** of an augmented dga is $\bar{A} := \ker(A \rightarrow k)$. Sending A to \bar{A} sets up an equivalence between augmented dgas and nonunital dgas. The inverse functor freely appends a unit, and indeed A is isomorphic to $\bar{A} \oplus k$ as an augmented dga.

Just like a dga is a monoid in the monoidal category of dg vector spaces over k , a **differential graded coalgebra** (or **dgc** for short) is a comonoid in this category. More concretely, a dgc is a dg k -vector space (C, d) equipped with a comultiplication $\Delta : C \rightarrow C \otimes C$ and a counit $\epsilon : C \rightarrow k$, satisfying the appropriate coassociativity and counitality identities, and such that d is a coderivation for Δ . A **coaugmentation** on a dgc is a section of ϵ ; if C is coaugmented then $\bar{C} := \ker \epsilon$ is the **coaugmentation coideal**. It is a dgc under the reduced coproduct $\bar{\Delta}x = \Delta x - x \otimes 1 - 1 \otimes x$, and C is isomorphic as a nonunital dgc to $\bar{C} \oplus k$. A morphism of coaugmented coalgebras must respect the coaugmentation. A coaugmented dgc C is **conilpotent** if every $x \in \bar{C}$ is annihilated by some suitably high power of Δ .

Definition 3.1.2. Let \mathbf{dgc}_k denote the category of dg coalgebras. Let $\mathbf{con.dgc}_k$ denote the category of conilpotent dg coalgebras.

Recall that if V is a dg vector space then V^* denotes its linear dual, which is also a dg vector space. Observe that if (C, Δ, ϵ) is a dgc, then Δ dualises to a map $(C \otimes C)^* \rightarrow C^*$, and composing Δ^* with the natural inclusion $C^* \otimes C^* \rightarrow (C \otimes C)^*$ turns $(C^*, \Delta^*, \epsilon^*)$ into a dga. If C is coaugmented, then C^* is augmented. We will almost exclusively be interested in the category $\mathbf{con.dgc}_k$.

Definition 3.1.3. If V is a dg vector space, then the tensor algebra $T^c(V) := k \oplus V \oplus V^{\otimes 2} \oplus \cdots$ is a dg coalgebra when equipped with the **deconcatenation coproduct** $T^c(V) \rightarrow T^c(V) \otimes T^c(V)$ which sends $v_1 \cdots v_n$ to $\sum_i v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$. The differential is induced from the differential on $V^{\otimes n}$.

It is easy to see that $T^c(V)$ is coaugmented and moreover conilpotent. Denote by $\bar{T}^c(V)$ the **reduced tensor coalgebra**: it is the coaugmentation coideal of the tensor coalgebra. The functor T^c is the cofree conilpotent coalgebra functor: if C is conilpotent then a morphism $C \rightarrow T^c(V)$ is determined completely by the composition $l : C \rightarrow T^c(V) \rightarrow V$. In particular, any morphism $f : \bar{T}^c(W) \rightarrow \bar{T}^c(V)$ is determined completely by its **Taylor coefficients** $f_n : W^{\otimes n} \rightarrow V$.

Definition 3.1.4. Let A be an augmented dga. Put $V := \bar{A}[1]$, the shifted augmentation ideal. Let d_V be the usual differential on the tensor coalgebra T^cV . Let d_B be the **bar differential**: send $a_1 \otimes \cdots \otimes a_n$ to the signed sum over i of the $a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$ and extend linearly. The signs come from the Koszul sign rule; see [LV12, 2.2] for a concrete formula. One can check that d_B is a degree 1 map from $V^{\otimes n+1} \rightarrow V^{\otimes n}$, and that it intertwines with d_V . Hence, one obtains a third and fourth quadrant bicomplex C with

rows $V^{\otimes n}[-n]$. By construction, the direct sum total complex of C is T^cV , with a new differential $\partial = d_V + d_B$. The **bar construction** of A is the complex $BA := (T^cV, \partial)$. One can check that the deconcatenation coproduct makes BA into a dgc.

Note that the degree 0 elements of A become degree -1 elements of BA .

Remark 3.1.5. If A is an augmented A_∞ -algebra, then one can define the A_∞ **bar construction** $B_\infty A$, which is a dgc, in an analogous manner (see [LH03] for a concrete formula). If A is a dga then $B_\infty A = BA$.

Example 3.1.6. Let A be the graded algebra $k[\epsilon]/\epsilon^2$, with ϵ in degree 0. Then $\bar{A}[1]$ is $k\epsilon$ placed in degree -1 . Since ϵ is square zero, the bar differential is identically zero. So BA is the tensor coalgebra $k[\epsilon]$, with ϵ in degree -1 .

Definition 3.1.7. Let C be a coaugmented dgc. One can analogously define a **cobar differential** d_Ω on the tensor algebra $T(\bar{C}[-1])$ by sending $c_1 \otimes \cdots \otimes c_n$ to the signed sum over i of the $c_1 \otimes \cdots \otimes \bar{\Delta}c_i \otimes \cdots \otimes c_n$, and the **cobar construction** on C is the dga $\Omega C := (T(\bar{C}[-1]), d_C + d_\Omega)$.

Bar and cobar are adjoints:

Theorem 3.1.8 ([LV12, 2.2.6]). *If A is an augmented dga and C is a conilpotent dgc, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{aug.dga}_k}(\Omega C, A) \cong \mathrm{Hom}_{\mathbf{con.dgc}_k}(C, BA).$$

Lemma 3.1.9. *The bar construction preserves quasi-isomorphisms.*

Proof. The idea is to filter BA by setting $F_p BA$ to be the elements of the form $a_1 \otimes \cdots \otimes a_n$ with $n \leq p$, and look at the associated spectral sequence. A proof for dgas is in [Pos11, §6.10] and a proof for A_∞ -algebras is in [LH03, Chapter 1]. \square

Remark 3.1.10. The cobar construction does not preserve quasi-isomorphisms in general.

Theorem 3.1.11 ([LV12, 2.3.2]). *Let A be an augmented dga. Then the counit $\Omega BA \rightarrow A$ is a quasi-isomorphism.*

Theorem 3.1.12 (Hinich [Hin97]). *The category \mathbf{dga}_k is a model category, with weak equivalences the quasi-isomorphisms and fibrations the degreewise surjections.*

Note that in \mathbf{dga}_k , every object is fibrant, and the cofibrant objects are precisely the semifree dgas (i.e. those that become free graded algebras after forgetting the differential).

Proposition 3.1.13. *The category $\mathbf{aug.dga}_k$ is a model category, with weak equivalences the quasi-isomorphisms and fibrations the levelwise surjections.*

Proof. This is true because $\mathbf{aug.dga}_k$ is the slice category \mathbf{dga}_k/k of dgas over k , and slice categories of model categories are themselves model categories under the same weak equivalences, fibrations, and cofibrations (see e.g. [DS95]). \square

Note that if C is a conilpotent coalgebra then ΩC is a cofibrant dga, and in particular we may think of ΩB as a cofibrant resolution functor on $\mathbf{aug.dga}_k$.

The failure of the cobar construction to preserve quasi-isomorphisms means that the ‘obvious’ model structure on $\mathbf{con.dgc}_k$, with weak equivalences quasi-isomorphisms, does not make $\Omega \dashv B$ into a Quillen adjunction. One way to fix this is to put a new model structure on $\mathbf{con.dgc}_k$ whose weak equivalences are created by Ω .

Theorem 3.1.14 ([Pos11, Theorem 9.3b and Theorem 6.10]). *The category $\mathbf{con.dgc}_k$ admits a model structure where the weak equivalences f are those maps for which Ωf is a dga quasi-isomorphism, and the cofibrations are the degreewise monomorphisms. Moreover, if C is a conilpotent dgc then the natural map $C \rightarrow B\Omega C$ is a fibrant resolution. The pair (Ω, B) is a Quillen equivalence between $\mathbf{con.dgc}_k$ and $\mathbf{aug.dga}_k$.*

Every object in \mathbf{cndgc}_k is cofibrant, and the fibrant objects are the semicofree conilpotent coalgebras, i.e. those that are tensor coalgebras after forgetting the differential.

Remark 3.1.15. Every weak equivalence in $\mathbf{con.dgc}_k$ is a quasi-isomorphism, but the converse is not true [LV12, 2.4.3]. One can think of the above model structure on $\mathbf{con.dgc}_k$ as a ‘Bousfield delocalisation’ of the obvious model structure, since it has less weak equivalences.

Lemma 3.1.16. *A quasi-isomorphism between coconnective coalgebras is a weak equivalence.*

Proof. This is [LH03, 1.3.1.5(e)]. Take $f : C \rightarrow D$ to be a quasi-isomorphism where C and D are both coconnective. Then the point is that the shifted augmentation ideals $\bar{C}[-1]$ and $\bar{D}[-1]$ are concentrated in strictly positive degrees, which causes f to be a filtered quasi-isomorphism; it is well known that such maps are weak equivalences (e.g. [LV12, 2.4.1]). \square

3.2. Pro-Artinian dgas.

Definition 3.2.1. Say that a dga $A \in \mathbf{aug.dga}_k$ is **Artinian local** or just **Artinian** if:

- (1) A has finite total dimension.
- (2) the augmentation ideal \bar{A} is nilpotent.

Equivalently, a dga is Artinian local if it is finite-dimensional and has a unique two-sided maximal ideal \mathfrak{m} which is nilpotent and has residue field k . Denote the category of Artinian local dgas by \mathbf{dgArt}_k ; it is a full subcategory of $\mathbf{aug.dga}_k$ in the obvious way. Let $\mathbf{dgArt}_k^{\leq 0}$ denote the category of connective Artinian dgas.

Note that if A is a finite-dimensional dga, then the multiplication dualises to a map $A^* \rightarrow A^* \otimes A^*$, and this makes $C := A^*$ into a dgc. In this case, A is augmented if and only if C is coaugmented, and A is Artinian if and only if C is conilpotent.

Definition 3.2.2 (e.g. [KS06, §6]). Let \mathcal{C} be a category. A **pro-object** in \mathcal{C} is a formal cofiltered limit, i.e. a diagram $J \rightarrow \mathcal{C}$ where J is a small cofiltered category. We denote such a pro-object by $\{C_j\}_{j \in J}$. The category of pro-objects $\mathbf{pro}\mathcal{C}$ has morphisms

$$\mathrm{Hom}_{\mathbf{pro}\mathcal{C}}(\{C_i\}_{i \in I}, \{D_j\}_{j \in J}) := \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(C_i, D_j).$$

If \mathcal{C} has cofiltered limits, then there is a ‘realisation’ functor $\varprojlim : \mathbf{pro}\mathcal{C} \rightarrow \mathcal{C}$. If C is a constant pro-object, then it is easy to see that one has $\mathrm{Hom}_{\mathbf{pro}\mathcal{C}}(C, \{D_j\}_{j \in J}) \cong \mathrm{Hom}_{\mathcal{C}}(C, \varprojlim_j D_j)$.

Definition 3.2.3. Let \mathcal{C} be a category. The **ind-category** of \mathcal{C} is $\mathbf{ind}\mathcal{C} := \mathbf{pro}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Less abstractly, an object of $\mathbf{ind}\mathcal{C}$ is a formal filtered colimit $J \rightarrow \mathcal{C}$, and the morphisms are

$$\mathrm{Hom}_{\mathbf{ind}\mathcal{C}}(\{C_i\}_{i \in I}, \{D_j\}_{j \in J}) := \varinjlim_i \varprojlim_j \mathrm{Hom}_{\mathcal{C}}(C_i, D_j)$$

If \mathcal{C} has filtered colimits, then there is a ‘realisation’ functor $\varinjlim : \mathbf{ind}\mathcal{C} \rightarrow \mathcal{C}$. In this situation, if $D \in \mathcal{C}$ is a constant ind-object then one has $\mathrm{Hom}_{\mathbf{ind}\mathcal{C}}(\{C_i\}_{i \in I}, D) \cong \mathrm{Hom}_{\mathcal{C}}(\varinjlim_i C_i, D)$.

Definition 3.2.4. We refer to an object of the procategory $\mathbf{pro}(\mathbf{dgArt}_k)$ as a **pro-Artinian dga** and an object of the procategory $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ as a **connective pro-Artinian dga**.

We caution that for the majority of this paper, “pro-Artinian” is shorthand for “pro-(Artinian local)”. Nonlocal profinite dgas will not be our main objects of study, although we remark on how to extend the theory to profinite multi-pointed dgas in §8. We list some standard results on the structure of $\mathbf{pro}(\mathbf{dgArt}_k)$:

Proposition 3.2.5 ([KS06, 6.1.14]). *Let \mathcal{C} be any category and let $f : A \rightarrow B$ be a morphism in $\mathbf{pro}(\mathcal{C})$. Then f is isomorphic to a level map: a collection of maps $\{f_\alpha : A_\alpha \rightarrow B_\alpha\}_{\alpha \in I}$ between objects of \mathcal{C} , where I is cofiltered.*

Proposition 3.2.6 ([Gro95, Corollary to 3.1]). *Every object of $\mathbf{pro}(\mathbf{dgArt}_k)$ is isomorphic to a **strict pro-object**, i.e. one for which the transition maps are surjections.*

Remark 3.2.7. To apply [Gro95] to some general procategory $\mathbf{pro}(\mathcal{C})$, it suffices that every object of \mathcal{C} is Artinian. In particular, every finite-dimensional dga is Artinian, and so it follows that every profinite dga is isomorphic to a strict profinite dga.

Definition 3.2.8. Let $A = \{A_\alpha\}_\alpha \in \mathbf{pro}(\mathbf{dgArt}_k)$. Let A° denote the dgc

$$A^\circ := \varinjlim_\alpha A_\alpha^* \in \mathbf{dgc}_k.$$

The assignment $A \mapsto A^\circ$ is (contravariantly) functorial, since we can represent every map in $\mathbf{pro}(\mathbf{dgArt}_k)$ by a level map. Because colimits of conilpotent dgc's are conilpotent, A° is a conilpotent dgc.

Proposition 3.2.9. *The functor $A \mapsto A^\circ$ gives an equivalence*

$$\mathbf{pro}(\mathbf{dgArt}_k) \xrightarrow{\cong} \mathbf{con.dgc}_k^{op}$$

that restricts to an equivalence

$$\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0}) \xrightarrow{\cong} (\mathbf{con.dgc}_k^{\geq 0})^{op}.$$

Proof. Via the linear dual, an Artinian local dga is the same thing as a finite-dimensional coaugmented conilpotent dgc. Taking procategories thus gives an equivalence between $\mathbf{pro}(\mathbf{dgArt}_k)^{op}$ and $\mathbf{ind}(\mathbf{fd-con.dgc}_k)$. A classical theorem of Sweedler says that an ungraded coalgebra is the filtered colimit of its finite-dimensional subcoalgebras. The same remains true for dgc's [GG99, 1.6]. In particular, the colimit functor $\varinjlim : \mathbf{ind}(\mathbf{fd-con.dgc}_k) \rightarrow \mathbf{con.dgc}_k$ is essentially surjective. By [GG99, 1.9], finite-dimensional conilpotent coalgebras are compact: given a finite-dimensional conilpotent dgc C , and $D = \{D_\alpha\}_\alpha$ a filtered system of finite-dimensional conilpotent dgc's, then there is a natural isomorphism $\varinjlim_\alpha \mathrm{Hom}(C, D_\alpha) \xrightarrow{\cong} \mathrm{Hom}(C, \varinjlim_\alpha D_\alpha)$. Hence if C and D are objects of $\mathbf{ind}(\mathbf{fd-con.dgc}_k)$, then one has an isomorphism $\mathrm{Hom}(C, D) \cong \mathrm{Hom}(\varinjlim C, \varinjlim D)$. Moreover, if A and B are finite-dimensional algebras, then one has an isomorphism $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(B^*, A^*)$. Putting these together we see that \varinjlim is fully faithful and hence an equivalence. Hence we obtain the first claim, and the second claim is clear. \square

Definition 3.2.10. If C is a conilpotent dgc, let $C^\sharp \in \mathbf{pro}(\mathbf{dgArt}_k)$ denote the levelwise dual of its filtered system of finite-dimensional sub-dgc's.

It is easy to see that $C \mapsto C^\sharp$ is the inverse functor to $A \mapsto A^\circ$.

Lemma 3.2.11.

- (1) Let $C \in \mathbf{con.dgc}_k$. Then C^* is functorially isomorphic to $\varprojlim C^\sharp$ as dgas.
(2) Let $A \in \mathbf{pro}(\mathbf{dgArt}_k)$. Then $A^{\circ*}$ is functorially isomorphic to $\varprojlim A$ as dgas.

Proof. For (1), pass the filtered limit through the linear dual and use that $C \cong \varinjlim C^{\sharp*}$, because $C^{\sharp*}$ is exactly the filtered system of finite-dimensional sub-dgcs of C . For (2), just apply (1) with $C = A^\circ$ and use $A^{\circ\sharp} \cong A$. The isomorphisms are clearly functorial. \square

Theorem 3.2.12. *The category $\mathbf{pro}(\mathbf{dgArt}_k)$ is a model category, with weak equivalences those maps f for which f° is a weak equivalence of coalgebras, and fibrations those maps for which $\varprojlim f$ is a degreewise surjection. The equivalence $\mathbf{pro}(\mathbf{dgArt}_k) \xrightarrow{\cong} \mathbf{con.dgc}_k^{\text{op}}$ is a Quillen equivalence.*

Proof. Transferring the model structure on $\mathbf{con.dgc}_k$ along the equivalence $\mathbf{pro}(\mathbf{dgArt}_k) \xrightarrow{\cong} \mathbf{con.dgc}_k^{\text{op}}$ puts a model category structure on $\mathbf{pro}(\mathbf{dgArt}_k)$ making it Quillen equivalent to $\mathbf{con.dgc}_k^{\text{op}}$. It is clear that the weak equivalences in $\mathbf{pro}(\mathbf{dgArt}_k)$ are exactly the claimed maps. For the fibrations, note that the cofibrations in $\mathbf{con.dgc}_k$, and hence the fibrations in $\mathbf{con.dgc}_k^{\text{op}}$, are exactly the degreewise injections. So we need to show that $\varprojlim f$ is a degreewise surjection if and only if f° is a degreewise injection. But this follows from 3.2.11(2). \square

Recall that there is a limit functor $\varprojlim : \mathbf{pro}(\mathbf{dgArt}_k) \rightarrow \mathbf{dga}_k$ which sends a cofiltered system to its limit. Note that this functor is right adjoint to the functor $\mathbf{dga}_k \rightarrow \mathbf{pro}(\mathbf{dgArt}_k)$ which sends a connective dga A to the cofiltered system \hat{A} of its Artinian local quotients (because any map $A \rightarrow \Gamma$ with Γ Artinian must factor through some such quotient). The unit map $A \rightarrow \varprojlim \hat{A}$ is pro-Artinian completion, and neither the unit nor the counit need be an isomorphism.

Proposition 3.2.13. *The functor $\varprojlim : \mathbf{pro}(\mathbf{dgArt}_k) \rightarrow \mathbf{dga}_k$ is right Quillen.*

Proof. Clearly \varprojlim preserves fibrations; we check that it preserves weak equivalences. Let f be a weak equivalence in $\mathbf{pro}(\mathbf{dgArt}_k)$, so that f° is a weak equivalence of coalgebras. In particular, f° is a quasi-isomorphism, and because the linear dual is exact, $f^{\circ*}$ is also a quasi-isomorphism. But $f^{\circ*}$ canonically agrees with $\varprojlim f$ by 3.2.11(2). Hence $\varprojlim f$ is a quasi-isomorphism too. \square

3.3. (Co)connective objects. We restrict our attention to connective pro-Artinian dgas and coconnective conilpotent coalgebras, where weak equivalences become simpler.

Theorem 3.3.1 (Pridham). *The category $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ is a model category, with weak equivalences those maps f for which each $H^n f$ is an isomorphism of profinite k -vector spaces, and fibrations those maps f for which $\varprojlim f$ is a degreewise surjection.*

Proof. The proof of [Pri10, 4.3] adapts to the noncommutative case. The idea of the proof is to use the Dold–Kan correspondence to identify $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ with the category of pro-objects in simplicial ungraded Artinian algebras. One then shows that this is equivalently the category of simplicial objects in pro-(ungraded Artinian algebras). One may as well put a model structure on the opposite category, which is the category of cosimplicial objects in noncommutative formal spectra. The result follows from an application of Bousfield’s theory of injective models [Bou03] to put model structures on categories of cosimplicial objects. \square

In fact, $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ is a simplicial model category, and the limit functor is a simplicial functor when $\mathbf{dga}_k^{\leq 0}$ is given its standard simplicial enrichment.

Lemma 3.3.2. *Let $V \in \mathbf{pro}(\mathbf{fd} - \mathbf{vect}_k)$ be a profinite vector space. The assignment $V \mapsto \varprojlim V^*$ is a contravariant equivalence $\mathbf{pro}(\mathbf{fd} - \mathbf{dgvect}_k) \rightarrow \mathbf{dgvect}_k$.*

Proof. The colimit functor $\varinjlim : \mathbf{ind}(\mathbf{fd} - \mathbf{dgvect}_k) \rightarrow \mathbf{dgvect}_k$ is an equivalence: it is essentially surjective because every vector space is the colimit of its finite-dimensional subspaces, and it is fully faithful because finite-dimensional vector spaces are compact in \mathbf{dgvect}_k . In other words, every vector space is canonically ind-finite. Clearly the linear dual is a contravariant autoequivalence of $\mathbf{fd} - \mathbf{dgvect}_k$, and the result follows. \square

Remark 3.3.3. The spectral algebra analogue of this result appears in [CI04].

