HOW TO INVERT WELL-POINTED ENDOFUNCTORS

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Abstract. In this short note we observe that Kelly's transfinite construction of free algebras yields a way to invert well-pointed endofunctors. In enriched settings, this recovers constructions of Keller, Seidel, and Chen–Wang. We also relate this procedure to localisation by spectra and to Heller's stabilisation.

1. Enriched preliminaries

Throughout we will let $(\mathcal{V}, \otimes, \mathbb{1})$ be a bicomplete closed symmetric monoidal category. We write the internal hom-objects as $\mathcal{V}(x, y) \in \mathcal{V}$ and the homsets as $\mathrm{Hom}_{\mathcal{V}}(x, y) \in \mathbf{Set}$. We will assume that $\mathrm{Hom}_{\mathcal{V}}(\mathbb{1}, -)$ is faithful, so that we can regard the objects of $\mathcal{V}$ as sets with extra structure (we call such monoidal categories **concrete**). We will moreover assume that $\mathbb{1}$ is compact, so that limits and filtered colimits in $\mathcal{V}$ are created in $\mathbf{Set}$.[1] The reader who does not care for generalities can imagine $\mathcal{V}$ to be $\mathbf{Set}$, $\mathbf{Vect}$, or $\mathbf{dgVect}$. If $\mathcal{C}$ is a $\mathcal{V}$-category, we denote the enriched hom-objects by $\mathcal{C}(x, y) \in \mathcal{V}$ and the underlying homsets by $\mathrm{Hom}_{\mathcal{C}}(x, y) \in \mathbf{Set}$.

If $\mathcal{D}$ is an ordinary category, recall that it has an **ind-category** $\mathrm{ind}\,\mathcal{D}$ whose objects are given by diagrams $X : J \to \mathcal{D}$ where J is small and filtered, and morphisms are given by $\mathrm{Hom}_{\mathrm{ind}\,\mathcal{D}}(X, Y) := \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{D}}(X_i, Y_j)$. Note that $\mathrm{ind}\,\mathcal{D}$ is an accessible category, and is locally finitely presentable provided that $\mathcal{D}$ is cocomplete (e.g. [Isa01, 11.1]). There is an embedding $\mathcal{D} \hookrightarrow \mathrm{ind}\,\mathcal{D}$ sending an object x to the diagram $* \xrightarrow{x} \mathcal{D}$. If $\mathcal{D}$ has filtered colimits, this has an adjoint given by $\varinjlim$.

If $\mathcal{C}$ is a $\mathcal{V}$-category, then since limits and filtered colimits in $\mathcal{V}$ are created in $\mathbf{Set}$ then the exact same formulas provide a canonical $\mathcal{V}$-enrichment for $\mathrm{ind}\,\mathcal{C}$. We denote this enriched ind-category by $\hat{\mathcal{C}}$, so that the underlying category of $\hat{\mathcal{C}}$ is $\mathrm{ind}\,\mathcal{C}$. Again, there is a $\mathcal{V}$-functor $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$, which is universal in the sense that any $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ extends to a $\mathcal{V}$-functor $\hat{F} : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ by requiring it to commute with formal filtered colimits. By construction, $\hat{F}$ is accessible (by which we mean simply that the underlying functor is accessible).

There is a deep theory of enriched accessible categories and the closely related notion of enriched ind-completions [Kel82, BQ96, LT22, LT23]. When the enriching category $\mathcal{V}$ has nontrivial homotopy theory, one also wants enriched ind-categories

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[1] If $\mathcal{V}$ is locally presentable, then $\mathrm{Hom}_{\mathcal{V}}(\mathbb{1}, -)$ has a left adjoint [AR94, 1.66]. If $\otimes$ in addition preserves compact objects, then we are in the setup of [Kel82].

that take this homotopy theory into account: when $\mathcal{V} = \mathbf{dgVect}$ such a *homotopy ind-dg-completion* is given in [GLSV24]. In this note we take a more naïve approach.

2. Well-pointed endofunctors

The arguments in this section are all essentially due to Kelly [Kel80], although we circumvent some of the issues encountered there by passing to ind-categories. Our presentation here was heavily influenced by [nLa25]. From now on, $\mathcal{V}$ is a concrete bicomplete closed symmetric monoidal category with compact unit. All categories, functors, etc. will be enriched over $\mathcal{V}$. A **pointed endofunctor** on a category $\mathcal{C}$ is a natural transformation $\theta : \mathrm{id} \to \Omega$ of functors on $\mathcal{C}$. Say that (Ω, θ) is **well-pointed** if $\theta\Omega = \Omega\theta$: for all X we have $\theta_{\Omega X} = \Omega(\theta_X)$ as maps $\Omega X \to \Omega^2 X$. An **Ω-algebra** is an object X together with a map $\Omega X \to X$ such that the composition $X \xrightarrow{\theta_X} \Omega X \to X$ is the identity. There is an evident category of Ω-algebras $\mathbf{Alg}(\Omega)$, constructed as a slice category.