Proposition 3.3.4. *The inclusion functor $\iota : \mathbf{pro}(\mathbf{dgArt}_k^{\leq 0}) \hookrightarrow \mathbf{pro}(\mathbf{dgArt}_k)$ is right Quillen.*

Proof. The functor ι is right adjoint to the levelwise truncation functor. It clearly preserves fibrations. It will hence be enough for us to show that it preserves weak equivalences. So suppose that $f : A \rightarrow B$ is a weak equivalence in $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$. We can assume that f is a level map $\{f_\alpha : A_\alpha \rightarrow B_\alpha\}_\alpha$, and then f° is the dgc map $\varinjlim_\alpha (f_\alpha^*)$. Fixing an $n \in \mathbb{Z}$, we have $H^n(f^\circ) \cong \varinjlim_\alpha H^n(f_\alpha^*) \cong \varinjlim_\alpha (H^{-n}(f_\alpha)^*)$, because filtered colimits and linear duals are exact. By 3.3.2, we know that a morphism g of profinite vector spaces is an isomorphism if and only if $\varinjlim g^*$ is. Applying this to $g = H^{-n}f$, we see that $H^n(f^\circ)$ is an isomorphism if and only if $H^{-n}f$ is. But by assumption f was a weak equivalence, so that every $H^{-n}f$ is an isomorphism. Hence f° is a quasi-isomorphism of coalgebras. By 3.1.16, a quasi-isomorphism between coconnective coalgebras is a weak equivalence. So f° is a weak equivalence, which means precisely that ιf is a weak equivalence. \square

One can push the above reasoning further to show that the weak equivalences in the procategory $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ are created by \varprojlim ; this will be useful to us later.

Lemma 3.3.5. *The functor $\varprojlim : \mathbf{pro}(\mathbf{fd} - \mathbf{vect}_k) \rightarrow \mathbf{dgvect}_k$ reflects isomorphisms.*

Proof. Let $g : U \rightarrow V$ be a map of profinite vector spaces. We may take g to be a level map $\{g_\alpha : U_\alpha \rightarrow V_\alpha\}_\alpha$. By 3.3.2, g is an isomorphism if and only if $\varinjlim_\alpha g_\alpha^* : \varinjlim_\alpha V_\alpha^* \rightarrow \varinjlim_\alpha U_\alpha^*$ is an isomorphism. But this map is an isomorphism if and only if its linear dual $\varprojlim_\alpha U_\alpha^{**} \rightarrow \varprojlim_\alpha V_\alpha^{**}$ is an isomorphism. But this latter map canonically agrees with $\varprojlim g = \varprojlim_\alpha g_\alpha$ since the U_α and V_α are finite-dimensional. So g is an isomorphism if and only if $\varprojlim g$ is. \square

Lemma 3.3.6. *The functor $\varprojlim : \mathbf{pro}(\mathbf{dgArt}_k) \rightarrow \mathbf{dga}_k$ is the homotopy limit functor.*

Proof. Let P be any object of $\mathbf{pro}(\mathbf{dgArt}_k)$. Without loss of generality, by 3.2.6 we may assume that P is strict. Let I be the indexing set of P . Because every cofiltered set I has a cofinal directed subset $I' \hookrightarrow I$ [AGV72, Exposé 1, 8.1.6], we may without loss of generality assume that I is directed, and in particular is a Reedy category. It is now easy to see that P is a Reedy fibrant diagram of dgas, and hence $\varprojlim P \simeq \mathbf{holim} P$, because one can compute homotopy limits as usual limits along resolved diagrams. \square

Remark 3.3.7. The above proof is a generalisation of the fact that \varprojlim is exact when restricted to Mittag-Leffler systems: indeed a Mittag-Leffler pro-object is precisely a strict pro-object indexed by a countable directed set.

Proposition 3.3.8. *The functor $\varprojlim : \mathbf{pro}(\mathbf{dga}_k^{\leq 0}) \rightarrow \mathbf{dga}_k$ both preserves and reflects weak equivalences.*

Proof. Let f be a morphism in $\mathbf{pro}(\mathbf{dga}_k^{\leq 0})$. By definition, f is a weak equivalence if and only if each $H^n f \in \mathbf{pro}(\mathbf{fd} - \mathbf{vect}_k)$ is an isomorphism. But \varprojlim reflects isomorphisms by 3.3.5 and so $H^n f$ is an isomorphism if and only if $\varprojlim H^n f$ is an isomorphism. But \varprojlim is exact by 3.3.6, and so $\varprojlim H^n f$ is an isomorphism if and only if $H^n \varprojlim f$ is an isomorphism. \square

Remark 3.3.9. The category $\mathbf{dga}_k^{\leq 0}$ is a model category, with weak equivalences quasi-isomorphisms and fibrations the surjections in strictly negative degrees. The model structure is transferred from that on simplicial vector spaces by the monoidal Dold–Kan correspondence [SS03]. From the above, it is easy to see that $\varprojlim : \mathbf{pro}(\mathbf{dga}_k^{\leq 0}) \rightarrow \mathbf{dga}_k^{\leq 0}$ is right Quillen. However, the inclusion $\mathbf{dga}_k^{\leq 0} \hookrightarrow \mathbf{dga}_k$ is not right Quillen as it does not preserve fibrations. Because we want to work with unbounded pro-Artinian dgas, we will not use the model structure on $\mathbf{dga}_k^{\leq 0}$.

We end this section by turning our attention to the coalgebra side of the story.

Theorem 3.3.10. *The category $\mathbf{con.dgc}_k^{\geq 0}$ of coconnective conilpotent dgc's admits a model structure where the weak equivalences are the quasi-isomorphisms and the cofibrations are the degreewise injections. The inclusion functor $\mathbf{con.dgc}_k^{\geq 0} \hookrightarrow \mathbf{con.dgc}_k$ is left Quillen. The equivalence $\mathbf{pro}(\mathbf{dga}_k^{\leq 0}) \xrightarrow{\cong} (\mathbf{con.dgc}_k^{\geq 0})^{op}$ is a Quillen equivalence.*

Proof. This is similar to 3.2.12 but transfers the Pridham model structure to the category of coalgebras. Clearly this transfer yields the claimed Quillen equivalence. The proof of 3.3.4 shows that a map f of connective dgas is a quasi-isomorphism if and only if f° is, and hence that a map g of coconnective dgc's is a quasi-isomorphism if and only if g^\sharp is. Hence the weak equivalences are as claimed. The proof of 3.2.12 shows that a map g is a degreewise injection if and only if $\varprojlim g^\sharp$ is a degreewise surjection. Hence the cofibrations are as claimed. It is easy to see that the inclusion is left Quillen since a quasi-isomorphism of coconnective dgc's is a weak equivalence. \square

4. KOSZUL DUALITY

In this section, k is any field. We study the Koszul dual of a dga, which is the linear dual of the bar construction. For a wide class of dgas, we show that the Koszul double dual can be interpreted as a sort of derived completion functor. We show that connective cohomologically locally finite dgas are derived complete; later this will give us extremely good control over prorepresenting objects. We apply this to the problem of lifting quasi-isomorphisms of limits of pro-Artinian dgas to weak equivalences.

4.1. The Koszul dual.

Definition 4.1.1. Let A be an augmented dga. The **Koszul dual** of A is $A^! := (BA)^*$.

Clearly $A^!$ is itself an augmented dga. Because both B and the linear dual preserve quasi-isomorphisms, so does $A \mapsto A^!$. Loosely, the differential $d(x^*)$ is the signed sum of the products $x_1^* \cdots x_r^*$ such that the x_i satisfy $\partial(x_1 | \cdots | x_r) = x$, where ∂ is the bar differential. The Koszul dual $A^!$ is a completed semifree dga, in the sense that the underlying graded algebra is a completed free algebra.

Proposition 4.1.2. *Let A be a connective augmented dga, and let S be the A -module k with A -action given by augmentation $A \rightarrow k$. Then there is a natural quasi-isomorphism of dgas $A^! \simeq \mathbb{R}\mathrm{End}_A(S)$.*

Proof. This is standard and appears as [FHT01, §19, Exercise 4] or [LV12, 2.2.2] for rings. The loose idea is that by taking the bar resolution of S , the bar construction BA becomes a model for the derived tensor product $S \otimes_A^{\mathbb{L}} S$, which is naturally a coalgebra. Taking the linear dual, one obtains the desired statement. \square

Because BA is a tensor coalgebra, it follows that $A^!$ is a completed tensor algebra. Under an additional finiteness hypothesis on A , we can identify the generators of $A^!$ as the duals of the cogenerators of BA . The finiteness hypothesis we will need is the following.

Definition 4.1.3. Say that a complex X is **raylike** if it is locally finite, and either bounded above or bounded below.

Clearly a locally finite bounded complex is raylike. It is also easy to see that if X is raylike then so is the linear dual X^* . The key property of a raylike complex is that the sum appearing in

$$(X^{\otimes n})^m \cong \bigoplus_{i_1 + \dots + i_n = m} X^{i_1} \otimes \dots \otimes X^{i_n}$$

is finite, and hence the tensor product $X^{\otimes n}$ is itself locally finite. Hence we have natural isomorphisms $(X^{\otimes n})^* \cong (X^*)^{\otimes n}$. In particular, the dual of a raylike dga is a dgc.

Definition 4.1.4. Say that a dga A is **1-coconnective** if it is coconnective and $A^0 \cong k$. Say that A is **2-coconnective** if it is 1-coconnective and $A^1 \cong 0$.

In other words, a 2-coconnective dga has nothing (except the unit) in degrees < 2 .

Remark 4.1.5. For $n \in \mathbb{Z}$, one can define n -coconnective similarly: 0-coconnective dgas are just coconnective dgas. For $n > 1$, n -coconnective dgas are 1-coconnective and are trivial in degrees $1, \dots, n-1$. For $n < 0$, an n -coconnective dga is simply one concentrated in degrees $\geq n$. One can make an analogous definition of n -connective dga.

Remark 4.1.6. In the graded algebra literature, a 1-(co)connective graded algebra is typically called **connected**. In the higher algebra literature, 1-connective dgas or ring spectra are often also called **connected**. In view of this we do not use ‘connected’ in this context.

Part (3) of the following is well known and appears as e.g. [FHT01, §19, Exercise 3]:

Proposition 4.1.7. *Let A be an augmented raylike dga. Then*

- (1) *There is a natural dga isomorphism $A^! \cong \widehat{\Omega}(A^*)$ where $\widehat{\Omega}$ denotes the completed cobar construction.*
- (2) *The natural map $\Omega(A^*) \rightarrow A^!$ is completion.*
- (3) *If A is connective then there is a natural dga isomorphism $A^! \cong \Omega(A^*)$.*
- (4) *If A is 2-coconnective then there is a natural dga isomorphism $A^! \cong \Omega(A^*)$.*

Proof. For brevity write V for the shifted augmentation ideal $V := \bar{A}[1]$, which is also raylike. For now, forget the differential and think about the underlying graded vector spaces. The dgc BA is the direct sum total complex of the double complex whose rows are $V^{\otimes n}$. Hence, $A^!$ is the direct product total complex of the double complex with rows $(V^{\otimes n})^*$. But because V was raylike, $A^!$ is the direct product total complex of the double

complex with rows $(V^*)^{\otimes n}$. But this is precisely the completed tensor algebra on V^* , which is by definition $\widehat{\Omega}(A^*)$. We have shown that $A^! \cong \widehat{\Omega}(A^*)$ as graded algebras. But is it not hard to check that the bar differential dualises to the cobar differential, and it now follows that $A^! \cong \widehat{\Omega}(A^*)$ as dgas, which is precisely the first claim. To see the second claim, just observe that the natural map $\Omega(A^*) \rightarrow A^!$ is induced by the identity on generators. For the third claim, first observe that if A is connective then V is concentrated in strictly negative degrees, from which it follows that in each degree BA is a finite direct sum. Hence $A^!$ is also the direct sum total complex of the double complex with rows $(V^*)^{\otimes n}$. In other words, $A^! \cong \Omega(A^*)$ as dgas; observe that the completion map is an isomorphism. The fourth claim is completely analogous to the third: because A is 1-coconnective, this time V is concentrated in strictly positive degrees. \square

Remark 4.1.8. The dga $A^!$ does not always agree with $\Omega(A^*)$. Indeed, let A be the graded algebra $\frac{k[\epsilon]}{\epsilon^2}$ with ϵ in degree 1. Then BA is the polynomial coalgebra $k[\epsilon]$ concentrated in degree zero, and hence $A^!$ is the complete polynomial algebra $k[[x]]$ on an element of degree zero. However, $\Omega(A^*)$ is $k[x]$. As expected, the completion of $\Omega(A^*)$ is $A^!$.

When BA is raylike, the double Koszul dual of A is its derived completion:

Proposition 4.1.9. *Let A be an augmented dga such that BA is raylike. Then*

- (1) *There is a natural isomorphism $A^{!!} \cong \widehat{\Omega}BA$.*
- (2) *If $A^!$ is connective then there is a natural quasi-isomorphism $A^{!!} \simeq A$.*
- (3) *If $A^!$ is 2-coconnective then there is a natural quasi-isomorphism $A^{!!} \simeq A$.*

Proof. Because BA is raylike so is $A^!$. We can hence apply 4.1.7(1) to conclude that $A^{!!} \cong \widehat{\Omega}(A^{!*})$. But because BA is locally finite, $A^{!*}$ is BA again. Hence $A^{!!} \cong \widehat{\Omega}BA$ as required. For the second statement, we can instead apply 4.1.7(3) and proceed as above to conclude that we have an isomorphism $A^{!!} \cong \Omega BA$. But ΩBA is naturally quasi-isomorphic to A . The third statement is the same but uses 4.1.7(4) instead. \square

Remark 4.1.10. We refer to $\widehat{\Omega}BA$ as the derived completion of A as it is the completion of a cofibrant resolution of A ; a priori completion need not actually have a derived functor as we have not shown that it is left Quillen. Derived completion functors for commutative rings have been studied by a number of authors [GM92, PSY14b, Sha19], and it is possible to show that in this setting, the (derived) completion of a noetherian commutative ring is its Koszul double dual [DGI06, PSY14a]. In the noncommutative setting, Efimov [Efi10] defines completion of dg categories via derived double centralisers; in our setting this reduces to the double Koszul dual.

Remark 4.1.11. Using the homotopy theory of A_∞ -coalgebras (as in [CPRNW19, §6] or [HLV19]), one can weaken the hypotheses of 4.1.9(1) to requiring only that BA is cohomologically raylike. Indeed, if A is a raylike A_∞ -algebra then A^* is an A_∞ -coalgebra, and vice versa. There is a completed cobar construction $\widehat{\Omega}$ (but not necessarily an uncompleted one) for A_∞ -coalgebras, and it is not hard to prove 4.1.7(1) when A is assumed to be a raylike A_∞ -algebra. Minimal models exist for A_∞ -coalgebras, so if BA is cohomologically raylike, take an A_∞ minimal model C , which is genuinely raylike. The completed cobar construction preserves quasi-isomorphisms (unlike the uncompleted one) and so using the above we have a quasi-isomorphism $\widehat{\Omega}BA \simeq \widehat{\Omega}BC$. By the A_∞ version of 4.1.7(1), we have an isomorphism $\widehat{\Omega}BC \cong (C^*)^!$. But $A^!$ is quasi-isomorphic to C^* and the result follows.

Proposition 4.1.12. *Let A be an augmented locally finite dga. Suppose that A is either connective or 2-coconnective. Then there is a natural quasi-isomorphism $A^{\dagger} \simeq A$.*

Proof. If A is connective, then it is raylike and 4.1.7(3) tells us that we have $A^{\dagger} \cong \Omega(A^*)$. Applying the Koszul dual to both sides we get an isomorphism $A^{\dagger\dagger} \cong \Omega(A^*)^{\dagger}$. But $\Omega(A^*)^{\dagger}$ is the linear dual of $B\Omega(A^*)$, which is weakly equivalent to A^* . A weak equivalence is a quasi-isomorphism, and applying the linear dual we get a quasi-isomorphism $A^{\dagger\dagger} \cong A$. On the other hand, if A is 2-coconnective, then one can use the same argument but with 4.1.7(4) instead. \square

Lemma 4.1.13. *Let V be a graded vector space. Assume $T(V)$, the tensor algebra on V , is locally finite. Then V is locally finite, $V^0 \cong 0$, and V is either connective or coconnective.*

Proof. The point is to look at the subalgebra $T(V)^0$. It is clear that V must be locally finite as we have an obvious linear embedding $V \hookrightarrow T(V)$ of rank 1 tensors. If the subalgebra $T(V)^0$ has an element x that is not a multiple of the unit, then x must generate a polynomial subalgebra $k[x]$, and hence $T(V)^0$ is infinite-dimensional. Hence $T(V)^0$ must be k . Because $T(V^0)$ embeds into $T(V)^0$ as a subalgebra, we must have $T(V^0) \cong k$ and hence $V^0 \cong 0$. Suppose that $V^n \not\cong 0$ for some $n > 0$. Then for any $m > 0$, $(V^n)^{\otimes m} \otimes (V^{-m})^{\otimes n}$ is a linear subspace of $T(V)^0$ not containing the unit. Hence it must be 0, and it follows that $V^m \cong 0$. In particular if V contains anything in positive degree, then it is coconnective. Hence V is either connective or coconnective. \square

Theorem 4.1.14 ([LPWZ08, Theorem A]). *Let A be an augmented dga such that BA is locally finite. Then there is a natural quasi-isomorphism $A^{\dagger} \simeq A$.*

Proof. By 4.1.13, it follows that $V := \bar{A}[1]$ is locally finite, either connective or coconnective, and $V^0 \cong 0$. It follows that A is locally finite, and either connective or 2-coconnective. In either case the theorem follows from 4.1.12. \square

Remark 4.1.15. The original proof of 4.1.14 appearing in [LPWZ08] is incorrect. The problem is with [LPWZ08, 1.15], which claims that if A is an augmented locally finite dga then $\Omega(A^*) \cong A^{\dagger}$ as dgas. This is not true, as we have already seen in 4.1.8. Moreover, A^* need not even be a dgc: consider the graded algebra $A = k[x, x^{-1}]$ with x in degree 1. Then A^* is not a graded coalgebra, since the multiplication fails to dualise to a map $A^* \rightarrow A^* \otimes A^*$.

4.2. Completions. One can improve 4.1.14 by being more careful: the basic idea is that if BA is a reasonable coalgebra, then ΩBA is quasi-complete, in the sense that the completion map is a quasi-isomorphism. The proof will require some use of A_{∞} methods: for basic facts about A_{∞} -algebras, we refer the reader to [Kel01, LH03]. All we really use is the existence of minimal models.

We begin with the following observation about the zeroth cohomology of a Koszul dual, which essentially appears in [Seg08].

Definition 4.2.1. Observe that if E is an A_{∞} -algebra, and m_n is one of the A_{∞} operations on E , then it restricts to a map $m_n : (E^1)^{\otimes n} \rightarrow E^2$. The **homotopy Maurer–Cartan function** is the direct sum $m = \bigoplus_n m_n : T(E^1) \rightarrow E^2$.

One can extend the terminology of 4.1.4 to A_{∞} -algebras in the obvious way; in particular we say that an A_{∞} -algebra E is 1-coconnective if it is coconnective and $E^0 \cong k$.

Proposition 4.2.2. *Let E be an augmented 1-coconnective A_∞ -algebra. Assume that E^1 is finite dimensional. Then there is an algebra isomorphism*

$$H^0(E^!) \cong \frac{\hat{T}(E^{1*})}{m^*(E^{2*})}$$

where m is the homotopy Maurer–Cartan function.

Proof. For brevity put $V := \bar{E}[1]$ the shifted augmentation ideal and $A := E^!$ the Koszul dual. Let ∂ be the A_∞ bar differential on the tensor coalgebra TV . Because E was 1-coconnective, V is coconnective. Hence BE is coconnective and so A is connective. In particular, we have

$$H^0(A) \cong \frac{A^0}{\partial^*(A^{-1})}.$$

For brevity, write G (‘generators’) for $V^0 \cong E^1$ and R (‘relations’) for $V^1 \cong E^2$; by assumption G is finite dimensional. Because V is coconnective it follows that $T^0(V) \cong T(G)$. Because G is finite dimensional it follows that A^0 is the completed tensor algebra $\hat{T}(G^*)$.

I claim that $\langle \text{im}(m^*) \rangle = \partial^*(A^{-1})$ as ideals of A^0 ; showing this claim will prove the theorem. For this, first observe that $T^1(V)$ is the double direct sum

$$T^1(V) \cong \bigoplus_{i,j} G^{\otimes i} \otimes R \otimes G^{\otimes j}$$

and hence dualising and using that G is finite dimensional we see that A^{-1} is

$$A^{-1} \cong \prod_{i,j} G^{*\otimes i} \otimes R^* \otimes G^{*\otimes j} \cong A^0 R^* A^0.$$

Because ∂^* vanishes on elements of A^0 , we see that $\partial^*(A^{-1}) = A^0 \partial^*(R^*) A^0$. So it will suffice to show that $\text{im}(m^*) = \partial^*(R^*)$. But it is easy to see that the restriction of ∂^* to R^* is precisely m^* , because an element $t \in T(G)$ satisfies $\partial t \in R$ precisely when $\partial t = mt$. \square

We will need a lemma on the completion of graded algebras. Recall that if A is an augmented graded algebra then \hat{A} denotes the completion along its augmentation ideal.

Lemma 4.2.3. *Let A be a connective augmented graded algebra with A^0 nilpotent. Then \hat{A} is complete.*

Proof. The idea is that in a large enough product of elements from A , one can group sufficiently many elements from A^0 together to make the product vanish. Let \mathfrak{m} be the augmentation ideal of A . Fix a ‘degree’ $i \leq 0$ and an ‘exponent’ $j > 0$. By connectivity, for sufficiently large n (depending on i and j) all monomials of $(\mathfrak{m}^n)^i$, the degree i part of the ideal \mathfrak{m}^n , must contain a product of at least j nonidentity elements from A^0 . But A^0 is nilpotent, so there is some fixed N such that if $j > N$ then a product of j nonidentity elements from A^0 must vanish. Hence it follows that for all sufficiently large n (depending on i) we have $(\mathfrak{m}^n)^i = 0$. So as vector spaces, we have

$$(\hat{A})^i \cong \varprojlim_n (A/\mathfrak{m}^n)^i \cong \varprojlim_n A^i / (\mathfrak{m}^n)^i \cong A^i$$

and hence \hat{A} is complete. \square

Remark 4.2.4. If A is a connective graded algebra, and A^0 is a central subalgebra, then A is complete if and only if A^0 is complete. More generally, A^0 itself need not be commutative; it suffices that the commutators $[A^0, A^i]$ vanish for $i < 0$.

Remark 4.2.5. This shows that the ‘forget the grading’ functor on graded algebras does not preserve limits. Indeed take for example $A = k[x]$ with x in degree -1. In the category of graded algebras, the limit $\varprojlim_n A/x^n$ is just A again by the above. But the limit in the category of ungraded algebras is $k[[x]]$, which does not even admit a nontrivial grading.

Our key technical result is the following:

Theorem 4.2.6. *Let E be an augmented 1-coconnective locally finite dga such that $H^0(\Omega(E^*))$ is nilpotent. Then the completion map $c : \Omega(E^*) \rightarrow \widehat{\Omega}(E^*)$ is a quasi-isomorphism.*

Proof. For brevity put $\Omega := \Omega(E^*)$ and $\widehat{\Omega} := \widehat{\Omega}(E^*)$. The rough idea of the proof is that $H\Omega$ is complete by 4.2.3, and moreover completion is flat so $\widehat{H\Omega} \cong H\widehat{\Omega}$. The above is literally true if we are using symmetric algebras rather than tensor algebras, but we do not have access to results from commutative algebra so we will have to be more careful.

We begin by studying H^0c . Because E is 1-coconnective we see that both Ω and $\widehat{\Omega}$ are connective. By 4.1.7 we have a dga isomorphism $\widehat{\Omega} \cong E^!$. As in the proof of 4.2.2, we see that $H^0\Omega \cong T(E^{1*})/\langle m^*(E^{2*}) \rangle$, where m denotes the multiplication on E . Hence by using 4.2.2 we see that H^0c is the completion map

$$H^0c : \frac{T(E^{1*})}{\langle m^*(E^{2*}) \rangle} \rightarrow \frac{\widehat{T}(E^{1*})}{\langle m^*(E^{2*}) \rangle}$$

which is an isomorphism because a nilpotent algebra is complete

First we show that c is a quasi-injection. Denote the differential in Ω by ∂ and denote the differential in $\widehat{\Omega}$ by $\widehat{\partial}$. Let $\bar{W} \subseteq H\Omega$ be a space of generators for the algebra $H\Omega$. Let $W \subseteq \ker \partial \subseteq \Omega$ be a space of lifts of \bar{W} , and let $K \subseteq \ker \partial \subseteq \Omega$ be the subalgebra of Ω generated by W . Because W lifts the generators of $H\Omega$, we see that the natural map $K \rightarrow H\Omega$ is a surjection. Complete this natural map to obtain a commutative diagram

$$\begin{array}{ccc} K & \longrightarrow & \widehat{K} \\ \downarrow & & \downarrow \\ H\Omega & \longrightarrow & \widehat{H\Omega}. \end{array}$$

Using 4.2.3 we see that $H\Omega$ is complete, and hence the bottom map is an isomorphism, which implies that $K \rightarrow H\Omega$ extends to a surjection $\widehat{K} \twoheadrightarrow H\Omega$. Viewing \widehat{K} as a subalgebra of $\widehat{\Omega}$, we obtain another commutative diagram

$$\begin{array}{ccc} \widehat{K} & \xrightarrow{\text{id}} & \widehat{K} \\ \downarrow & & \downarrow \\ H\Omega & \longrightarrow & H\widehat{\Omega}. \end{array}$$

Suppose that $x \in H^i\Omega$ satisfies $Hc(x) = 0$. Lift x to an element \tilde{x} of \widehat{K} . Because \tilde{x} maps to 0 under the projection $\widehat{K} \rightarrow H\widehat{\Omega}$, we must have $\tilde{x} = \widehat{\partial}y$ for some y . But then \tilde{x} maps to 0 under the projection $\widehat{K} \twoheadrightarrow H\Omega$. Because \tilde{x} lifted x , we see that x was 0. Hence Hc is a levelwise injection, meaning precisely that c is a quasi-injection.

The proof that c is a quasi-surjection is in some sense dual. The inclusion $\Omega \hookrightarrow \widehat{\Omega}$ gives maps of graded algebras

$$\frac{\Omega}{\langle \text{im } \partial \rangle} \twoheadrightarrow \frac{\Omega}{\langle \text{im } \hat{\partial} \rangle \cap \Omega} \hookrightarrow \frac{\widehat{\Omega}}{\langle \text{im } \hat{\partial} \rangle}.$$

Moreover, these maps are compatible with cohomology, in the sense that we have a commutative diagram

$$\begin{array}{ccc} H\Omega & \hookrightarrow & \frac{\Omega}{\langle \text{im } \partial \rangle} \\ \downarrow & & \downarrow \\ H\widehat{\Omega} & \hookrightarrow & \frac{\widehat{\Omega}}{\langle \text{im } \hat{\partial} \rangle}. \end{array}$$

But because $H^0(\widehat{\Omega})$ is nilpotent, it follows that $H\widehat{\Omega}$ is a polynomial algebra, in the sense that the image of $H\widehat{\Omega}$ is contained in the subalgebra $\frac{\Omega}{\langle \text{im } \partial \rangle \cap \Omega}$. Now we can lift: take $x \in H\widehat{\Omega}$, regarded as a subspace of $\frac{\Omega}{\langle \text{im } \partial \rangle \cap \Omega}$. Then find $\tilde{x} \in \Omega$ mapping to x . Because x was represented by a coboundary, \tilde{x} must be a coboundary, and its class in $H\Omega$ is a lift of x . Hence Hc is a levelwise surjection, i.e. c is a quasi-surjection.

So c is a quasi-isomorphism, as required. \square

Remark 4.2.7. We see that 4.2.6 may fail to be true if $H^0(\Omega(E^*))$ is not complete, for the same reason as in 4.1.8.

Theorem 4.2.8. *Let A be a connective augmented cohomologically locally finite dga with $H^0(A)$ Artinian local. Then there is a natural quasi-isomorphism $A^{\text{!}} \simeq A$.*

Proof. Let H be a minimal A_∞ -algebra model for A , the existence of which is ensured by [Kad80]. In other words, as a graded vector space H is the cohomology algebra of A , and we equip H with higher multiplications making it A_∞ -quasi-isomorphic to A . Let BH be the A_∞ bar construction on H , which is a dgc; as a graded vector space it is $T(\bar{H}[1])$ and the differential incorporates the higher multiplications into the usual bar differential. Because H is connective and locally finite, we see that BH is also connective and locally finite. Moreover, $(BH)^0$ is just k (i.e. BH is 1-connective). In particular, $(\bar{B}H)[-1]$ is connective, and it follows that ΩBH and $\widehat{\Omega}BH$ are connective. Because BH is raylike, the dga $H^{\text{!}}$ is raylike and so by 4.1.7 we have a dga isomorphism $H^{\text{!}} \cong \widehat{\Omega}BH$. Because the A_∞ Koszul dual sends A_∞ -quasi-isomorphisms to dga quasi-isomorphisms, we see that $H^{\text{!}} \simeq A^{\text{!}}$ as dgas. Similarly, because ΩB sends A_∞ -quasi-isomorphisms to dga quasi-isomorphisms, we see that $\Omega BH \simeq \Omega BA \simeq A$ as dgas. It is now easy to see that $H^{\text{!}}$ satisfies the hypotheses of 4.2.6 and, using that $H^{1*} \cong BH$, we hence see that the completion map $\Omega BH \rightarrow \widehat{\Omega}BH$ is a quasi-isomorphism. We hence have a chain of quasi-isomorphisms

$$A \simeq \Omega BH \simeq \widehat{\Omega}BH \cong H^{\text{!}} \simeq A^{\text{!}}$$

and the claim is proved. \square

Remark 4.2.9. In the above, one does not necessarily need A to be connective; it suffices that $H(A^1)$ is 1-coconnective, as this is what is required to make the cobar construction on its dual connective.

Remark 4.2.10. Andrey Lazarev has suggested to the author that the ‘correct’ version of the preceding theorem should be something like the following. Let $A \rightarrow k$ be an augmented dga. Then under some mild conditions on A , the Koszul double dual $A \rightarrow A^{!!}$ is quasi-isomorphic to the Bousfield localisation of the right A -module A with respect to the homology theory $M \mapsto \mathrm{Tor}_*^A(M, k)$. The idea is that the formal completion $A^!$, the nilpotent completion \widehat{A}_k , and the Bousfield localisation $L_k(A)$ should all agree. Moreover, if A is cohomologically locally finite with $H^0(A)$ Artinian local, then the Bousfield localisation of A is A again. To prove this second statement, he suggests that one should show that the cobar spectral sequence associated to $A^!$ converges to the cohomology of the localisation of A , in a similar manner to the convergence of the E -Adams spectral sequence. The relevant computations ought to be similar to those of [Bou79] or [Dwy75]. This approach may also extend to the setting of ring spectra.

Remark 4.2.11. Let A be any augmented connective dga. Let $B^\sharp(A)$ denote the **continuous Koszul dual**: one takes the bar construction on A and then applies the $(-)^{\sharp}$ functor of 3.2.10 to obtain a pro-Artinian dga. If one takes the levelwise Koszul dual to obtain a pro-Artinian dga $(B^\sharp(A))^!$, it is clear that $\varprojlim (B^\sharp(A))^!$ is quasi-isomorphic to A by applying 4.1.12 levelwise. However, it is far from clear that the same applies when we forget that $B^\sharp A$ is pro-Artinian; i.e. take $(\varprojlim B^\sharp(A))^!$ instead.

4.3. Lifting weak equivalences. Suppose that A and A' are limits of two connective pro-Artinian dgas. When does a quasi-isomorphism $A \simeq A'$ lift to a weak equivalence of pro-Artinian dgas? We show that this can always be done when A and A' are cohomologically locally finite. The key observation is that the Koszul double dual is closely related to cofibrant resolutions for pro-Artinian algebras. We begin with the following useful observation:

Lemma 4.3.1. *Let A be a connective augmented dga. If A is cohomologically locally finite then so are BA and $A^!$.*

Proof. Since the linear dual is exact, the statement for $A^!$ is implied by the statement for BA . To prove the latter, filter BA by the tensor powers of A to obtain a spectral sequence with E_1 page $H^p(A^{\otimes q}) \Rightarrow H^{p-q}(BA)$. Since there are only finitely many nonzero $H^p(A^{\otimes q})$ with $p - q$ fixed, and they are all finite-dimensional, $H^{p-q}(BA)$ must also be finite-dimensional. \square

Remark 4.3.2. One can also prove 4.3.1 by applying the A_∞ bar construction to an A_∞ minimal model for A , which yields a locally finite model for BA .

Theorem 4.3.3. *Let $\mathcal{A} \in \mathbf{pro}(\mathbf{dGArt}_k)$ be a pro-Artinian dga and put $A := \varprojlim \mathcal{A}$.*

- (1) *The natural map $B^\sharp \Omega(\mathcal{A}^\circ) \rightarrow \mathcal{A}$ is a cofibrant resolution of pro-Artinian dgas.*
- (2) *The natural map $\Omega(\mathcal{A}^\circ)^! \rightarrow A$ is a dga quasi-isomorphism.*
- (3) *If \mathcal{A} is connective then there is a natural dga isomorphism $\Omega \mathcal{A}^\circ \cong \varprojlim \mathcal{A}^!$, where $\mathcal{A}^!$ is the ind-dga obtained from \mathcal{A} by applying the Koszul dual levelwise.*
- (4) *If \mathcal{A} is connective and BA is cohomologically locally finite then there is a natural dga quasi-isomorphism $\varprojlim \mathcal{A}^! \rightarrow A^!$.*
- (5) *If \mathcal{A} is connective and \overline{BA} is cohomologically locally finite then the natural map $B^\sharp(\mathcal{A}^!) \rightarrow \mathcal{A}$ is a cofibrant resolution of pro-Artinian dgas.*
- (6) *If \mathcal{A} is connective and BA is cohomologically locally finite then the natural map $A^{!!} \rightarrow A$ is a cofibrant resolution.*

Proof. For (1), just use that $B\Omega$ is a fibrant dgc resolution along with the fact that \sharp and \circ are inverse Quillen equivalences. For (2), use that \varprojlim preserves weak equivalences (by 3.2.13) to conclude that the natural map $\Omega(\mathcal{A}^\circ)^! \cong \varprojlim B^\sharp \Omega(\mathcal{A}^\circ) \rightarrow \varprojlim \mathcal{A} \cong A$ is a quasi-isomorphism. For (3), first note that by construction, $\mathcal{A}^\circ = \varinjlim \mathcal{A}^*$. Because Ω is a left adjoint, it is cocontinuous, and so we have an isomorphism $\Omega \mathcal{A}^\circ \cong \varinjlim \Omega(\mathcal{A}^*)$. But if A is a connective Artinian dga then $\Omega(\mathcal{A}^*) \cong A^!$ by 4.1.7. So $\Omega(\mathcal{A}^*) \cong \mathcal{A}^!$ as ind-dgas and the claim follows.

To prove (4) will take some more work. For brevity put $C := BA$ and $\mathcal{C} := BA$; note that because B is a right adjoint it is continuous and hence $C \cong \varprojlim \mathcal{C}$ as coalgebras. Let ϕ be the natural map $\phi : \varinjlim C^* \rightarrow C^*$ that we wish to prove is a quasi-isomorphism.

For $n \in \mathbb{Z}$, consider the induced linear map

$$\psi_n : \quad \varinjlim (H^n(C^*)) \xrightarrow{\cong} H^n(\varinjlim C^*) \xrightarrow{H^n \phi} H^n(C^*) \xrightarrow{\cong} H^{-n}(C)^*$$

where we have used exactness of filtered colimits and the linear dual, and dualise it to obtain a map

$$\chi_n : \quad H^{-n}(C) \rightarrow H^{-n}(C)^{**} \xrightarrow{\psi_n^*} (\varinjlim (H^n(C^*)))^* \xrightarrow{\cong} \varprojlim H^n(C^*)^* \xrightarrow{\cong} \varprojlim H^{-n}(C^{**})$$

where we have used exactness of the linear dual again along with the fact that Hom preserves limits. Because C is cohomologically locally finite, $H^{-n}(C) \rightarrow H^{-n}(C)^{**}$ is an isomorphism. Similarly, each level \mathcal{C}_α of \mathcal{C} is locally finite, since it is the bar construction on a connective Artinian dga. In particular, the natural map $H^{-n}(\mathcal{C}_\alpha) \rightarrow H^{-n}(\mathcal{C}_\alpha^{**})$ which sends $[v]$ to $[\text{ev}_v]$ is an isomorphism. Let $[u] \in H^{-n}(C)$; one can compute that $\chi_n([u]) = ([\text{ev}_{u_\alpha}])_\alpha$, where u_α is the image of u under the natural map $C \rightarrow \mathcal{C}_\alpha$. Hence, the composition $H^{-n}(C) \xrightarrow{\chi_n} \varprojlim H^{-n}(C^{**}) \xrightarrow{\cong} \varprojlim H^{-n}(C)$ of χ_n with the inverse to the natural isomorphism sends $[u]$ to $[u_\alpha]_\alpha$. But this is precisely the natural map $H^{-n}(C) \rightarrow \varprojlim H^{-n}(C)$. But as in 3.3.6, this natural map is an isomorphism: because we may choose \mathcal{A} to be strict, we may assume that \mathcal{A} is a Reedy fibrant diagram of dgas. Because B is right Quillen, it follows that \mathcal{C} is a Reedy fibrant diagram of dgcs and hence $C \simeq \text{holim } \mathcal{C}$. Now it follows that χ_n , ψ_n , and $H^n \phi$ are isomorphisms for all n . Hence, ϕ is a quasi-isomorphism as claimed.

For (5), first combine (4) and (3) to get a natural dga quasi-isomorphism $\Omega \mathcal{A}^\circ \rightarrow A^!$. Apply B^\sharp to this to get a natural weak equivalence $B^\sharp(A^!) \rightarrow B^\sharp \Omega(\mathcal{A}^\circ)$. Compose with the weak equivalence of (1) to see that there is a natural weak equivalence $B^\sharp(A^!) \rightarrow \mathcal{A}$, which is a cofibrant resolution because dgcs in the image of B are fibrant.

The proof of (6) is the same as (2): just take limits of (5). \square

Proposition 4.3.4. *Let \mathcal{A} and \mathcal{A}' be connective pro-Artinian dgas and put $A := \varprojlim \mathcal{A}$, $A' := \varprojlim \mathcal{A}'$. Suppose that A is quasi-isomorphic to A' and that moreover both are cohomologically locally finite. Then \mathcal{A} and \mathcal{A}' are weakly equivalent.*

Proof. Because A is cohomologically locally finite, so is BA by 4.3.1. So applying 4.3.3(5) we see that $B^\sharp(A^!) \rightarrow \mathcal{A}$ is a cofibrant resolution. Similarly, $B^\sharp(A'^!) \rightarrow \mathcal{A}'$ is a cofibrant resolution. But the functor $X \mapsto B^\sharp(X^!)$ sends quasi-isomorphisms to weak equivalences, and so the result follows. \square

4.4. Derived mapping spaces. Our result 4.3.3 is quite powerful, and will allow us to compare derived mapping spaces between connective pro-Artinian dgas and derived mapping

spaces between their limits. Our main theorem here will later be used to give us a more concrete description of representing objects for framed deformations.

Lemma 4.4.1. *Let C be a coaugmented conilpotent dgc and let Γ be an Artinian local dga.*

(1) *There is an isomorphism*

$$\mathrm{Hom}_{\mathbf{pro}(\mathbf{dgArt}_k)}(C^\sharp, \Gamma) \cong \mathrm{Hom}_{\mathbf{aug.dga}_k}(C^*, \Gamma).$$

(2) *There is an isomorphism $\widehat{C^*} \cong C^\sharp$ of pro-Artinian dgas.*

(3) *The statements (1) and (2) remain true if C is coconnective and Γ is connective.*

(4) *The statements (1) and (2) remain true if both C and Γ are ungraded.*

Proof. The point is that Γ is finite-dimensional. We begin with (1). By 3.2.9 we have an isomorphism

$$\mathrm{Hom}_{\mathbf{pro}(\mathbf{dgArt}_k)}(C^\sharp, \Gamma) \cong \mathrm{Hom}(\Gamma^*, C)$$

where we take the right-hand hom in the category of conilpotent coalgebras, so it suffices to show that there is an isomorphism $\mathrm{Hom}(\Gamma^*, C) \cong \mathrm{Hom}_{\mathbf{aug.dga}_k}(C^*, \Gamma)$. Any coalgebra morphism $\Gamma^* \rightarrow C$ gives an algebra morphism $C^* \rightarrow \Gamma$ by dualising, and it is clear that the corresponding map $\mathrm{Hom}(\Gamma^*, C) \rightarrow \mathrm{Hom}_{\mathbf{aug.dga}_k}(C^*, \Gamma)$ is injective. For surjectivity, suppose given a morphism of augmented algebras $C^* \rightarrow \Gamma$, and dualise to obtain a linear map $\Gamma^* \rightarrow C^{**}$. The functionals in the image of Γ^* all have finite support since they vanish on the cofinite subspace $\ker(C^* \rightarrow \Gamma)$ of C^* . Hence the map $\Gamma^* \rightarrow C^{**}$ factors through the canonical map $C \hookrightarrow C^{**}$. Dualising the resulting map $\Gamma^* \rightarrow C$, which is a coalgebra morphism, obtains the desired morphism $C^* \rightarrow \Gamma$.

For (2), first recall that if A is a dga, then \widehat{A} is the left adjoint of the \varprojlim functor; concretely \widehat{A} is the cofiltered set of Artinian quotients of A . If C is a coalgebra, the proof of (1) shows that the dual is an order-reversing bijection from the set of finite-dimensional subdgs of C to the set of Artinian quotients of C^* . Because the pro-Artinian dga C^\sharp is exactly the levelwise dual of the cofiltered system of finite-dimensional subdgs of C , the claim follows.

For (3) and (4), just observe that the proofs of (1) and (2) work just as well when connectivity conditions are imposed. \square

Recall that both $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ and $\mathbf{dga}_k^{\leq 0}$ are simplicial model categories; their simplicial structure comes from the monoidal Dold–Kan correspondence identifying connective k -dgas with simplicial k -algebras. It is easy to see that the limit functor is simplicial. Recall that for a simplicial category, we write Map for the simplicial mapping spaces.

Lemma 4.4.2. *Let C be a coconnective coaugmented conilpotent dgc and let \mathcal{B} be a pro-Artinian dga.*

(1) *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(C^\sharp, \mathcal{B}) \cong \mathrm{Hom}_{\mathbf{aug.dga}_k^{\leq 0}}(C^*, \varprojlim \mathcal{B}).$$

(2) *There is a natural isomorphism*

$$\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(C^\sharp, \mathcal{B}) \cong \mathrm{Map}_{\mathbf{aug.dga}_k^{\leq 0}}(C^*, \varprojlim \mathcal{B}).$$

Proof. For the first statement, we have a natural isomorphism

$$\mathrm{Hom}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(C^\sharp, \mathcal{B}) \cong \mathrm{Hom}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\widehat{C^*}, \mathcal{B}) \cong \mathrm{Hom}_{\mathbf{aug.dga}_k^{\leq 0}}(C^*, \varprojlim \mathcal{B})$$

where the first isomorphism is 4.4.1(2) and the second isomorphism is the adjunction $(-)\dashv\varprojlim$. The second statement follows from the first along with the fact that the above adjunction is simplicial. \square

Theorem 4.4.3. *Let \mathcal{A} and \mathcal{A}' be connective pro-Artinian dgas and put $A := \varprojlim \mathcal{A}$, $A' := \varprojlim \mathcal{A}'$. Suppose that the bar construction BA is cohomologically locally finite. Then there is a weak equivalence of derived mapping spaces*

$$\mathbb{R}\mathrm{Map}_{\mathrm{pro}(\mathrm{d}\mathbf{g}\mathbf{A}\mathbf{r}\mathbf{t}_k^{\leq 0})}(\mathcal{A}, \mathcal{A}') \simeq \mathbb{R}\mathrm{Map}_{\mathrm{aug.}\mathrm{d}\mathbf{g}\mathbf{a}_k^{\leq 0}}(A, A').$$

Proof. Because both model categories are simplicial, with all objects fibrant, we can compute derived mapping spaces as $\mathbb{R}\mathrm{Map}(X, Y) \simeq \mathrm{Map}(QX, Y)$ where QY denotes a cofibrant replacement of Y . Hence we can write

$$\begin{aligned} \mathbb{R}\mathrm{Map}_{\mathrm{pro}(\mathrm{d}\mathbf{g}\mathbf{A}\mathbf{r}\mathbf{t}_k^{\leq 0})}(\mathcal{A}, \mathcal{A}') &\simeq \mathrm{Map}_{\mathrm{pro}(\mathrm{d}\mathbf{g}\mathbf{A}\mathbf{r}\mathbf{t}_k^{\leq 0})}(B^\sharp(A^\dagger), \mathcal{A}') && \text{by 4.3.3(5)} \\ &\cong \mathrm{Map}_{\mathrm{aug.}\mathrm{d}\mathbf{g}\mathbf{a}_k^{\leq 0}}(A^{\dagger\dagger}, A') && \text{by 4.4.2(2)} \\ &\simeq \mathbb{R}\mathrm{Map}_{\mathrm{aug.}\mathrm{d}\mathbf{g}\mathbf{a}_k^{\leq 0}}(A, A') && \text{by 4.3.3(6)} \end{aligned}$$

as required. \square

Corollary 4.4.4. *Let A be a connective augmented cohomologically locally finite dga with $H^0(A)$ Artinian local. Let \mathcal{A}' be any connective pro-Artinian dga and put $A' := \varprojlim \mathcal{A}'$. Then there is a weak equivalence of derived mapping spaces*

$$\mathbb{R}\mathrm{Map}_{\mathrm{pro}(\mathrm{d}\mathbf{g}\mathbf{A}\mathbf{r}\mathbf{t}_k^{\leq 0})}(B^\sharp(A^\dagger), \mathcal{A}') \simeq \mathbb{R}\mathrm{Map}_{\mathrm{aug.}\mathrm{d}\mathbf{g}\mathbf{a}_k^{\leq 0}}(A, A').$$

Proof. For brevity put $\mathcal{A} := B^\sharp(A^\dagger)$. By 4.2.8, we have a natural quasi-isomorphism $\varprojlim \mathcal{A} \cong A^{\dagger\dagger} \simeq A$. By 4.3.1 and the invariance of B under quasi-isomorphisms, BA is cohomologically locally finite. Hence we may apply 4.4.3 to conclude that we have a weak equivalence

$$\mathbb{R}\mathrm{Map}_{\mathrm{pro}(\mathrm{d}\mathbf{g}\mathbf{A}\mathbf{r}\mathbf{t}_k^{\leq 0})}(\mathcal{A}, \mathcal{A}') \simeq \mathbb{R}\mathrm{Map}_{\mathrm{aug.}\mathrm{d}\mathbf{g}\mathbf{a}_k^{\leq 0}}(A^{\dagger\dagger}, A').$$

But because $\mathbb{R}\mathrm{Map}$ is invariant under quasi-isomorphisms, we may replace $A^{\dagger\dagger}$ by A in the right hand side, and this is precisely the claim. \square

5. DEFORMATIONS AND PROREPRESENTABILITY

In this section, we need to work over a field k of characteristic zero. In some sense, commutative formal deformation theory in this setting is about the Koszul duality between the commutative and Lie operads. Indeed, given a commutative deformation problem, one expects it to be ‘controlled’ in some way by a differential graded Lie algebra (**dgl** \mathbf{a}). This philosophy is originally due to Deligne, and first appears in print in a paper of Goldman and Millson [GM88]. Hinich [Hin01] viewed this philosophy through the lens of Koszul duality, via which coalgebras become important objects. The correspondence between commutative deformation problems and dglas was made precise by later work of Pridham [Pri10] and Lurie [Lur11]. Correspondingly, in view of the Koszul self-duality of the associative algebra operad, one should expect noncommutative deformation problems to be controlled by noncommutative dgas, and indeed this is true [Lur11, §3].

Our goal in this section is to use Koszul duality to explicitly identify prorepresenting objects for the ‘algebraic’ deformation problem of deforming a module over a dga. We will work with deformation functors valued in simplicial sets, and our representability statement

will really be a homotopy representability statement giving a weak equivalence between our functor and a derived mapping complex. We work with nonunital dgas throughout; this will help in the next section when we rigidify by considering framed deformations, which on the side of representing objects corresponds loosely to forgetting about the unit.

The input to our derived deformation functors will be connective Artinian dgas. If Γ is an Artinian local dga, denote its maximal ideal by \mathfrak{m}_Γ . If one wants to consider non-connective dgas as input then one needs to consider some sort of stacky deformations, because non-connective dgas may have nontrivial Maurer–Cartan elements (5.3.4).

We will mention dglas for motivational purposes only; these are dg vector spaces together with a graded Lie bracket satisfying the graded Leibniz identity with respect to the differential. For more about commutative deformation theory in characteristic zero via dglas, one should refer to the papers of Manetti [Man04, Man99].

We remark that many of the results of this chapter are true when k has positive characteristic: key here is that we are deforming over connective noncommutative dgas, which are Quillen equivalent to simplicial k -algebras. In positive characteristic the equivalence between connective cdgas and simplicial commutative algebras breaks down. We note that in positive characteristic, a version of the Lurie–Pridham correspondence for commutative formal moduli problems has recently been given by Brantner and Mathew [BM19]. The only place we really need k to be characteristic zero is in the polynomial Poincaré Lemma 5.1.13, which allows us to build simplicial resolutions of dgas via $\Omega(\Delta^\bullet)$, the dga of polynomial differential forms. One might be able to reproduce all of our results in arbitrary characteristic using a simplicial resolution for k instead of $\Omega(\Delta^\bullet)$.

5.1. The Maurer–Cartan and Deligne functors. It is known that deformations of modules are given by a certain functor called the **Deligne functor**, which is constructed as a homotopy quotient. Our representability statement will consist of showing that the Deligne functor is (homotopy) representable. In this section we construct the Deligne functor.

Definition 5.1.1. Let U be a (possibly nonunital) dga. The set of **Maurer–Cartan elements** (or just **MC elements**) of U is the set

$$\mathrm{MC}(U) := \{x \in U^1 : dx + x^2 = 0\}.$$

Remark 5.1.2. Note that if U is unital, with differential d , then $x \in \mathrm{MC}(U)$ if and only if the map $u \mapsto d(u) + xu$ is a differential on U .

Remark 5.1.3. A (possibly nonunital) dga U canonically becomes a dgla when equipped with the commutator bracket, and the set of MC elements of the dgla U is the same as the set of MC elements of the dga U .

Definition 5.1.4. Let E be a dga. The **Maurer–Cartan functor**

$$\mathcal{MC}(E) : \mathrm{dgArt}_k^{\leq 0} \rightarrow \mathbf{Set}$$

sends an Artinian dga Γ to the set $\mathcal{MC}(E)(\Gamma) := \mathrm{MC}(E \otimes \mathfrak{m}_\Gamma)$.

Definition 5.1.5. Let E be a dga. The **gauge group functor**

$$\mathcal{Gg}(E) : \mathrm{dgArt}_k^{\leq 0} \rightarrow \mathbf{Grp}$$

sends Γ to the set $1 + (E \otimes \mathfrak{m}_\Gamma)^0$, which is a group under multiplication.

Proposition 5.1.6. *Let E be a nonunital dga and Γ an Artinian local dga. Then $\mathcal{Gg}(E)(\Gamma)$ acts on $\mathcal{MC}(E)(\Gamma)$ via the formula $g.x = gxg^{-1} + gd(g^{-1})$.*

Proof. This is an easy verification. \square

Remark 5.1.7. Regarding $d + x$ as a twisted differential on $E \otimes \mathfrak{m}_\Gamma$, the action of the gauge group is the conjugation action on the space of differentials.

Remark 5.1.8. If L is a dgla, its gauge group has as elements formal symbols $\exp(a)$ for $a \in L^0$, and multiplication given by the Baker–Campbell–Hausdorff formula [Man99]. In characteristic zero, if $E \otimes \mathfrak{m}_\Gamma$ is made into a dgla using the commutator bracket then its dgla gauge group is isomorphic to the gauge group defined above, via the map that sends each formal exponential $\exp(a)$ to the sum $\sum_n \frac{a^n}{n!}$, which exists because \mathfrak{m}_Γ is nilpotent. Note that in positive characteristic the sum may not be defined. The exponential of the dgla gauge action [Man04, V.4] is the gauge action described above.

Definition 5.1.9. Let E be a dga. The **Deligne functor** is the quotient functor

$$\mathcal{D}el(E) := \mathcal{MC}(E)/\mathcal{G}g(E).$$

Often $\mathcal{D}el(E)$ is called the deformation functor associated to E . We will not provide an axiomatic treatment of (derived) deformation functors here; see e.g. [Pri10] in the commutative case.

Remark 5.1.10. By taking the groupoid quotient rather than the set quotient, one can immediately enhance $\mathcal{D}el$ to a groupoid-valued functor. However, we will see that $\mathcal{D}el$ has a natural enhancement to a functor $\underline{\mathcal{D}el}$ valued in simplicial sets, and we would like the groupoid $\mathcal{D}el$ to be the 1-truncation of $\underline{\mathcal{D}el}$. However, nontrivial 2-simplices in $\underline{\mathcal{D}el}$ induce homotopies between gauges, and one has to quotient these out to get the correct fundamental groupoid; see [ELO09, §5] or [Man99, Proof of 3.2] in the commutative setting. In the literature, both groupoid-valued functors are referred to as the **Deligne groupoid**. When deforming along ungraded Artinian algebras the two definitions coincide, so the difference between them only becomes apparent when deforming along genuinely derived objects.

Proposition 5.1.11 ([ELO09, 8.1]). *If E and E' are quasi-isomorphic dgas then the functors $\mathcal{D}el(E)$ and $\mathcal{D}el(E')$ are isomorphic.*

Now we enhance all of our constructions to functors valued in simplicial sets. The loose idea is to take (co)simplicial resolutions to get derived mapping spaces, which are simplicial sets. We use the explicit simplicial enhancement of Hinich [Hin97], who is generalising the work of Bousfield and Gugenheim [BG76].

Definition 5.1.12. Let $\Omega(\Delta^\bullet)$ denote the simplicial cdga of polynomial differential forms on the standard cosimplicial space Δ^\bullet ; in simplicial level n it is freely generated as a cdga by indeterminates t_0, \dots, t_n in degree zero, modulo the relation $t_0 + \dots + t_n = 1$. See e.g. [BG76, §1] or [Hin97, 4.8.1] for an explicit definition.

Proposition 5.1.13 (Polynomial Poincaré Lemma [BG76, 1.3]). *The simplicial dga $\Omega(\Delta^\bullet)$ is quasi-isomorphic to the constant simplicial dga k .*

Proof. Because we have an isomorphism $\Omega(\Delta^q) \cong \Omega(\Delta^1)^{\otimes q}$, it suffices to check that $\Omega(\Delta^1) \cong k[t, dt]$ is quasi-isomorphic to k . To do this, it is enough to show that the cohomology of $k[t, dt]$ is k , because then the unit map $k \rightarrow k[t, dt]$ is a quasi-isomorphism. \square

Remark 5.1.14. In characteristic p , the polynomial Poincaré Lemma does not hold. Indeed, one can check that $\Omega(\Delta^1) \cong k[t, dt]$ has cohomology $k[t^p]$, concentrated in degree zero:

indeed the degree i cocycles look like $Q(t^p)dt^i$, where Q is any polynomial. Since $tQ(t^p)$ maps to $Q(t^p)dt$ under the differential, if $i > 0$ then these cocycles are all coboundaries.

Definition 5.1.15. Let E be a dga. The **simplicial Maurer–Cartan functor** $\underline{\mathcal{M}C}$ sends E to the simplicial set $\underline{\mathcal{M}C}(E) := \mathcal{M}C(E \otimes \Omega(\Delta^\bullet))$.

Unwinding the definitions, we hence have $\underline{\mathcal{M}C}(E)(\Gamma) = \text{MC}(E \otimes \Omega(\Delta^\bullet) \otimes \mathfrak{m}_\Gamma)$.

Remark 5.1.16. It is not true that $\mathcal{M}C \cong \pi_0 \underline{\mathcal{M}C}$, because the right-hand side has elements identified by homotopies coming from 1-simplices in $\underline{\mathcal{M}C}$. All we have is a quotient map $\mathcal{M}C \rightarrow \pi_0 \underline{\mathcal{M}C}$. In fact, $\pi_0 \underline{\mathcal{M}C}$ is $\mathcal{D}el$, which follows by combining 5.1.19 and 5.1.20 below.

Proposition 5.1.17 ([Pri15, 2.18]). *If E and E' are quasi-isomorphic dgas then the functors $\mathcal{M}C(E)$ and $\underline{\mathcal{M}C}(E')$ are weakly equivalent.*

Definition 5.1.18. The **simplicial Deligne functor** is the homotopy quotient²

$$\mathcal{D}el(E) := [\underline{\mathcal{M}C}(E)/\mathcal{G}g(E)].$$

Lemma 5.1.19. *There is an isomorphism $\mathcal{D}el \cong \pi_0 \underline{\mathcal{D}el}$.*

Proof. As in [Pri15, 1.27], this follows by considering the long exact sequence of homotopy groups associated to the fibration $X \rightarrow [X/G] \rightarrow BG$. \square

Proposition 5.1.20 ([Pri15, 2.21]). *The quotient map $\underline{\mathcal{M}C} \rightarrow \underline{\mathcal{D}el}$ is a weak equivalence.*

Proof. The idea is that the classifying space of the gauge group is contractible and so taking the homotopy quotient does not affect the weak equivalence type of $\underline{\mathcal{M}C}$. \square

5.2. DG and simplicial categories. We provide a very brief introduction to the theory of dg categories, simplicially enriched categories, and the interplay between them, primarily to set notation. Survey articles on dg categories include [Kel06] and [Toë11]. For the relevant material on simplicial categories, see [Ber07].

Definition 5.2.1. A (k -linear) **dg category** is a category \mathcal{C} enriched over the monoidal category $(\mathbf{dgvect}_k, \otimes)$ of dg vector spaces with the usual tensor product. In other words, to every pair of elements $(x, y) \in \mathcal{C}^2$ we assign a chain complex $\underline{\text{Hom}}_{\mathcal{C}}(x, y)$, to every triple (x, y, z) we assign a chain map $\mu_{xyz} : \underline{\text{Hom}}_{\mathcal{C}}(x, y) \otimes \underline{\text{Hom}}_{\mathcal{C}}(y, z) \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(x, z)$ satisfying associativity, and for every $x \in \mathcal{C}$ we assign a map $\eta_x : k \rightarrow \underline{\text{Hom}}_{\mathcal{C}}(x, x)$ which is a unit with respect to composition.

Definition 5.2.2. A **dg functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ between two dg categories is an enriched functor; i.e. a map of objects $\mathcal{C} \rightarrow \mathcal{D}$ together with, for every pair $(x, y) \in \mathcal{C}^2$, a map of complexes $F_{xy} : \underline{\text{Hom}}_{\mathcal{C}}(x, y) \rightarrow \underline{\text{Hom}}_{\mathcal{D}}(Fx, Fy)$. We require that F satisfies the associativity condition $\mu_{F_x F_y F_z} \circ (F_{xy} \otimes F_{yz}) = F_{xz} \circ \mu_{xyz}$ and the unitality condition $F_{xx} \circ \eta_x = \eta_{F_x}$.

Definition 5.2.3. Let \mathcal{C} be a dg category. The **homotopy category** of \mathcal{C} is the k -linear category $[\mathcal{C}]$ whose objects are the same as \mathcal{C} and whose hom-spaces are given by $\text{Hom}_{[\mathcal{C}]}(x, y) := H^0(\underline{\text{Hom}}_{\mathcal{C}}(x, y))$. Composition is inherited from \mathcal{C} . We sometimes write $[x, y] := \text{Hom}_{[\mathcal{C}]}(x, y)$.

Definition 5.2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a dg functor.

- F is **quasi-fully faithful** if all of its components F_{xy} are quasi-isomorphisms.

²See [GJ09, Chapter V] or [Pri15, 1.23] for the definition of homotopy quotients.

- F is **quasi-essentially surjective** if the induced functor $[F] : [\mathcal{C}] \rightarrow [\mathcal{D}]$ is essentially surjective.
- F is a **quasi-equivalence** if it is quasi-fully faithful and quasi-essentially surjective.

Definition 5.2.5. If A is a dga, then $D_{\text{dg}}(A)$ denotes the dg category of cofibrant dg modules over A .

The dg category $D_{\text{dg}}(A)$ is **pretriangulated**, meaning essentially that its homotopy category is canonically triangulated. It enhances the usual derived category, in the sense that one has an equivalence of triangulated categories $[D_{\text{dg}}(A)] \cong D(A)$. Note that $\underline{\text{Hom}}$ is a model for the derived hom $\mathbb{R}\text{Hom}$. The category of all dg categories admits a model structure whose weak equivalences are the quasi-equivalences:

Theorem 5.2.6 (Tabuada [Tab05]). *The category of all small dg categories admits a (cofibrantly generated) model structure where the weak equivalences are the quasi-equivalences. The fibrations are the objectwise levelwise surjections that lift isomorphisms. Every dg category is fibrant.*

Now we turn to simplicial categories.

Definition 5.2.7. A **simplicial category** is a category enriched in \mathbf{sSet} . A functor between simplicial categories is an enriched functor. The category of all simplicial categories is denoted $\mathbf{sSetCat}$.

Remark 5.2.8. Note that every simplicial category is a simplicial object in \mathbf{Cat} , but not every simplicial object in \mathbf{Cat} is a simplicial category.

Definition 5.2.9. Let \mathcal{D} be a simplicial category. The **homotopy category** of \mathcal{D} is the category whose objects are the same as \mathcal{D} and whose homsets are given by taking taking π_0 of the morphism complexes in \mathcal{D} . Composition is inherited from \mathcal{D} .

Bergner proved that $\mathbf{sSetCat}$ admits a model structure analogous to Tabuada's (equivalently, a many-object version of the classical model structure on simplicial sets):

Theorem 5.2.10 (Bergner [Ber07]). *The category $\mathbf{sSetCat}$ admits a model structure where the weak equivalences are the DK^3 -equivalences: those functors which induce weak equivalences on derived mapping spaces and which induce isomorphisms on π_0 . The fibrant simplicial categories are precisely those enriched in Kan complexes.*

If \mathcal{A} is an abelian category, the Dold–Kan correspondence [GJ09, III.2] gives a Quillen equivalence between simplicial objects in \mathcal{A} and connective complexes in \mathcal{A} . We describe a many-object version due to Tabuada:

Definition 5.2.11. Let \mathcal{C} be a dg category. Let $\mathcal{C}_{\leq 0}$ denote the associated dg category obtained by taking the good truncation to nonpositive degrees of the morphism complexes. Applying the Dold–Kan correspondence to the connective morphism complexes of $\mathcal{C}_{\leq 0}$ gives a category enriched in simplicial vector spaces⁴, and we may forget the linear structure to obtain a simplicial category we denote by \mathcal{C}_s . Similarly, given a simplicial category \mathcal{D} we may linearise the simplicial homsets to obtain a category enriched in simplicial vector spaces, and the Dold–Kan correspondence gives us a connective dg category⁵, which we denote by \mathcal{D}_{dg} .

³Dwyer–Kan.

⁴Composition is given by the Alexander–Whitney map.

⁵Composition is now given by the Eilenberg–Zilber map.

Theorem 5.2.12 (Tabuada [Tab10a]). *The functors $\mathcal{C} \mapsto \mathcal{C}_s$ and $\mathcal{D} \mapsto \mathcal{D}_{\text{dg}}$ give a Quillen adjunction $\mathbf{dgCat} \leftarrow \mathbf{sSetCat}$, where the left hand side has the Tabuada model structure and the right hand side has the Bergner model structure.*

Remark 5.2.13. In fact, the above Quillen adjunction restricts to a Quillen equivalence between connective dg categories and categories enriched in simplicial vector spaces, when both are equipped with modifications of the above model structures. In the nonconnective world, one can view a dg vector space as a spectrum object in connective vector spaces; applying Dold–Kan levelwise one then gets a spectrum object in simplicial k -modules, or (Quillen) equivalently an Hk -module spectrum. This gives a Quillen equivalence between dg categories and Hk -module spectrally enriched categories [Tab10b].

We will be interested only in whether two deformations are (quasi)-isomorphic, and not in the whole complex of morphisms between them. Hence, in our applications, it will suffice to work with simplicial (or dg) groupoids. We finish by describing a core functor for simplicial categories, as well as an enriched nerve construction for turning simplicial categories into simplicial sets.

Definition 5.2.14. If \mathcal{D} is a simplicial category, let $c(\mathcal{D})$ denote the subcategory on those morphisms which induce isomorphisms on $\pi_0\mathcal{D}$; in [Ber07] these are called homotopy equivalences.

We think of c as a sort of core functor. Note that $c(\mathcal{D})$ is DK-equivalent to a simplicial groupoid in the sense of [GJ09, V.7]. For the next construction, observe that the nerve of a simplicial category is naturally a bisimplicial set, with the enrichment in one direction and the nerve in the other. One can totalise this bisimplicial set to obtain a simplicial set, and doing this in the homotopically correct manner gives us a homotopy coherent nerve for simplicial categories.

Definition 5.2.15. Let $\bar{W} : \mathbf{ssSet} \rightarrow \mathbf{sSet}$ denote the right adjoint of Illusie’s décalage functor; see [Pri12a, 1.6] for a concrete definition. Given a simplicial category \mathcal{D} let $\bar{W}\mathcal{D}$ denote \bar{W} applied to the nerve of \mathcal{D} .

Remark 5.2.16. In [CR05] it is proved that for a bisimplicial set X , the canonical morphism $\text{diag}X \rightarrow \bar{W}X$ is a weak equivalence; see [Pri12a, 1.7] for further discussion. If \mathcal{G} is a simplicial groupoid, it follows that $\bar{W}\mathcal{G}$ is weakly equivalent to the homotopy coherent nerve $N(\mathcal{G})$ [Hin15], which is a quasicategory in the sense of [Lur09]. See [Lur09, §1.1.5] for the details of the homotopy coherent nerve; we note that $N(\mathcal{C}_s)$ is weakly equivalent to the dg nerve of \mathcal{C} [Lur17, §1.3.1]. We use $\bar{W}\mathcal{G}$ instead of $N(\mathcal{G})$ for consistency with [Pri12a].

Definition 5.2.17. Let \mathcal{C} be a dg category. Write $\mathcal{W}(\mathcal{C}) := \bar{W}(c(\mathcal{C}_s))$.

In other words, $\mathcal{W}(\mathcal{C})$ is a model for the homotopy coherent nerve of the underlying simplicial groupoid of \mathcal{C}_s .

Remark 5.2.18. To define $\mathcal{W}(\mathcal{C})$, we could have instead taken the ‘dg core’ of \mathcal{C} , converted to a simplicial category, and applied \bar{W} to obtain something weakly equivalent.

Remark 5.2.19. The functor \mathcal{W} is a right derived functor. Indeed, because every dg category is fibrant, $\mathcal{C} \mapsto \mathcal{C}_s$ is its own right derived functor. Moreover, the composition $\bar{W}c$ is a right derived functor, because it is weakly equivalent to the derived mapping space functor $\mathbb{R}\text{Map}(\bullet, -)$ from the initial simplicial category. Hence, \mathcal{W} is a composition of right derived functors and so is itself a right derived functor.

5.3. Deformations of modules. Now we may define our deformation functors. Recall that a classical deformation of an A -module X over an Artinian local ring Γ is an $A \otimes \Gamma$ -module \tilde{X} , flat over Γ , that reduces to X modulo \mathfrak{m}_Γ . A derived deformation is defined similarly:

Definition 5.3.1. Let A be a dga and X an A -module. Let Γ be an Artinian local dga. A **derived deformation** of X over Γ is a pair (\tilde{X}, f) where \tilde{X} is an $A \otimes \Gamma$ -module and $f : \tilde{X} \otimes_\Gamma^{\mathbb{L}} k \rightarrow X$ is an isomorphism in $D(A)$. An **isomorphism** of derived deformations is an isomorphism $\phi : \tilde{X}_1 \rightarrow \tilde{X}_2$ in $D(A \otimes \Gamma)$ such that $f_1 = f_2 \circ (\phi \otimes_\Gamma^{\mathbb{L}} k)$.

Deformations are functorial with respect to algebra maps: given a map $\Gamma \rightarrow \Gamma'$ of Artinian local dgas, and a derived deformation \tilde{X} of X over Γ , then the derived base change $\tilde{X} \otimes_\Gamma^{\mathbb{L}} \Gamma'$ is a derived deformation of X over Γ' .

Definition 5.3.2. Let A be a dga and X an A -module. The functor $\mathcal{D}\text{ef}_A(X) : \mathbf{dgArt}_k^{\leq 0} \rightarrow \mathbf{Set}$ sends an Artinian local dga Γ to the set

$$\mathcal{D}\text{ef}_A(X)(\Gamma) := \frac{\{\text{derived deformations of } X \text{ over } \Gamma\}}{(\text{isomorphisms})}.$$

We will just write $\mathcal{D}\text{ef}(X)$ if there is no ambiguity. The functor of derived deformations is well-known to be a Deligne functor:

Theorem 5.3.3. *Let A be a dga and X an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then the functors $\mathcal{D}\text{ef}_A(X)$ and $\mathcal{D}\text{el}(E)$ are isomorphic.*

Proof. This appears in Efimov–Lunts–Orlov [ELO09, 6.1]; in fact they consider the natural enhancement of $\mathcal{D}\text{ef}_A(X)$ to a groupoid-valued functor and show that it is equivalent to the (non-naïve) Deligne groupoid of E ; our desired statement follows by taking π_0 . The basic idea of the proof is that if X is cofibrant then a deformation of X is a deformation of the differential on $X \otimes_k \Gamma$, which is exactly a Maurer–Cartan element of $E \otimes \mathfrak{m}_\Gamma$. Quasi-isomorphisms between cofibrant deformations are homotopies, and the homotopy class of a deformation is its orbit under the gauge action. The result follows. \square

We note that the statement of 5.3.3 makes sense because $\mathcal{D}\text{el}$ is invariant under quasi-isomorphisms by 5.1.11.

Remark 5.3.4. Take $A = X = k$, so that the derived endomorphism algebra of X is k . If Γ is any Artinian dga (not necessarily connective), it is easy to see that $\mathcal{D}\text{el}(k)(\Gamma)$ is $\text{MC}(\mathfrak{m}_\Gamma)/(1 + \mathfrak{m}_\Gamma^0)$. In particular, putting $\Gamma = \frac{k[\epsilon]}{\epsilon^2}$ with ϵ in degree 1, we have $\mathcal{D}\text{el}(k)(\Gamma) \cong k$. Hence if 5.3.3 were to remain true when we take unbounded Artinian dgas as input, we would require the k -vector space k to have nontrivial deformations over Γ . Intuitively these should be some sort of stacky deformations: loosely, the connective part of Γ records information about the derived geometry of Γ , whilst the coconnective part of Γ records stacky information.

Now we may define our \mathbf{sSet} -valued deformation functors.

Definition 5.3.5. Let A be a dga and let X be an A -module. Let X_{dg} denote the dga $\mathbb{R}\text{End}_A(X)$ considered as a one-object dg category. There is an obvious inclusion dg functor $X_{\text{dg}} \hookrightarrow D_{\text{dg}}(A)$ whose image is X . If $\Gamma \in \mathbf{dgArt}_k^{\leq 0}$ then there is a ‘reduce modulo \mathfrak{m}_Γ ’ dg

functor $D_{\text{dg}}(A \otimes \Gamma) \rightarrow D_{\text{dg}}(A)$ which sends an $A \otimes \Gamma$ -module M to the A -module $M \otimes_{\Gamma}^{\mathbb{L}} k$. Let $\text{dgDef}_A(X)(\Gamma)$ be the homotopy fibre product of dg categories

$$\text{dgDef}(X)(\Gamma) := X_{\text{dg}} \times_{D_{\text{dg}}(A)}^h D_{\text{dg}}(A \otimes \Gamma).$$

Let $\underline{\mathcal{D}ef}_A(X)(\Gamma)$ denote the simplicial set $\mathcal{W}(\text{dgDef}_A(X)(\Gamma))$.

It is easy to check that $\underline{\mathcal{D}ef}_A(X)$ is a functor from $\mathbf{dgArt}_k^{\leq 0}$ to \mathbf{sSet} . Observe that the dg category $\text{dgDef}(X)$ is clearly ‘too big’, since it contains many maps which are not isomorphisms after passing to the homotopy category. This is why we take the core.

Remark 5.3.6. If A and A' are dgas with a fully faithful dg functor $D_{\text{dg}}(A) \hookrightarrow D_{\text{dg}}(A')$, then we obtain a weak equivalence $\underline{\mathcal{D}ef}_A(X) \xrightarrow{\simeq} \underline{\mathcal{D}ef}_{A'}(X)$.

$\underline{\mathcal{D}ef}$ enhances $\mathcal{D}ef$:

Proposition 5.3.7. *There is an isomorphism of functors $\mathcal{D}ef \cong \pi_0 \underline{\mathcal{D}ef}$.*

Proof. This follows by combining 5.3.8 below with 5.1.19 and 5.3.3 to see that

$$\pi_0 \underline{\mathcal{D}ef} \cong \pi_0 \underline{\mathcal{D}el} \cong \mathcal{D}el \cong \mathcal{D}ef.$$

A more direct proof goes as follows: chasing the definitions, we see that $\pi_0(\underline{\mathcal{D}ef}_A(X)(\Gamma))$ agrees with the set of isomorphism classes in the homotopy category $[\text{dgDef}_A(X)(\Gamma)]$. But the objects of $[\text{dgDef}_A(X)(\Gamma)]$ are precisely the derived deformations of X over Γ , and two objects are isomorphic in $[\text{dgDef}_A(X)(\Gamma)]$ if and only if the corresponding derived deformations are isomorphic. \square

The enhanced functor of derived deformations is also known to be a Deligne functor:

Theorem 5.3.8. *Let A be a dga and X an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . Then there is a weak equivalence of \mathbf{sSet} -valued functors $\underline{\mathcal{D}ef}_A(X) \xrightarrow{\simeq} \underline{\mathcal{D}el}(E)$.*

Proof. This appears as e.g. [DN15, 4.2.6] or [Pri15, 4.6]. The basic idea of the proof is the same as that of the set-valued case 5.3.3; we give a sketch. The proof of [Pri15, 3.13] shows that the simplicial groupoid of deformations of X is the same as the simplicial subgroupoid of $\underline{\mathcal{D}el}(E)$ on constant objects, which we will denote \mathcal{C} . By [Pri15, §4.1], \mathcal{C} and $\underline{\mathcal{D}el}(E)$ are both derived deformation functors. The inclusion induces a map on tangent spaces which is a weak equivalence and so the two functors are weakly equivalent by an obstruction-theoretic argument. \square

We note that the statement of 5.3.8 makes sense because $\mathcal{D}el$ is invariant under quasi-isomorphisms by 5.1.17 and 5.1.20.

Remark 5.3.9. Taking π_0 of 5.3.8 recovers 5.3.3, and taking fundamental groupoids recovers the groupoid-valued version appearing in [ELO09].

Remark 5.3.10. As in 5.3.4, to extend 5.3.8 to non-connective dgas would require the incorporation of some sort of stacky deformations.

Remark 5.3.11. Abstractly, homotopy representability for deformation functors follows from a generalised Brown representability theorem [Jar11]; this perspective appears in [GLST20]. Indeed, in the commutative setting Pridham’s deformation functors [Pri10] are precisely the homotopy prorepresentable functors.

Corollary 5.3.12. *Let A be a dga and X an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . Then the \mathbf{sSet} -valued functors $\underline{\mathcal{D}}\text{ef}_A(X)$ and $\underline{\mathcal{M}}\mathcal{C}(E)$ are weakly equivalent.*

Proof. Combine 5.3.8 with 5.1.20. □

5.4. Prorepresentability. We prove a prorepresentability statement for set-valued functors, and then we enhance this to \mathbf{sSet} -valued functors. Essentially everything we use here (at least for set-valued functors) can be found in Loday–Vallette [LV12, Chapter 2] or Positselski [Pos11]. We will need to use nonunital dgas in order to get the correct prorepresentability statements; we will later see that the use of nonunital dgas can be avoided if one rigidifies to consider framed deformations.

Definition 5.4.1 (see e.g. [Pos11, 6.2]). Let E be a nonunital dga and let C be a nonunital dgc. Then the complex $\text{Hom}_k(C, E)$ of k -vector spaces is a nonunital dga under the product given by $fg := \mu_E \circ (f \otimes g) \circ \Delta_C$. This dga is the **convolution algebra**. A Maurer–Cartan element of the convolution algebra is a **nonunital twisting morphism**; the set of all nonunital twisting morphisms is denoted $\text{Tw}(C, E)$.

Remark 5.4.2. In the (co)augmented setting, one should add the condition that twisting morphisms are zero when composed with the augmentation or coaugmentation.

Lemma 5.4.3. *Let E, Z be nonunital dgas, with Z finite-dimensional. Then there is a natural isomorphism*

$$\text{Tw}(Z^*, E) \cong \text{MC}(E \otimes Z).$$

Proof. There is a standard linear isomorphism $E \otimes Z \rightarrow \text{Hom}_k(Z^*, E)$, and one can check that this is a map of nonunital dgas after equipping $\text{Hom}_k(Z^*, E)$ with the convolution product. Hence the MC elements of both sides agree. □

Definition 5.4.4. Let E be a nonunital dga. The **nonunital bar construction** is the (coaugmented) dgc

$$B_{\text{nu}}(E) := B(E \oplus k).$$

We caution that if E is an augmented dga, then $B_{\text{nu}}(E)$ does not agree with $B(E)$ as $B_{\text{nu}}(E)$ will contain elements corresponding to the unit of E .

Definition 5.4.5. Let C be a noncounital conilpotent dgc. The **nonunital cobar construction** is the (augmented) dga

$$\Omega_{\text{nu}}(C) := \Omega(C \oplus k).$$

The functor of twisting morphisms is (up to units) representable on either side:

Theorem 5.4.6 ([LV12, 2.2.6]). *If E is a nonunital dga and C is a noncounital conilpotent dgc, then there are natural isomorphisms*

$$\text{Hom}_{\mathbf{aug.dga}_k}(\Omega_{\text{nu}}C, E \oplus k) \cong \text{Tw}(C, E) \cong \text{Hom}_{\mathbf{con.dgc}_k}(C \oplus k, B_{\text{nu}}E).$$

We recall from 3.2.10 that if C is a (counital) conilpotent dgc then C^\sharp denotes the pro-Artinian dga constructed by levelwise dualising the filtered system of finite sub-dgcs of C . If E is a nonunital dga, write $B_{\text{nu}}^\sharp E := (B_{\text{nu}}E)^\sharp$ for the **continuous nonunital Koszul dual**.

Proposition 5.4.7. *Let E be a nonunital dga. Then the functor $\mathcal{MC}(E)$ is prorepresented by $B_{\text{nu}}^\sharp E$, in the sense that $\mathcal{MC}(E)$ and $\text{Hom}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, -)$ are naturally isomorphic.*

Proof. Let Γ be a connective Artinian local dga. It is easy to see that Γ^* is a conilpotent dgc. By 3.2.9, we have an isomorphism

$$\text{Hom}_{\text{con.dgc}_k}(\Gamma^*, B_{\text{nu}}E) \cong \text{Hom}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, \Gamma^{*\sharp}).$$

Because Γ^* is Artinian we have an isomorphism $\Gamma^{*\sharp} \cong \Gamma$ and it follows that we have isomorphisms

$$\text{Hom}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, \Gamma) \cong \text{Hom}_{\text{con.dgc}_k}(\Gamma^*, B_{\text{nu}}E) \cong \text{Tw}(\mathfrak{m}_\Gamma^*, E)$$

where the second isomorphism is 5.4.6. By 5.4.3 we have an isomorphism

$$\text{Tw}(\mathfrak{m}_\Gamma^*, E) \cong \text{MC}(E \otimes \mathfrak{m}_\Gamma)$$

and so we are done. \square

Now that we have our prorepresentability result, we will enhance it to **sSet**-valued functors. This will not be too hard; we just need to identify the correct simplicial mapping spaces in $\text{pro}(\text{dgArt}_k)$. Note that if Γ is an Artinian dga, then $\Gamma \otimes \Omega(\Delta^\bullet)$ will not be a simplicial Artinian dga, so the answer is not as simple as ‘replace Γ by the simplicial resolution $\Gamma \otimes \Omega(\Delta^\bullet)$ ’. We get around this by using the Quillen adjunction $\Omega \dashv B$.

Theorem 5.4.8. *Let E be a nonunital dga. Then there is a weak equivalence of functors*

$$\underline{\mathcal{MC}}(E) \simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, -).$$

Proof. If E is a nonunital dga write E^\bullet for the simplicial nonunital dga $E^\bullet := E \otimes \Omega(\Delta^\bullet)$. By definition, we have

$$\underline{\mathcal{MC}}(E)(\Gamma) := \text{MC}(E^\bullet \otimes \mathfrak{m}_\Gamma).$$

Applying 5.4.6 levelwise we see that we have an isomorphism

$$\text{MC}(E^\bullet \otimes \mathfrak{m}_\Gamma) \cong \text{Hom}_{\text{aug.dga}_k}(\Omega_{\text{nu}}(\mathfrak{m}_\Gamma^*), E^\bullet \oplus \text{const}(k)) \cong \text{Hom}_{\text{aug.dga}_k}(\Omega(\Gamma^*), E^\bullet \oplus \text{const}(k))$$

where $\text{const}(k)$ denotes the constant simplicial dga on k . But because $\text{const}(k) \rightarrow \Omega(\Delta^\bullet)$ is a levelwise quasi-isomorphism by 5.1.13, we have a quasi-isomorphism of simplicial dgas $E^\bullet \oplus \text{const}(k) \simeq (E \oplus k)^\bullet$. Because $\Omega(\Gamma^*)$ is cofibrant and all dgas are fibrant, it follows that we have a weak equivalence

$$\text{Hom}_{\text{aug.dga}_k}(\Omega(\Gamma^*), E^\bullet \oplus \text{const}(k)) \simeq \mathbb{R}\text{Map}_{\text{aug.dga}_k}(\Omega(\Gamma^*), E \oplus k).$$

Now because Ω is part of a Quillen equivalence by 3.1.14 we have

$$\mathbb{R}\text{Map}_{\text{aug.dga}_k}(\Omega(\Gamma^*), E \oplus k) \simeq \mathbb{R}\text{Map}_{\text{con.dgc}_k}(\Gamma^*, B_{\text{nu}}E)$$

and by 3.2.12 we have

$$\mathbb{R}\text{Map}_{\text{con.dgc}_k}(\Gamma^*, B_{\text{nu}}E) \simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, \Gamma^{*\sharp}).$$

As before we have $\Gamma^{*\sharp} \cong \Gamma$ and hence we are done. \square

We obtain our desired representability theorem:

Theorem 5.4.9 (Prorepresentability). *Let A be a dga and X an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then the **sSet**-valued functors $\underline{\mathcal{D}\text{ef}}_A(X)$ and $\mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, -)$ are weakly equivalent.*

Proof. Combine 5.3.12 with 5.4.8. \square

Remark 5.4.10. Yekutieli obtains commutative analogues of several of our results in [Yek12]. More specifically, he works with pronilpotent dg Lie algebras and deforms over complete local commutative noetherian algebras (with maximal ideal \mathfrak{m}). These methods ought to adapt to our setting to give more direct proofs of some of our statements about set-valued functors and classical groupoid-valued functors. More specifically, [Yek12] shows that a quasi-isomorphism between pronilpotent dglas of the form $\mathfrak{m}\widehat{\otimes}\mathfrak{g} \rightarrow \mathfrak{m}\widehat{\otimes}\mathfrak{h}$ induces an isomorphism $\mathrm{MC}(\mathfrak{m}\widehat{\otimes}\mathfrak{g}) \rightarrow \mathrm{MC}(\mathfrak{m}\widehat{\otimes}\mathfrak{h})$, as well as an equivalence of (naïve) Deligne groupoids. Presumably this is a set-level manifestation of the existence of a model structure on the category of pronilpotent dglas, for which the levelwise Chevalley–Eilenberg functor is a Quillen equivalence to the category of commutative dgs. One would have to repeat our arguments in the commutative/Lie setting; the goal would be to obtain a weak equivalence roughly of the form $\mathcal{MC}(\mathfrak{g}) \simeq \mathbb{R}\mathrm{Map}(\mathrm{CE}^\sharp(\mathfrak{g}), -)$.

5.5. Inclusion-truncation adjunctions. We have just seen that $\mathcal{D}\mathrm{ef}_A(X)$ is (homotopy) prorepresented by the possibly unbounded pro-Artinian dga $B_{\mathrm{nu}}^\sharp E$. But because we only allow connective Artinian dgas as input, one can use the inclusion-truncation adjunction to deduce a sharper result:

Proposition 5.5.1. *Let A be a dga and X an A -module. Let $E := \mathbb{R}\mathrm{End}_A(X)$ be the derived endomorphism dga of X . Then there is a weak equivalence of \mathbf{sSet} -valued functors*

$$\mathcal{D}\mathrm{ef}_A(X) \simeq \mathbb{R}\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\tau_{\leq 0}B_{\mathrm{nu}}^\sharp E, -).$$

Proof. By 3.3.4, the inclusion functor $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0}) \hookrightarrow \mathbf{pro}(\mathbf{dgArt}_k)$ is right Quillen; hence its left adjoint, the (levelwise) truncation functor, is left Quillen. Because every object in $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ is fibrant, the inclusion functor is its own right derived functor. Hence for $\Gamma \in \mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$ we have weak equivalences

$$\mathbb{R}\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\mathrm{nu}}^\sharp E, \Gamma) \simeq \mathbb{R}\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\mathbb{L}\tau_{\leq 0}B_{\mathrm{nu}}^\sharp E, \Gamma).$$

By 4.3.3(1), we may compute left derived functors as

$$\mathbb{L}\tau_{\leq 0}B_{\mathrm{nu}}^\sharp E \simeq \tau_{\leq 0}B^\sharp \Omega(B_{\mathrm{nu}}^{\circ} E).$$

But \sharp and \circ are inverses, so we have a quasi-isomorphism of dgas

$$\Omega(B_{\mathrm{nu}}^{\circ} E) \simeq \Omega B_{\mathrm{nu}}(E) = \Omega B(E \oplus k) \simeq E \oplus k$$

by the definition of the nonunital bar construction. Because B^\sharp sends quasi-isomorphisms to weak equivalences, we see that

$$\mathbb{L}\tau_{\leq 0}B_{\mathrm{nu}}^\sharp E \simeq \tau_{\leq 0}B_{\mathrm{nu}}^\sharp E$$

and so we have a weak equivalence

$$\mathbb{R}\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\mathrm{nu}}^\sharp E, \Gamma) \simeq \mathbb{R}\mathrm{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\tau_{\leq 0}B_{\mathrm{nu}}^\sharp E, \Gamma).$$

Now the result follows from 5.4.9. \square

Similarly, one may restrict the input to ungraded Artinian algebras. We can prove more here, but this will require more work. We begin with two preliminary lemmas.

Lemma 5.5.2. *Let $\mathbf{pro}(\mathbf{Art}_k)$ be the category of pro-objects in ungraded Artinian algebras, and give it the trivial model structure. Then there is a Quillen adjunction*

$$H^0 : \mathbf{pro}(\mathbf{dgArt}_k^{\leq 0}) \leftarrow \mathbf{pro}(\mathbf{Art}_k) : \iota$$

where the inclusion functor ι is the right adjoint.

Proof. First we verify that the claimed adjunction is actually an adjunction. The point is that if $A \rightarrow A'$ is a map of Artinian dgas, where A is connective and A' is ungraded, then it must factor through $H^0(A)$. Hence we get an adjunction on the level of Artinian dgas, which prolongs to an adjunction between procategories. To complete the proof we just need to verify that H^0 is left Quillen. It obviously preserves cofibrations. It preserves weak equivalences, because a weak equivalence between connective Artinian dgas gives an isomorphism (of ungraded pro-Artinian algebras) on H^0 by the definition of the Pridham model structure 3.3.1. \square

Lemma 5.5.3. *Let E be a nonunital dga. There is an isomorphism of pro-Artinian dgas*

$$(H^0 B_{\text{nu}} E)^\sharp \cong H^0(B_{\text{nu}}^\sharp E).$$

Proof. Let C be the coconnective dgc $C := \tau_{\geq 0} B_{\text{nu}} E$; observe that $H^0 C$ is a subcoalgebra of C . The subdgc's of C that are concentrated in degree zero are precisely the subcoalgebras of $H^0 C$. Hence it follows that $\tau_{\geq 0}(C^\sharp) \cong (H^0 C)^\sharp$. But $\tau_{\geq 0}(C^\sharp) = \tau_{\geq 0}(\tau_{\geq 0} B_{\text{nu}} E)^\sharp \cong \tau_{\geq 0} \tau_{\leq 0}(B_{\text{nu}}^\sharp E) \cong H^0(B_{\text{nu}}^\sharp E)$. Hence we have $H^0(B_{\text{nu}} E)^\sharp \cong H^0(B_{\text{nu}}^\sharp E)$ as required. \square

Proposition 5.5.4. *Let E be a nonunital dga and let Γ be an ungraded Artinian local algebra. Then there is an isomorphism*

$$\text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma) \cong \text{Hom}_{\mathbf{aug. alg}_k}(H^0(B_{\text{nu}}^* E), \Gamma).$$

Proof. First apply 5.5.3 to get an isomorphism

$$\text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma) \cong \text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}((H^0 B_{\text{nu}} E)^\sharp, \Gamma)$$

and then apply 4.4.1(1) to get an isomorphism

$$\text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}((H^0 B_{\text{nu}} E)^\sharp, \Gamma) \cong \text{Hom}_{\mathbf{aug. alg}_k}((H^0 B_{\text{nu}} E)^*, \Gamma).$$

The linear dual is exact, so we have $(H^0 B_{\text{nu}} E)^* \cong H^0(B_{\text{nu}}^* E)$ and the result follows. \square

Theorem 5.5.5 (Representability for ungraded algebras). *Let A be a dga and X an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then the functors $\mathcal{D}\text{ef}_A(X)|_{\mathbf{Art}_k}$ and $\text{Hom}_{\mathbf{aug. alg}_k}(H^0(B_{\text{nu}}^* E), -)$ are isomorphic.*

Proof. Let Γ be an ungraded Artinian algebra. Then 5.5.1 gives a weak equivalence

$$\mathcal{D}\text{ef}_A(X)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\tau_{\leq 0} B_{\text{nu}}^\sharp E, \Gamma).$$

Applying the Quillen adjunction of 5.5.2, and noting that H^0 is its own left derived functor by definition of the Pridham model structure 3.3.1, we get weak equivalences

$$\mathcal{D}\text{ef}_A(X)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})}(\tau_{\leq 0} B_{\text{nu}}^\sharp E, \Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma).$$

Taking π_0 of this weak equivalence gets us isomorphisms

$$\pi_0 \mathcal{D}\text{ef}_A(X)(\Gamma) \cong \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{Art}_k))}(H^0(B_{\text{nu}}^\sharp E), \Gamma) \cong \text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma).$$

By 5.3.7, we have an isomorphism $\pi_0 \mathcal{D}ef_A(X)(\Gamma) \cong \mathcal{D}ef_A(X)(\Gamma)$, and by 5.5.4, we have an isomorphism $\text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma) \cong \text{Hom}(H^0(B_{\text{nu}}^* E), \Gamma)$. Putting it all together, we get isomorphisms

$$\mathcal{D}ef_A(X)(\Gamma) \cong \text{Hom}_{\mathbf{pro}(\mathbf{Art}_k)}(H^0(B_{\text{nu}}^\sharp E), \Gamma) \cong \text{Hom}_{\mathbf{aug. alg}_k}(H^0(B_{\text{nu}}^* E), \Gamma)$$

as required. \square

When deforming modules over rings, a derived deformation over an ungraded algebra is the same thing as a classical deformation; we briefly recall the classical deformation functor.

Definition 5.5.6. Let A be a k -algebra and let X be an A -module. Let $\Gamma \in \mathbf{Art}_k$ be an ungraded Artinian algebra. A **classical deformation** of X over Γ is an $A \otimes \Gamma$ -module \tilde{X} , flat over Γ , such that $\tilde{X} \otimes_\Gamma k \cong X$. An **isomorphism** of classical deformations is an isomorphism $\phi : \tilde{X}_1 \rightarrow \tilde{X}_2$ of $A \otimes \Gamma$ -modules such that $f_1 = f_2 \circ (\phi \otimes_\Gamma k)$. The functor of classical deformations of X sends an ungraded Artinian algebra Γ to the set

$$\mathcal{D}ef_A^{\text{cl}}(X)(\Gamma) := \frac{\{\text{classical deformations of } X \text{ over } \Gamma\}}{(\text{isomorphisms})}.$$

Proposition 5.5.7. *Let A be a k -algebra and let X be an A -module. There is a natural isomorphism $\mathcal{D}ef_A^{\text{cl}}(X) \cong \mathcal{D}ef_A(X)|_{\mathbf{Art}_k}$.*

Proof. Let $\Gamma \in \mathbf{Art}_k$. Let \tilde{X} be a classical deformation of X over Γ . It is easy to see that $\tilde{X} \otimes_\Gamma^\mathbb{L} k \cong X$ inside the derived category $D(A \otimes \Gamma)$. Hence \tilde{X} is a derived deformation of X . Moreover, if two classical deformations are isomorphic, they are clearly isomorphic as derived deformations, and hence we obtain a map of sets $\Phi : \mathcal{D}ef_A^{\text{cl}}(X)(\Gamma) \rightarrow \mathcal{D}ef_A(X)(\Gamma)$. It is injective, because $A \otimes \Gamma\text{-Mod}$ embeds in $D(A \otimes \Gamma)$; here is where we are using that A is ungraded. Observe that if $\tilde{X} \in D(A \otimes \Gamma)$ is a derived deformation of X over Γ , then it must actually be (quasi-isomorphic to) an $A \otimes \Gamma$ -module concentrated in degree zero. Because we have $\tilde{X} \otimes_\Gamma^\mathbb{L} k \simeq X$, we have $\text{Tor}_i^\Gamma(\tilde{X}, k) \cong 0$ for $i > 0$. Because Γ is Artinian local, this implies Tor-vanishing for all Γ -modules, and hence \tilde{X} is a flat Γ -module. Hence \tilde{X} is in the image of Φ , and so Φ is a surjection and thus an isomorphism of sets. \square

Finally, we can deduce a representability theorem for classical deformations:

Theorem 5.5.8 (Representability for classical deformations). *Let A be a k -algebra and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the derived endomorphism dga of X . Then the functors $\mathcal{D}ef_A^{\text{cl}}(X)$ and $\text{Hom}_{\mathbf{aug. alg}_k}(H^0(B_{\text{nu}}^* E), -)$ are isomorphic.*

Proof. This follows immediately from 5.5.5 and 5.5.7. \square

6. FRAMED DEFORMATIONS

Let A be a dga and let X be an A -module. By 5.4.9, the functor of deformations of X is prorepresented by the pro-Artinian dga $B_{\text{nu}}^\sharp E$, where $E := \mathbb{R}\text{End}_A(X)$. If E happens to be augmented, does the functor prorepresented by $B^\sharp E$ admit a deformation-theoretic interpretation? In this section we will show that when deforming a one-dimensional module over a ring, one can interpret $\mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B^\sharp E, -)$ in terms of rigidified deformations: the data we will need to add to deformations to rigidify will be that of a framing. A framed deformation of a module X over a dga A is essentially a deformation of X that restricts to the trivial deformation of the underlying vector space of X . We give a prorepresentability result for framed deformations of a point, as well as a concrete identification of the set-valued

functor of framed deformations. Restricting our prorepresentability result to the classical case, we obtain a new proof of a representability result of Segal. In the final part, we give an application involving Braun–Chuang–Lazarev’s derived quotient, and give a new proof of a representability theorem of Donovan–Wemyss.

6.1. Framings. In this part we will investigate framed deformations in generality; in the next part we will restrict to one-dimensional modules to get a prorepresentability statement.

Let A be a dga and let X be an A -module. Via the forgetful functor from A -modules to vector spaces, a deformation of the A -module X gives rise to a deformation of the vector space X ; this gives us a natural transformation $\underline{\mathcal{D}ef}_A(X) \rightarrow \underline{\mathcal{D}ef}_k(X)$. Observe that the functor $\underline{\mathcal{D}ef}_k(X)$ is pointed by the trivial deformation.

Definition 6.1.1. Let A be a dga and let X be an A -module. The functor of **framed deformations** of X is the homotopy fibre

$$\underline{\mathcal{D}ef}_A^{\text{fr}}(X) := \text{hofib}(\underline{\mathcal{D}ef}_A(X) \rightarrow \underline{\mathcal{D}ef}_k(X)).$$

In other words, one restricts to those deformations of X which are trivial deformations of the underlying dg-vector space. As for the unframed case, we would like to check that $\pi_0 \underline{\mathcal{D}ef}_A^{\text{fr}}(X)$ has a natural interpretation as ‘framed deformations modulo framed isomorphisms’. We start by defining the desired quotient functor.

Definition 6.1.2. Let A be a dga and let X be an A -module. Let Γ be a connective Artinian dga. Let \tilde{X} be a deformation of X over Γ . A **framing** of \tilde{X} is a quasi-isomorphism $\nu : U(\tilde{X}) \rightarrow X \otimes_k^{\mathbb{L}} \Gamma$, where $U : D(A \otimes \Gamma) \rightarrow D(k \otimes \Gamma)$ is the forgetful functor. A **framed deformation** of X is a pair (\tilde{X}, ν) consisting of a deformation and a framing. A **framed isomorphism** $F : (X, \nu_X) \rightarrow (Y, \nu_Y)$ is an isomorphism $F : X \rightarrow Y$ of deformations satisfying $\nu_X = \nu_Y \circ UF$. The set-valued functor of framed deformations of X is

$$\mathcal{D}ef_A^{\text{fr}}(X)(\Gamma) := \frac{\{\text{framed deformations of } X \text{ over } \Gamma\}}{(\text{framed isomorphisms})}.$$

Our proof that $\pi_0 \underline{\mathcal{D}ef}_A^{\text{fr}}(X) \cong \mathcal{D}ef_A^{\text{fr}}(X)$ will reduce to the case of groupoid-valued functors, where the situation is better understood. We will need a few facts about the homotopy theory of groupoids, which we take from [Str00, §6].

Proposition 6.1.3. *The category of groupoids admits a model structure where the weak equivalences are the equivalences of categories, the fibrations are the isofibrations, and the cofibrations are the functors injective on objects.*

Proposition 6.1.4. *Suppose that $* \rightarrow B$ is a pointed groupoid, and $F : A \rightarrow B$ is a functor between groupoids. The homotopy fibre of F is the groupoid with objects the pairs (a, u) with $a \in A$ and $u : Fa \rightarrow *$, and morphisms $(a, u) \rightarrow (a', u')$ those maps $v : a \rightarrow a'$ such that $u = u' \circ Fv$.*

The set of connected components of a homotopy fibre of simplicial sets can be computed by taking the homotopy fibre on the level of fundamental groupoids:

Lemma 6.1.5. *Let Z be a pointed simplicial set, and let $f : Y \rightarrow Z$ be a map of simplicial sets. Let Π_1 be the fundamental groupoid functor. Then there is an isomorphism*

$$\pi_0(\text{hofib}(f)) \cong \pi_0(\text{hofib}(\Pi_1 f)).$$

Proof. Factorise $f = Y \xrightarrow{f'} Y' \xrightarrow{f''} Z$ into a weak equivalence followed by a fibration. Because Π_1 preserves weak equivalences and fibrations, the diagram $\Pi_1 Y \xrightarrow{\Pi_1 f'} \Pi_1 Y' \xrightarrow{\Pi_1 f''} \Pi_1 Z$ is a factorisation of $\Pi_1 f$ into a weak equivalence followed by a fibration. Because the classical model structure on simplicial sets is right proper [GJ09, II.9.6] the homotopy fibre of f is weakly equivalent to the fibre of f'' . Moreover because every groupoid is fibrant, the model structure on groupoids is also right proper, and so the homotopy fibre of $\Pi_1 f$ is weakly equivalent to the fibre of $\Pi_1 f''$. It is not hard to see that if g is a map of simplicial sets with pointed codomain, then there is an isomorphism $\pi_0 \text{fib}(g) \cong \pi_0 \text{fib}(\Pi_1 g)$. Hence we have isomorphisms

$$\pi_0(\text{hofib}(f)) \cong \pi_0(\text{fib}(f'')) \cong \pi_0(\text{fib}(\Pi_1 f'')) \cong \pi_0(\text{hofib}(\Pi_1 f))$$

as required. \square

Theorem 6.1.6. *Let A be a dga and let X be an A -module. Then there is an isomorphism of functors $\pi_0 \underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X) \cong \mathcal{D}\text{ef}_A^{\text{fr}}(X)$.*

Proof. Let \mathcal{H} denote the homotopy fibre

$$\mathcal{H} := \text{hofib}(\Pi_1 \underline{\mathcal{D}\text{ef}}_A(X) \rightarrow \Pi_1 \underline{\mathcal{D}\text{ef}}_k(X)).$$

By the definition of $\underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X)$ along with 6.1.5, we have an isomorphism $\pi_0 \underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X) \cong \pi_0 \mathcal{H}$. As in [ELO09], the groupoid $\Pi_1 \underline{\mathcal{D}\text{ef}}_A(X)$ has objects the A -deformations of X , and the morphisms are certain homotopy classes of isomorphisms of deformations (the precise structure of the homotopies is not important here). Using 6.1.4 it is not hard to see that \mathcal{H} is the groupoid whose objects are the framed deformations, and whose morphisms are the homotopy classes of framed isomorphisms. Certainly if two framed deformations are linked by a framed isomorphism, they are linked in \mathcal{H} , and conversely if two framed deformations are linked in \mathcal{H} there must be some framed isomorphism linking them. So it follows that $\pi_0 \mathcal{H}$ is isomorphic to $\mathcal{D}\text{ef}_A^{\text{fr}}(X)$, which is precisely the claim. \square

Remark 6.1.7. The groupoid \mathcal{H} appearing in the proof of 6.1.6 is not the fundamental groupoid of $\underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X)$; it is rather just a groupoid with the same π_0 . We could use the equivalence

$$\Pi_1 \underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X) \simeq \Pi_1 \text{hofib}(\Pi_2 \underline{\mathcal{D}\text{ef}}_A(X) \rightarrow \Pi_2 \underline{\mathcal{D}\text{ef}}_k(X))$$

to compute $\Pi_1 \underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X)$, but this would require a detailed analysis of homotopies.

6.2. Prorepresentability. We specialise to the situation when X is a one-dimensional module, where we can obtain a prorepresentability result. We specialise further to the set-valued and classical deformation functors, where we can recover a result of Segal.

Lemma 6.2.1. *Let A be a connective dga and let S be a one-dimensional A -module. Then $\text{Ext}_A^0(S, S) \cong k$. In particular, $\mathbb{R}\text{End}_A(S)$ is augmented.*

Proof. The idea is to use the t-structure on $D(A)$, which exists because A is connective. By definition we have an isomorphism $\text{Ext}_A^0(S, S) \cong \text{End}_{D(A)}(S)$. By connectivity, the derived category $D(A)$ admits a t-structure such that the inclusion $\mathbf{Mod}\text{-}H^0(A) \hookrightarrow D(A)$ is an equivalence onto the heart, with inverse given by taking zeroth cohomology [HKM02, Ami09, KY11, KY16]. In particular, S is in the heart of this t-structure, and so we have $\text{End}_{D(A)}(S) \cong \text{End}_{H^0(A)}(S)$. But it is clear that $\text{End}_{H^0(A)}(S)$ is just k . For the second statement, let E be any model for $\mathbb{R}\text{End}_A(S)$ and observe that the natural dga map $E \rightarrow$

$H^0 E \cong \text{Ext}_A^0(S, S) \cong k$ is an augmentation. Since it did not matter which model we chose, this is quasi-isomorphism invariant and so we can say that $\mathbb{R}\text{End}_A(S)$ is augmented. \square

Theorem 6.2.2 (Prorepresentability for framed deformations). *Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the derived endomorphism dga of S . Then there is a weak equivalence*

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S) \simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B^\sharp E, -).$$

Proof. The idea is that $\bar{E} \rightarrow E \rightarrow k$ is a homotopy fibre sequence of dgas. By 6.2.1, E is augmented. Because the augmentation $E \rightarrow k$ is surjective, there is a weak equivalence of nonunital dgas $\text{hofib}(E \rightarrow k) \simeq \bar{E}$ where \bar{E} denotes the augmentation ideal. The homotopy fibre sequence

$$\bar{E} \rightarrow E \rightarrow k$$

gives us a homotopy fibre sequence

$$BE \rightarrow B_{\text{nu}}E \rightarrow B_{\text{nu}}k$$

of coalgebras, because B is right Quillen and all dgas are fibrant. Applying \sharp now gives us a homotopy cofibre sequence

$$B_{\text{nu}}^\sharp k \rightarrow B_{\text{nu}}^\sharp E \rightarrow B^\sharp E$$

of pro-Artinian dgas, and taking derived mapping spaces now gives a homotopy fibre sequence

$$\mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B^\sharp E, -) \rightarrow \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, -) \rightarrow \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp k, -)$$

of representable functors. The weak equivalences of 5.4.9 assemble into a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp E, -) & \longrightarrow & \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp k, -) \\ \downarrow \simeq & & \downarrow \simeq \\ \underline{\mathcal{D}}\text{ef}_A(S) & \longrightarrow & \underline{\mathcal{D}}\text{ef}_k(S) \end{array}$$

where the upper horizontal map is induced by $E \rightarrow k$ and the lower horizontal map is the forgetful map. It follows that the homotopy fibres of the rows are weakly equivalent, which is the desired claim. \square

Remark 6.2.3. It is not hard to compute that $B_{\text{nu}}^\sharp k$ is $k[[x]]$ with x in degree 1, equipped with its obvious pro-Artinian structure. In particular, if Γ is connective Artinian then there is a weak equivalence

$$\mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(B_{\text{nu}}^\sharp k, \Gamma) \simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgArt}_k)}(k, \Gamma)$$

by the inclusion-truncation adjunction.

Now we have a prorepresentability result, we can specialise to what happens on π_0 .

Proposition 6.2.4 (Set-valued prorepresentability for framed deformations). *Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the derived endomorphism dga of S . Then there is an isomorphism*

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S) \simeq \text{Hom}_{\text{Ho}(\text{pro}(\text{dgArt}_k))}(B^\sharp E, -).$$

Proof. Take π_0 of 6.2.2 and appeal to 6.1.6. \square

The functors $\mathcal{D}ef^{\text{fr}}$ and $\mathcal{D}ef$ are very similar, and in fact the latter is the quotient of the former by inner automorphisms:

Theorem 6.2.5 (Set-valued representability up to automorphism). *Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the derived endomorphism dga of S . Then there is an isomorphism*

$$\mathcal{D}ef_A(S)(\Gamma) \cong \frac{\text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k))}(B^\sharp E, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

Proof. Note that this is a prorepresentability result for unframed deformations. The idea is that an analysis of the set-valued Deligne functor shows that $\mathcal{D}el(E)$ is the quotient of $\mathcal{D}el(\bar{E})$ by inner automorphisms, and the result then follows via known equivalences. Combining 6.2.2 with the weak equivalence of 5.3.8, we obtain a commutative square

$$\begin{array}{ccc} \mathcal{D}ef_A^{\text{fr}}(S) & \longrightarrow & \mathcal{D}ef_A(S) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{D}el(\bar{E}) & \longrightarrow & \mathcal{D}el(E) \end{array}$$

where the vertical maps are weak equivalences. Taking π_0 of this square we get a commutative square

$$\begin{array}{ccc} \mathcal{D}ef_A^{\text{fr}}(S) & \longrightarrow & \mathcal{D}ef_A(S) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{D}el(\bar{E}) & \longrightarrow & \mathcal{D}el(E) \end{array}$$

with vertical maps isomorphisms, where the identification of the top line is 6.1.6 and 5.3.7 and the identification of the bottom line is 5.1.19. If Γ is a connective Artinian dga, we see that $\text{MC}(\bar{E})(\Gamma) = \text{MC}(E)(\Gamma)$. Moreover, it is not hard to check that the sequence of gauge groups $\mathcal{G}g(\bar{E}) \rightarrow \mathcal{G}g(E) \rightarrow \mathcal{G}g(k)$ is split exact. It follows that $\mathcal{D}el(\bar{E}) \rightarrow \mathcal{D}el(E)$ is a surjection, and moreover we can identify $\mathcal{D}el(E)$ as the quotient of $\mathcal{D}el(\bar{E})$ by the gauge group $\mathcal{G}g(k)$. The weak equivalences $\mathcal{D}el \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k)}(B_{\text{nu}}^\sharp, -)$ give isomorphisms $\mathcal{D}el \cong \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k))}(B_{\text{nu}}^\sharp, -)$, compatible with the action of $\mathcal{G}g(k)$, and it follows that we have an isomorphism

$$\mathcal{D}ef_A(S) \cong \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k))}(B^\sharp E, -) / \mathcal{G}g(k).$$

Because Γ is connective the units of Γ are the same as the units of Γ^0 , and because Γ^0 is an Artinian local algebra its units are $1 + \mathfrak{m}_\Gamma^0$. It follows by definition that $\mathcal{G}g(k)(\Gamma) \cong \Gamma^\times$. Moreover, Γ^\times acts on $\text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k))}(B^\sharp E, \Gamma)$ by inner automorphisms, and the result follows. \square

Remark 6.2.6. The homotopy fibre sequence

$$\mathcal{D}ef_A^{\text{fr}}(S) \rightarrow \mathcal{D}ef_A(S) \rightarrow \mathcal{D}ef_k(S)$$

yields a long exact sequence on homotopy groups

$$\cdots \rightarrow \pi_1 \mathcal{D}ef_k(S) \rightarrow \mathcal{D}ef_A^{\text{fr}}(S) \rightarrow \mathcal{D}ef_A(S) \rightarrow *$$

from which we can immediately see that the natural map $\mathcal{D}ef_A^{\text{fr}}(S) \rightarrow \mathcal{D}ef_A(S)$ is a surjection: in other words, every deformation admits a framing. The proof above identifies the action of $\pi_1 \mathcal{D}ef_k(S)$ on $\mathcal{D}ef_A^{\text{fr}}(S)$ as the gauge action of $\mathcal{G}g(k)$.

Remark 6.2.7. One can probably show more directly, without the use of deformation theory, that we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{pro}(\mathrm{dgArt}_k))}(B_{\mathrm{nu}}^\sharp, \Gamma) \cong \frac{\mathrm{Hom}_{\mathrm{Ho}(\mathrm{pro}(\mathrm{dgArt}_k))}(B^\sharp E, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

Heuristically, when taking the nonunital bar construction of E , all that one is doing is taking BE and adding an extra idempotent, nontrivial in cohomology. A map $B^\sharp(E) \oplus k \rightarrow \Gamma$ is the same thing as a pair of a map $B^\sharp E \rightarrow \Gamma$ and an invertible element of Γ . Hence one has to mod out by the invertible elements of Γ , which act as inner automorphisms.

We finish by recovering a theorem of Segal:

Theorem 6.2.8 (cf. [Seg08, 2.13]). *Let A be a k -algebra and let S be a one-dimensional A -module. Assume that $\mathrm{Ext}_A^1(S, S)$ is finite dimensional. Let T be the k -algebra*

$$T := \frac{\hat{T}(\mathrm{Ext}_A^1(S, S)^*)}{m^*(\mathrm{Ext}_A^2(S, S)^*)}$$

where m is the homotopy Maurer–Cartan function (4.2.1). Then there is an isomorphism

$$\mathcal{D}_{ef_A}^{\mathrm{cl}}(S)(\Gamma) \cong \frac{\mathrm{Hom}_{\mathrm{aug. alg}_k}(T, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

Proof. Let E be the derived endomorphism algebra of S and let E' be an A_∞ minimal model for E . Then E' satisfies the hypotheses of 4.2.2, and using that along with the quasi-isomorphism invariance of the Koszul dual we see that $T \cong H^0(E^!)$. As in the proof of 5.5.5, if Γ is ungraded then we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{pro}(\mathrm{dgArt}_k))}(B^\sharp E, \Gamma) \cong \mathrm{Hom}_{\mathrm{aug. alg}_k}(H^0(E^!), \Gamma) \cong \mathrm{Hom}_{\mathrm{aug. alg}_k}(T, \Gamma).$$

By 5.5.7 and 6.2.5 we have

$$\mathcal{D}_{ef_A}^{\mathrm{cl}}(S)(\Gamma) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathrm{pro}(\mathrm{dgArt}_k))}(B^\sharp E, \Gamma) / (\text{inner automorphisms})$$

and so we are done. \square

Remark 6.2.9. In particular, if A is noetherian then S admits some finitely generated projective resolution \tilde{S} , from which it is easy to see that $\mathrm{Hom}_A(\tilde{S}, S)$ is finite dimensional in each degree. But the cohomology of $\mathrm{Hom}_A(\tilde{S}, S)$ is exactly the Ext-algebra of S , and it follows that the conditions of the theorem are satisfied.

6.3. The derived quotient. We explore in more detail the links between one-dimensional modules and idempotents. We will give a deformation-theoretic meaning to Braun–Chuang–Lazarev’s derived quotient [BCL18], which is a way to quotient an algebra (or a dga) by an idempotent in a homotopically well-behaved manner.

Let A be a k -algebra and let S be a one-dimensional A -module. Let a be the corresponding algebra map $a : A \rightarrow \mathrm{End}_k(S) \cong k$. As vector spaces, we have $A \cong \ker(a) \oplus k$; let $e \in A$ be the element in A corresponding to $1 \in k$, so that $a(e) = 1$. It is easy to see that e is an idempotent in A ; let $e_S := 1 - e$ be its orthogonal idempotent. Then a factors through the quotient $A/Ae_S A$, and hence S is naturally an $A/Ae_S A$ -module. The corresponding algebra map $A/Ae_S A \rightarrow k$ is an augmentation.

Definition 6.3.1. Let A be a k -algebra and let S be a one-dimensional A -module. We call the idempotent e_S of the previous construction the **idempotent associated to S** .

If A is a dga then we call a dga B with a map $A \rightarrow B$ an A -dga.

Definition 6.3.2 ([BCL18]). Let A be a k -algebra and let $e \in A$ be an idempotent. The **derived quotient** $A/\mathbb{L}AeA$ is the universal A -dga homotopy annihilating e . It is defined up to quasi-isomorphism of A -dgas.

Proposition 6.3.3 ([BCL18, 3.10 and 3.4]). *The derived quotient exists and is unique up to quasi-isomorphism of A -dgas.*

We summarise the key properties we will need.

Proposition 6.3.4 ([BCL18, 4.15]). *The map $A \rightarrow A/\mathbb{L}AeA$ induces an embedding of derived categories $D(A/\mathbb{L}AeA) \hookrightarrow D(A)$, with image those complexes X such that eX is acyclic.*

Proposition 6.3.5 ([Boo20, 3.2.4]). *The derived quotient of an ungraded algebra A by an idempotent e is a connective dga, with $H^0(A/\mathbb{L}AeA) \cong A/AeA$.*

Remark 6.3.6. The derived quotient is a special case of the derived localisation, which can be constructed much more generally: see [BCL18] for the full theory.

The following is a generalisation of an argument given in the proof of [KY16, 5.5].

Proposition 6.3.7. *Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Then $A/\mathbb{L}Ae_S A$ is augmented, and there is a quasi-isomorphism $\mathbb{R}\text{End}_A(S) \simeq A/\mathbb{L}Ae_S A^!$.*

Proof. Consider S as an object of the derived category $D(A)$. It is easy to see that $e_S S$ is acyclic. Hence S is naturally an object of the full subcategory $D(A/\mathbb{L}Ae_S A) \hookrightarrow D(A)$ by 6.3.4. We may hence compute $\mathbb{R}\text{End}_A(S) \simeq \mathbb{R}\text{End}_{A/\mathbb{L}Ae_S A}(S)$. By 6.3.5, $A/\mathbb{L}Ae_S A$ is augmented (via the map to H^0) and connective, so we can use 4.1.2 to conclude that we have a quasi-isomorphism $\mathbb{R}\text{End}_{A/\mathbb{L}Ae_S A}(S) \simeq A/\mathbb{L}Ae_S A^!$ as required. \square

Corollary 6.3.8. *Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Then there is a weak equivalence*

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S) \simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgaArt}_k)}(B^\sharp(A/\mathbb{L}Ae_S A^!), -).$$

Proof. Combine 6.2.2 with 6.3.7. \square

Theorem 6.3.9 (Prorepresentability by $A/\mathbb{L}Ae_S A$). *Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Suppose that $A/\mathbb{L}Ae_S A$ is cohomologically locally finite and $A/Ae_S A$ is Artinian local. Then there is a weak equivalence*

$$\underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S) \simeq \mathbb{R}\text{Map}_{\text{aug.dga}_k^{\leq 0}}(A/\mathbb{L}Ae_S A, -).$$

Proof. Let Γ be a connective Artinian dga. We have natural weak equivalences

$$\begin{aligned} \underline{\mathcal{D}}\text{ef}_A^{\text{fr}}(S)(\Gamma) &\simeq \mathbb{R}\text{Map}_{\text{pro}(\text{dgaArt}_k)}(B^\sharp(A/\mathbb{L}Ae_S A^!), \Gamma) && \text{by 6.3.8} \\ &\simeq \mathbb{R}\text{Map}_{\text{aug.dga}_k^{\leq 0}}(A/\mathbb{L}Ae_S A, \Gamma) && \text{by 4.4.4} \end{aligned}$$

and so we are done. \square

Restricting to classical deformations, we obtain the following:

Theorem 6.3.10 (cf. [DW19, 3.9]). *Let A be a k -algebra and let S be a one-dimensional k -module with associated idempotent e_S . Suppose that $A/\mathbb{L}Ae_S A$ is cohomologically locally finite and $A/Ae_S A$ is Artinian local. Then there is an isomorphism*

$$\mathcal{D}ef_A^{\text{cl}}(S)(\Gamma) \cong \frac{\text{Hom}_{\mathbf{aug. alg}_k}(A/Ae_S A, \Gamma)}{(\text{inner automorphisms of } \Gamma)}.$$

Proof. Using 6.3.8 then as in the proof of 6.2.8 we can identify $\mathcal{D}ef_A^{\text{cl}}(S)$ as the quotient of $\text{Hom}_{\mathbf{aug. alg}_k}(H^0(A/\mathbb{L}Ae_S A^!), -)$ by the inner automorphisms. But by 6.3.5 and 4.2.8 we have $H^0(A/\mathbb{L}Ae_S A^!) \cong H^0(A/\mathbb{L}Ae_S A) \cong A/Ae_S A$. \square

Remark 6.3.11. The cohomology spaces of $A/\mathbb{L}AeA$ are completely known: aside from H^0 and H^{-1} , they are all of the form $\text{Tor}_*^{eAe}(Ae, eA)$ [Boo20, 3.2.4]. Under some geometric hypotheses, we get cohomological finiteness of $A/\mathbb{L}AeA$ for free: if $A = \text{End}_R(R \oplus M)$ is a noncommutative partial resolution of a Gorenstein ring R , and across this isomorphism the idempotent e corresponds to id_R , then the results of [Boo20] show that $A/\mathbb{L}AeA$ is cohomologically locally finite (indeed, its cohomology spaces are all Ext groups in the singularity category of R , which is hom-finite).

7. THE UNIVERSAL PRODEFORMATION

By taking homotopy limits, the formalism of functors valued in \mathbf{sSet} allows us to deform modules over pro-Artinian dgas. We set up the theory of prodeformations, and show that our prorepresentability statements give us a universal prodeformation. We do some explicit computations towards the identification of the universal prodeformation, and we finish by tracking this universal prodeformation across quasi-equivalences.

7.1. Prodeformations.

Definition 7.1.1. Let $F : \mathbf{dgArt}_k^{\leq 0} \rightarrow \mathbf{sSet}$ be any functor. Denote by \hat{F} the functor $\mathbf{pro}(\mathbf{dgArt}_k^{\leq 0}) \rightarrow \mathbf{sSet}$ which sends an inverse system $\{\Gamma_\alpha\}_\alpha$ to the homotopy limit $\mathop{\text{holim}}\limits_{\leftarrow \alpha} F(\Gamma_\alpha)$. Call \hat{F} the **pro-completion** of F .

Definition 7.1.2. Let A be a dga and let X be an A -module. The functor of **prodeformations of X** is the functor $\widehat{\mathcal{D}ef}_A(X)$. The **set-valued functor of prodeformations of X** is the functor $\widehat{\mathcal{D}ef}_A(X) := \pi_0 \widehat{\mathcal{D}ef}_A(X)$. A **prodeformation of X** is an element of the set $\widehat{\mathcal{D}ef}_A(X)$.

It is easy to see that if Γ is a connective Artinian local dga then $\widehat{\mathcal{D}ef}_A(X)(\Gamma) \simeq \mathcal{D}ef_A(X)(\Gamma)$ and hence $\widehat{\mathcal{D}ef}_A(X)(\Gamma) \cong \mathcal{D}ef_A(X)(\Gamma)$. The reader is warned that $\widehat{\mathcal{D}ef}_A(X)(\Gamma)$ may not agree with the inverse limit $\mathop{\text{lim}}\limits_{\leftarrow \alpha} \mathcal{D}ef_A(X)(\Gamma)$, as π_0 need not commute with homotopy limits; prodeformations are intrinsically homotopical in nature.

Remark 7.1.3. If $\Gamma = \{\Gamma_\alpha\}_\alpha$ is pro-Artinian then there is a weak equivalence

$$\widehat{\mathcal{D}ef}_A(X)(\Gamma) \simeq \mathcal{W} \left(X_{\text{dg}} \times_{D_{\text{dg}}(A)}^h \mathop{\text{holim}}\limits_{\leftarrow \alpha} D_{\text{dg}}(A \otimes \Gamma_\alpha) \right)$$

given by passing the homotopy limit through \mathcal{W} using 5.2.19 and commuting homotopy limits with homotopy pullbacks. There is an obvious system of maps of dgas $\mathop{\text{lim}}\limits_{\leftarrow \alpha} \Gamma \rightarrow \Gamma_\alpha$

and this gives a dg functor

$$D_{\text{dg}}(A \otimes \varprojlim \Gamma) \rightarrow \varprojlim_{\alpha} D_{\text{dg}}(A \otimes \Gamma_{\alpha}).$$

Although $\varprojlim \Gamma$ may not be Artinian, it is still augmented, so one can use it as a base for deformations. Extending the notation of 5.3.5 in the obvious way, we hence obtain a map in the homotopy category of simplicial sets

$$\underline{\mathcal{D}ef}_A(X)(\varprojlim \Gamma) \rightarrow \widehat{\underline{\mathcal{D}ef}}_A(X)(\Gamma)$$

and so a deformation of X over $\varprojlim \Gamma$ gives a prodeformation of X over Γ . In particular, letting Λ be a complete local augmented dga, we get a natural morphism

$$\underline{\mathcal{D}ef}_A(X)(\Lambda) \rightarrow \varprojlim_n \underline{\mathcal{D}ef}_A(X)(\Lambda/\mathfrak{m}^n)$$

which should be relevant for noncommutative Artin–Lurie representability [Lur12, Pri12b].

We are about to give a representability statement for prodeformations; before we do so we prove a subsidiary lemma.

Lemma 7.1.4. *Let $\Gamma = \{\Gamma_{\alpha}\}_{\alpha}$ be a pro-Artinian dga. Then there is a weak equivalence*

$$\mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(-, \Gamma) \simeq \varprojlim_{\alpha} \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(-, \Gamma_{\alpha}).$$

Proof. Let Γ' be the filtered diagram of Artinian dgas that defines Γ , but regarded as a filtered diagram in $\mathbf{pro}(\mathbf{dgArt}_k)$. The Yoneda lemma gives us an isomorphism $\varprojlim \Gamma' \cong \Gamma$. Moreover, in the exact same manner as the proof of 3.3.6 we see that $\varprojlim \Gamma' \simeq \Gamma$ as well, because homotopy limits are just limits. Now use that $\mathbb{R}\text{Map}$ commutes with homotopy limits in the second variable. \square

Proposition 7.1.5. *Let A be a dga and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . Let $\Gamma \in \mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$. Then there is a canonical weak equivalence*

$$\widehat{\underline{\mathcal{D}ef}}_A(X)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\text{nu}}^{\sharp} E, \Gamma).$$

Proof. Put $\Gamma = \{\Gamma_{\alpha}\}_{\alpha}$. We have

$$\begin{aligned} \widehat{\underline{\mathcal{D}ef}}_A(X)(\Gamma) &:= \varprojlim_{\alpha} \underline{\mathcal{D}ef}_A(X)(\Gamma_{\alpha}) && \text{by definition} \\ &\simeq \varprojlim_{\alpha} \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\text{nu}}^{\sharp} E, \Gamma_{\alpha}) && \text{by 5.4.9 levelwise} \\ &\simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{dgArt}_k)}(B_{\text{nu}}^{\sharp} E, \Gamma) && \text{by 7.1.4} \end{aligned}$$

as required. \square

Corollary 7.1.6. *Let A be a dga and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . Let $\Gamma \in \mathbf{pro}(\mathbf{dgArt}_k^{\leq 0})$. Then there is an isomorphism*

$$\widehat{\underline{\mathcal{D}ef}}_A(X)(\Gamma) \cong \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{dgArt}_k))}(B_{\text{nu}}^{\sharp} E, \Gamma).$$

There are analogous versions of the previous theorems for framed prodeformations.

Definition 7.1.7. Let A be a dga and let X be an A -module. The functor of **framed prodeformations of X** is the functor $\widehat{\mathcal{D}ef}_A^{\text{fr}}(X)$. The **set-valued functor of framed prodeformations of X** is the functor $\widehat{\mathcal{D}ef}_A^{\text{fr}}(X) := \pi_0 \widehat{\mathcal{D}ef}_A(X)$. A **framed prodeformation of X** is an element of the set $\widehat{\mathcal{D}ef}_A^{\text{fr}}(X)$.

Again, $\mathcal{D}ef^{\text{fr}}$ agrees with $\widehat{\mathcal{D}ef}^{\text{fr}}$ when the input is an Artinian local dga.

Lemma 7.1.8. *Let A be a dga and let X be an A -module. Then there is a weak equivalence*

$$\widehat{\mathcal{D}ef}_A^{\text{fr}}(X) \simeq \text{hofib} \left(\widehat{\mathcal{D}ef}_A(X) \rightarrow \widehat{\mathcal{D}ef}_k(X) \right).$$

Proof. Commute the homotopy limit in the definition of $\widehat{\mathcal{D}ef}_A^{\text{fr}}(X)$ through the homotopy fibre in the definition of $\mathcal{D}ef_A^{\text{fr}}(X)$. \square

Proposition 7.1.9. *Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the endomorphism dga of S . Then E is augmented. Let $\Gamma \in \mathbf{pro}(\mathbf{d}g\mathbf{Art}_k^{\leq 0})$. Then there is a canonical weak equivalence*

$$\widehat{\mathcal{D}ef}_A^{\text{fr}}(S)(\Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k)}(B^\sharp E, \Gamma).$$

Proof. This is similar to the proof of 7.1.5: apply 6.2.2 levelwise and then use 7.1.4. \square

Corollary 7.1.10. *Let A be a dga and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . Let $\Gamma \in \mathbf{pro}(\mathbf{d}g\mathbf{Art}_k^{\leq 0})$. Then there is an isomorphism*

$$\widehat{\mathcal{D}ef}_A^{\text{fr}}(X)(\Gamma) \cong \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k))}(B^\sharp E, \Gamma).$$

7.2. Universal elements. Now that we are able to work with prodeformations on the level of prorepresenting objects, we can immediately obtain a universal prodeformation. Since our deformation functors accept connective dgas as input, we need to truncate the representing object. Observe that if P is any pro-Artinian dga and Γ is a connective pro-Artinian dga then the proof of 5.5.1 gives a natural equivalence

$$\mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k)}(P, \Gamma) \simeq \mathbb{R}\text{Map}_{\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k^{\leq 0})}(\tau_{\leq 0}P, \Gamma)$$

and hence there is no danger in truncation.

Definition 7.2.1. Let A be a dga and let X be an A -module. Let $E := \mathbb{R}\text{End}_A(X)$ be the endomorphism dga of X . The **universal prodeformation of X** is the prodeformation of X over $\tau_{\leq 0}B_{\text{nu}}^\sharp E$ corresponding to the element $\text{id} \in \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k^{\leq 0}))}(\tau_{\leq 0}B_{\text{nu}}^\sharp E, \tau_{\leq 0}B_{\text{nu}}^\sharp E)$ across the isomorphism of 7.1.6.

One can also define the universal framed prodeformation:

Definition 7.2.2. Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the endomorphism dga of S , which is augmented. The **universal framed prodeformation of S** is the framed prodeformation of S over $\tau_{\leq 0}B^\sharp E$ corresponding to the element $\text{id} \in \text{Hom}_{\text{Ho}(\mathbf{pro}(\mathbf{d}g\mathbf{Art}_k^{\leq 0}))}(\tau_{\leq 0}B^\sharp E, \tau_{\leq 0}B^\sharp E)$ across the isomorphism of 7.1.10.

Remark 7.2.3. When $\tau_{\leq 0}B_{\text{nu}}^\sharp E$ is weakly equivalent to an Artinian dga, then one could refer to the corresponding universal prodeformation as simply the **universal deformation**, and similarly for framed deformations.

We will make some progress towards identifying the universal framed prodeformation.

Proposition 7.2.4. *Let A be a connective dga, let S be a one-dimensional A -module, and let $E := \mathbb{R}\mathrm{End}_A(S)$ be its derived endomorphism dga. Let Γ be any Artinian dga and let $B^\sharp E \rightarrow \Gamma$ be a map of pro-Artinian dgas. Then the corresponding framed deformation of S is the A - Γ -bimodule Γ .*

Before we begin the proof, we observe that this statement makes sense because $E^!$ is an A -bimodule, and hence the induced map $E^! \rightarrow \Gamma$ makes Γ into an A - Γ -bimodule, which is clearly a deformation of S . The canonical map $\Gamma \rightarrow k$ provides the framing.

Proof. Choose a resolution $\tilde{S} \rightarrow S$ and replace E by the quasi-isomorphic $\mathrm{End}_A(\tilde{S})$. The identity map $B^\sharp E \rightarrow B^\sharp E$ gives the universal twisting cochain $\pi : BE \rightarrow E$. Dualising the map $B^\sharp E \rightarrow \Gamma$ and composing hence gives us a twisting cochain $\Gamma^* \rightarrow E$, which we also denote by π . By [Pos11, §6.5], twisting the differential on $\mathrm{Hom}_k(BE, \tilde{S})$ gives us a quasi-isomorphism $\mathrm{Hom}_k^\pi(BE, \tilde{S}) \simeq \mathbb{R}\mathrm{Hom}_E(k, \tilde{S})$. By 4.1.2, $\mathbb{R}\mathrm{Hom}_E(k, \tilde{S})$ is quasi-isomorphic to $E^!$, and hence we get a quasi-isomorphism $\mathrm{Hom}_k^\pi(BE, \tilde{S}) \simeq E^!$. Regarding BE as an ind-coalgebra lets us regard $\mathrm{Hom}_k^\pi(BE, \tilde{S})$ as a pro-object, and using that \varprojlim reflects weak equivalences we hence obtain a weak equivalence $\mathrm{Hom}_k^\pi(BE, \tilde{S}) \simeq B^\sharp E$. Base changing this weak equivalence along $B^\sharp E \rightarrow \Gamma$ (the base change is the complex that computes $\mathrm{Cotor}^{BE}(\Gamma^*, \tilde{S}^*)$; see [Pos11, §2.5]) we obtain a weak equivalence $\mathrm{Hom}_k^\pi(\Gamma^*, \tilde{S}) \simeq \Gamma$. But because Γ is Artinian, the left-hand side is isomorphic to the twist $\tilde{S} \otimes_\pi \Gamma$. But this twist is precisely the deformation corresponding to the map $B^\sharp E \rightarrow \Gamma$. \square

Remark 7.2.5. One has an isomorphism of pro-objects $\mathrm{Hom}_k(BE, E) \cong E \otimes B^\sharp E$. Taking the limit, one gets an isomorphism

$$\mathrm{Hom}_k(BE, E) \cong \varprojlim (E \otimes B^\sharp E) =: E \widehat{\otimes} E^!$$

between the convolution algebra and the completed tensor product. Loosely, this isomorphism sends the universal twisting cochain π to the Casimir element

$$\sum_e e \otimes e^* \in \mathrm{MC}(E \widehat{\otimes} E^!)$$

where we sum over a (possibly infinite!) basis of E and e^* denotes the dual basis vector corresponding to e . We can twist the differentials by π to obtain an isomorphism

$$\mathrm{Hom}_k^\pi(BE, E) \cong \varprojlim (E \otimes_\pi B^\sharp E) =: E \widehat{\otimes}_\pi E^!.$$

As before, we can twist the differential on $\tilde{S} \widehat{\otimes} E^!$, and it is possible to show that $\tilde{S} \widehat{\otimes}_\pi E^! \simeq E^!$ as bimodules. In some sense, this is a computation of the universal prodeformation. As in [Neg17] or [Her19], the dga $\mathrm{Hom}_k^\pi(BE, E)$ is quasi-isomorphic to the dga $\mathbb{R}\mathrm{Hom}_{E^e}(E, E)$ which computes the Hochschild cohomology of E . This Hochschild dga should be thought of as the universal algebra deformation of E , and so we see that loosely, the universal deformation of S is obtained by base change from the universal deformation of E .

It is not even clear what sort of object the universal framed prodeformation should be. However, as in 7.1.3, the set $\widehat{\mathcal{D}\mathrm{ef}}_A^{\mathrm{fr}}(S)(B^\sharp E)$ of which it is an element receives a map from the set $\mathcal{D}\mathrm{ef}_A^{\mathrm{fr}}(S)(E^!)$. The A - $E^!$ -bimodule $E^!$ is a (framed) deformation of S over $E^!$, and in view of the above we make the following conjecture:

Conjecture 7.2.6. Let A be a connective dga and let S be a one-dimensional A -module. Let $E := \mathbb{R}\text{End}_A(S)$ be the derived endomorphism dga of S . Then the image of $E^! \in \mathcal{D}\text{ef}_A^{\text{fr}}(S)(E^!)$ along the map $\mathcal{D}\text{ef}_A^{\text{fr}}(S)(E^!) \rightarrow \widehat{\mathcal{D}\text{ef}}_A^{\text{fr}}(S)(B^\sharp E)$ is the universal framed prodeformation of S as an A -module.

A proof of this conjecture would presumably involve a careful analysis of the image of the dg functor $S \times_{D(A)}^h D(A \otimes E^!) \rightarrow \mathop{\text{holim}}\limits_{\leftarrow} S \times_{D(A)}^h D(A \otimes B^\sharp E)$.

7.3. Deformations and quasi-equivalences. Let A and B be two dgas. Suppose that $F : D(A) \rightarrow D(B)$ is a derived equivalence given by tensoring with an A - B -bimodule (this is not really a restriction as up to homotopy, all quasi-equivalences $D(A) \rightarrow D(B)$ are of this form [Toë07, Corollary 4.8]). We analyse how deformations behave under F . Let X be an A -module and put $Y := F(X)$. Observe that we can enhance F to a quasi-equivalence $F : D_{\text{dg}}(A) \rightarrow D_{\text{dg}}(B)$ of dg categories.

Lemma 7.3.1. F induces a weak equivalence

$$F : \underline{\mathcal{D}\text{ef}}_A(X) \xrightarrow{\simeq} \underline{\mathcal{D}\text{ef}}_B(Y).$$

Proof. For each Artinian dga Γ , the quasi-equivalence F induces a commutative diagram of dg categories

$$\begin{array}{ccccc} X_{\text{dg}} & \hookrightarrow & D_{\text{dg}}(A) & \longleftarrow & D_{\text{dg}}(A \otimes \Gamma) \\ \downarrow F & & \downarrow F & & \downarrow F \\ Y_{\text{dg}} & \hookrightarrow & D_{\text{dg}}(B) & \longleftarrow & D_{\text{dg}}(B \otimes \Gamma) \end{array}$$

where the vertical maps are quasi-equivalences, and in the third column we have used that F is tensoring by a bimodule. So we get a quasi-equivalence

$$F : \left(X_{\text{dg}} \times_{D_{\text{dg}}(A)}^h D_{\text{dg}}(A \otimes \Gamma) \right) \xrightarrow{\simeq} \left(Y_{\text{dg}} \times_{D_{\text{dg}}(B)}^h D_{\text{dg}}(B \otimes \Gamma) \right)$$

between the homotopy pullbacks and hence a weak equivalence $F : \underline{\mathcal{D}\text{ef}}_A(X) \xrightarrow{\simeq} \underline{\mathcal{D}\text{ef}}_B(Y)$. \square

Lemma 7.3.2. F induces a weak equivalence $F : \widehat{\mathcal{D}\text{ef}}_A(X) \xrightarrow{\simeq} \widehat{\mathcal{D}\text{ef}}_B(Y)$.

Proof. Take homotopy limits of 7.3.1. \square

Put $E_A := \mathbb{R}\text{End}_A(X)$ and $E_B := \mathbb{R}\text{End}_B(Y)$. The component of F at X gives us a quasi-isomorphism $E_A \rightarrow E_B$, and hence a weak equivalence $\widehat{\mathcal{D}\text{ef}}_B(Y)(B^\sharp E_B) \rightarrow \widehat{\mathcal{D}\text{ef}}_B(Y)(B^\sharp E_A)$. Combined with the weak equivalence of 7.3.2, we get a weak equivalence $\widehat{\mathcal{D}\text{ef}}_A(X)(B^\sharp E_A) \simeq \widehat{\mathcal{D}\text{ef}}_B(Y)(B^\sharp E_B)$ and it is easy to see that this preserves the universal prodeformation. We repeat our analysis in the framed setting.

Lemma 7.3.3. F induces a weak equivalence $F : \widehat{\mathcal{D}\text{ef}}_A^{\text{fr}}(X) \xrightarrow{\simeq} \widehat{\mathcal{D}\text{ef}}_B^{\text{fr}}(Y)$.

Proof. The weak equivalences of 7.3.2 commute with the forgetful functors and using 7.1.8 it follows that we get an induced weak equivalence between framed prodeformations. \square

Remark 7.3.4. We could also have proved this by checking that we get a weak equivalence $F : \underline{\mathcal{D}\text{ef}}_A^{\text{fr}}(X) \xrightarrow{\simeq} \underline{\mathcal{D}\text{ef}}_B^{\text{fr}}(Y)$ and taking homotopy limits.

In exactly the same manner as before, one can check that universal framed prodeformations are preserved by F .

Theorem 7.3.5. *Let A and B be connective dgas. Let S_A and S_B be one-dimensional A and B -modules respectively. Suppose that there is a derived equivalence $F : D(A) \rightarrow D(B)$ given by tensoring with a complex of bimodules, satisfying $F(S_A) \simeq S_B$. Put $U_A := \mathbb{R}\mathrm{End}_A(S_A)^\dagger$ and $U_B := \mathbb{R}\mathrm{End}_B(S_B)^\dagger$. Then there is a quasi-isomorphism of B -modules $F(U_A) \simeq U_B$.*

Proof. Morally, this is true because U_A is the universal framed prodeformation of S_A , and similarly for U_B , and F preserves the universal prodeformation. Indeed, this easily follows from the results above if 7.2.6 is true. As above, put $E_A := \mathbb{R}\mathrm{End}_A(X)$ and $E_B := \mathbb{R}\mathrm{End}_B(Y)$. Put $V_A := B^\sharp E_A$ and $V_B := B^\sharp E_B$, so that one has $U_A \simeq \varprojlim V_A$ and $U_B \simeq \varprojlim V_B$. Because F is a quasi-equivalence, one gets a quasi-isomorphism $E_A \rightarrow E_B$, hence a weak equivalence $V_B \rightarrow V_A$ and hence a quasi-isomorphism $U_B \rightarrow U_A$. If Γ is an Artinian dga appearing in V_A , then one has a map $U_A \rightarrow \Gamma$, and this induces a dg functor $D(A \otimes \Gamma) \rightarrow D(A \otimes U_A)$. Similarly one has a dg functor $D(B \otimes \Gamma) \rightarrow D(B \otimes U_B)$. These functors fit into a commutative diagram

$$\begin{array}{ccc} D(A \otimes U_A) & \xrightarrow{G} & D(B \otimes U_B) \\ \uparrow & & \uparrow \\ D(A \otimes \Gamma) & \xrightarrow{F} & D(B \otimes \Gamma) \end{array}$$

with horizontal arrows quasi-equivalences, and where G is induced by F and the morphism $U_B \rightarrow U_A$. As Γ varies, the images of the vertical arrows assemble to give pro-objects $P_A \in \mathbf{pro}(D(A \otimes U_A))$ and $P_B \in \mathbf{pro}(D(B \otimes U_B))$ such that $F(P_A) = P_B$. The homotopy limit of P_A is the A - U_A -bimodule $\mathop{\mathrm{holim}}_{V_A \rightarrow \Gamma} \Gamma \simeq \mathop{\mathrm{holim}} V_A \simeq U_A$, and similarly the homotopy limit of P_B is U_B . But because G is a quasi-equivalence, it preserves homotopy limits, and so one has $G(U_A) \simeq U_B$. Forgetting the U_B -module structure, one simply has $G \simeq F$, and so we get a quasi-isomorphism of B -modules $F(U_A) \simeq U_B$. \square

Remark 7.3.6. In some sense, this is a computation of the universal framed prodeformation. What we have done is to take the pro-object in dg categories defining the universal framed prodeformation, picked out the pieces of the universal prodeformation in some appropriate ‘universal target’ category, and taken the homotopy limit of those pieces inside that category. Whereas to find the universal framed prodeformation, we take the homotopy limit of the system of dg categories and look at the object corresponding to the pieces we had. The problem with lifting this proof to an actual computation of the universal prodeformation is again the need to understand homotopy limits of dg categories.

8. MULTI-POINTED DEFORMATIONS

In this section, we adapt our arguments to work in the setting of multi-pointed deformation theory [Lau02, Eri06, Kaw18, DW19]; for brevity we will tend to drop the modifier ‘multi-’. Here, our base ring k is no longer a field, but a semisimple ring of the form $\mathbb{F}^{\oplus n}$ with \mathbb{F} a field and $n > 1$. Note that a k -module has finite rank if and only if it is finite-dimensional over \mathbb{F} . Following Laudal [Lau02], we refer to an augmented k -algebra as an n -**pointed** \mathbb{F} -algebra; note that the n -pointed \mathbb{F} -algebras are precisely the (non-local) Artinian \mathbb{F} -algebras with precisely n isomorphism classes of simple modules.

In essence, any result that uses only the homological properties of k remains true, as semisimple rings are precisely the rings over which every module is both projective and injective. Any result which uses the internal structure of k probably becomes false as stated in the pointed setting, and requires some alterations to remain true.

8.1. Model structures and Koszul duality. Firstly we consider §3 and §4, the results of which generalise straightforwardly. Everything in §3 goes through verbatim: the only thing that one must really check is that Pridham’s proof [Pri10, 4.3] of 3.3.1 adapts to the pointed case. In §4, there are a few more things to check. One must first check that 4.1.2 holds in the pointed setting: however [FHT01, §19, Exercise 4] holds over any base ring, so (in the notation used there) it suffices to check that the bar construction $B(A, A)$ is a cofibrant A -module, for which the usual proof suffices. At the level of generality in which we need it, the relevant A_∞ -algebra theory all works in the pointed setting; for a useful reference see [KY20, §2.4]. Everything else goes through verbatim.

8.2. Deformations and prorepresentability. Now we consider §5, where things start to get a little harder. We will not necessarily check that all of the statements made in §5 remain true in the pointed setting: we will only use what we need to get analogues of the main results.

We first consider §5.1. All of the basic definitions up to 5.1.11 adapt to the pointed case. We won’t check directly that 5.1.11 or 5.1.17 adapt, as homotopy invariance for $\mathcal{D}el$ and $\mathcal{M}C$ will follow from our later prorepresentability statements. The definition of $\Omega(\Delta^\bullet)$ remains the same, and the Polynomial Poincaré Lemma 5.1.13 remains true. It is easy to see that 5.1.19 adapts. The important thing for us to check is 5.1.20, which remains true since it is really a statement about the homotopy theory of simplicial groups [Pri15, 2.20].

Everything in §5.2 goes through in the pointed case; indeed the only k -linear results we use are Tabuada’s [Tab05, Tab10a], where they are stated over arbitrary commutative base rings. As for §5.3, observe that the only theorem here that needs to be checked is 5.3.8; everything else follows. But it is not too hard to see that [Pri15, 4.6] adapts to the pointed setting; the key is that every k -module is flat.

As for §5.4, it is not hard to check that the basic theory of twisting cochains adapts to the pointed case; the main thing to check is 5.4.6, for which Loday–Vallette’s proof [LV12, 2.2.6] adapts verbatim to the pointed setting. The proofs of 5.4.7, 5.4.8, and 5.4.9 all work in the pointed setting. Finally, all of the arguments of §5.5 work in the pointed setting.

8.3. Framed deformations. Here is where we need to start making some changes. The results of §6.1 hold in the pointed setting, as nowhere do we really use any special hypothesis on A or S . If we keep the hypothesis that S is one-dimensional, then Lemma 6.2.1 fails; we see that $\text{Ext}_A^0(S, S) \cong \mathbb{F}$. Hence $\mathbb{R}\text{End}_A(S)$ is \mathbb{F} -augmented, but not k -augmented (indeed if it were k -augmented then $\text{Ext}_A^0(S, S)$ must contain a copy of k). As a consequence, the rest of §6.2 and §6.3 both fail. However, in this situation, we may forget that everything is a k -algebra, and go back to regarding all our objects as simply \mathbb{F} -algebras. Then one can apply all of our original results.

The natural condition to put on S in the n -pointed setting is that it is a direct sum of n one-dimensional modules, each acted on nontrivially by exactly one of the obvious orthogonal idempotents of k . We’ll show that in fact, S must be k itself; we need to begin by introducing some terminology. Let $e_i \in k = \mathbb{F}^n$ be the idempotent whose i^{th} component is 1 and all of whose other components are zero. Hence $\{e_1, \dots, e_n\}$ are a complete set of

primitive orthogonal idempotents for k . Let S_i be the one-dimensional k -module spanned over \mathbb{F} by e_i ; we call S_i the **vertex simple module** associated to i . If A is an n -pointed \mathbb{F} -algebra, then the augmentation $A \rightarrow k$ allows us to regard the S_i as A -modules.

Lemma 8.3.1. *Let A be an n -pointed \mathbb{F} -algebra, and let S be a one-dimensional A -module. Then S is isomorphic to one of the vertex simple modules S_i .*

Proof. The data of an A -module structure on S is the same as the data of an \mathbb{F} -augmentation of A . Because A is k -augmented, any \mathbb{F} -augmentation of A must factor through k . Hence all such S arise as base changes of one-dimensional k -modules along $A \rightarrow k$. In particular it will suffice to prove the lemma when $A = k$. First note that the map $k \rightarrow \text{End}_k(S) \cong \mathbb{F}$ is necessarily a \mathbb{F} -augmentation of k . Hence it corresponds to an idempotent $e \in k$ by the argument at the beginning of §6.3. An easy calculation shows that the idempotents in k are all of the form $\sum_{i \in I} e_i$, where I is any subset of $\{1, \dots, n\}$. By an automorphism of k permuting the e_i we may as well assume that $e = e_1 + \dots + e_k$ for some $1 \leq k \leq n$. Not all of the e_i can act trivially on S , so by another automorphism permuting the e_i we may assume that e_1 acts nontrivially on S ; in fact because it is an idempotent it must act as id_S . For $i > 1$, if e_i acts nontrivially then it must act as id_S , but then one has $s = e_1 s = e_i e_1 s = 0 s = 0$ for all $s \in S$, a contradiction. Hence $S \cong S_1$. Since this is up to an automorphism of k permuting the e_i , it must be the case that S was one of the S_i . \square

Remark 8.3.2. This says that when $n = 1$, a one-dimensional module over an augmented \mathbb{F} -algebra is necessarily isomorphic to \mathbb{F} with the module structure induced from the augmentation map $A \rightarrow \mathbb{F}$.

Proposition 8.3.3. *Let $S = S^1 \oplus \dots \oplus S^n$ be an n -fold direct sum of k -modules, with each S^i one-dimensional. If for each $0 < i \leq n$ there exists a unique $0 < j \leq n$ such that the action of e_i on S^j is nontrivial, then $S \cong k$.*

Proof. Applying 8.3.1 to each S^j , we see that $S^j \cong S_{\sigma(j)}$ for some endofunction σ of $\{1, \dots, n\}$. By the existence and uniqueness condition in the hypothesis, S must be a permutation, and so $S \cong S_1 \oplus \dots \oplus S_n$. But it is not hard to see that the k -module $S_1 \oplus \dots \oplus S_n$ is isomorphic to k itself. \square

So in what follows, we will take $S = k$, with the understood A -module structure. Lemma 6.2.1 holds in this setting: the t-structure argument gives $\text{Ext}_A^0(S, S) \cong \text{End}_{H^0(A)}(S)$. But because $H^0(A)$ is k -augmented, every $H^0(A)$ -linear endomorphism of k necessarily factors through $H^0(A) \rightarrow k$. So one has $\text{End}_{H^0(A)}(S) \cong \text{End}_k(S) \cong k$, and the rest of the argument goes through. It is easy to now check that 6.2.2 and 6.2.4 hold in the pointed setting.

One must be more careful with 6.2.5. The proof works up until one identifies $\mathcal{G}g(k)(\Gamma) \cong 1 + \mathfrak{m}_\Gamma^0$ with the group of units Γ^\times : this no longer holds in the pointed setting. Indeed, everything in $1 + \mathfrak{m}_\Gamma^0$ is a unit, but the converse is not true: for example if $n = 2$ then $(e_1 - e_2)^2 = 1$, and hence $e_1 - e_2$ is a unit outside of $1 + \mathfrak{m}_\Gamma^0$. In general the group of units is $\Gamma^\times = (\Gamma^0)^\times = k^\times + \mathfrak{m}_\Gamma^0$, where in the first equation we use connectivity. The group of units of k consists exactly of those elements $x = \sum_i \lambda_i e_i$ with all $\lambda_i \neq 0$; the inverse of x is $\sum_i \lambda_i^{-1} e_i$. Say that a unit $u \in \Gamma^\times$ is **pointed** if it lies in the subgroup $1 + \mathfrak{m}_\Gamma^0$; we similarly say that an inner automorphism of Γ is **pointed** if its corresponding unit is pointed. Then the proof of 6.2.5 gives in the pointed setting an isomorphism

$$\mathcal{D}\text{ef}_A(S)(\Gamma) \cong \frac{\text{Hom}_{\text{Ho}(\text{pro}(\text{dgArt}_k))}(B^\sharp E, \Gamma)}{(\text{pointed inner automorphisms of } \Gamma)}.$$

Theorem 6.2.8 holds in the pointed setting if one again replaces ‘inner automorphisms’ with ‘pointed inner automorphisms’.

We move on to §6.3. Let A be a k -algebra and let S be a A -module with endomorphism ring $\text{End}_k(S) \cong k$ (this condition is needed to rule out situations like $S = S_1^{\oplus n}$, or possibly more pathological modules). Then the argument at the start of §6.3 goes through, and one obtains an idempotent $e_S \in A$; one can apply 8.3.3 to conclude that in fact $S \cong k$ as $A/Ae_S A$ -modules. Braun–Chuang–Lazarev’s derived quotient [BCL18] is defined over any base commutative ring, and so we have no problem defining $A/\mathbb{L}Ae_S A$ in the pointed setting, and it has all of the properties we list. The proof of 6.3.5 goes through in the pointed setting: it follows easily from [Boo20, 3.2.2], which holds over any commutative base ring. It is easy to check that 6.3.7, 6.3.8, and 6.3.9 hold in the pointed setting. Moreover, 6.3.10 holds in the pointed setting if, as before, one replaces ‘inner automorphisms’ with ‘pointed inner automorphisms’.

8.4. Prodeformations. Finally, we consider §7. The definition of prodeformations and framed deformations are the same as in the unpointed setting, and it is easy to see that all of the statements in §7.1 hold in the pointed setting. The definition of the universal prodeformation in §7.2 also remains the same. The proof of 7.2.4 goes through in the pointed setting: one only needs to check that the relevant parts of [Pos11] hold over semisimple rings. It is easy to check that all of the arguments of §7.3 adapt to the pointed setting.

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