Lemma 1. *If θ is well-pointed then an object X admits the structure of an Ω-algebra if and only if θ_X is invertible; in this case the algebra structure is unique.*

Proof. This is [Kel80, Proposition 5.2]. If θ_X is invertible then one takes the algebra structure map $\Omega X \to X$ to be its inverse. Conversely, if $f : \Omega X \to X$ is any morphism then well-pointedness yields a commutative diagram

$$\begin{array}{ccc} \Omega X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle \Omega\theta_X} & & \downarrow{\scriptstyle \theta_X} \\ \Omega^2 X & \xrightarrow{\Omega f} & \Omega X \end{array}$$

which shows that $\theta_X f = \Omega(f\theta_X)$. If f is an algebra then this shows that f is both a left and right inverse of θ_X, and thus θ_X is invertible. It is clear that the algebra structure must be unique. $\square$

In particular, if θ is well-pointed then the category $\mathbf{Alg}(\Omega)$ is naturally a full subcategory of $\mathcal{C}$. By extending (Ω, θ) to a well-pointed endofunctor $(\hat{\Omega}, \hat{\theta})$ of $\hat{\mathcal{C}}$, we see that we may define a functor $\hat{\Omega}^\infty : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ by

$$\hat{\Omega}^\infty(X) \coloneqq \varinjlim \left(X \xrightarrow{\hat{\theta}_X} \hat{\Omega} X \xrightarrow{\hat{\theta}_{\Omega X}} \hat{\Omega}^2 X \xrightarrow{\hat{\theta}_{\Omega^2 X}} \cdots \right)$$

where we take the colimit in the ind-category.[2]

Theorem 2. *If $\mathcal{C}$ is cocomplete, then $\hat{\Omega}^\infty$ is a reflection of $\hat{\mathcal{C}}$ into $\mathbf{Alg}(\hat{\theta})$.*

Proof. This is [Kel80, Remark 6.3], which applies since $\hat{\mathcal{C}}$ is locally presentable and in particular well-copowered. The idea is simple: by construction $\hat{\Omega}$ is accessible, so for any ind-object X we obtain a natural map $\hat{\Omega}\hat{\Omega}^\infty(X) \to \hat{\Omega}^\infty(X)$ that makes $\hat{\Omega}^\infty(X)$ into an $\hat{\Omega}$-algebra. It follows that $\hat{\Omega}^\infty$ is a reflection of $\hat{\mathcal{C}}$ into $\mathbf{Alg}(\hat{\mathcal{C}})$. $\square$

From now on we assume that $\mathcal{C}$ is cocomplete. The following definition, at least in the enriched setting, is due to Wolff [Wol73, Wol74]:

[2] If $X : J \to \mathcal{C}$ is a filtered diagram, then the colimit of the associated diagram $J \xrightarrow{X} \mathcal{C} \to \hat{\mathcal{C}}$ is precisely the ind-object X. One can easily prove this using the Yoneda lemma.

Definition 3. A functor $F : \mathcal{C} \to \mathcal{D}$ **inverts** a natural transformation θ between endofunctors of $\mathcal{C}$ if for every X in $\mathcal{C}$, the morphism $F(\theta_X)$ is an isomorphism. The **localisation of** $\mathcal{C}$ **along** θ is the initial functor that inverts θ; i.e. it is a functor $\gamma : \mathcal{C} \to \mathcal{C}'$ such that if $F : \mathcal{C} \to \mathcal{D}$ inverts θ then there exists a unique $F' : \mathcal{C}' \to \mathcal{D}$ such that $F = F'\gamma$.

Let Ω^∞ denote the composition $\mathcal{C} \hookrightarrow \hat{\mathcal{C}} \xrightarrow{\hat{\Omega}^\infty} \mathbf{Alg}(\hat{\Omega})$. We have an isomorphism $\mathbf{Alg}(\hat{\Omega})(\Omega^\infty X, \Omega^\infty Y) \cong \varprojlim_n \varinjlim_m \mathcal{C}(\Omega^n X, \Omega^m Y)$, since $\mathbf{Alg}(\hat{\Omega})$ is a full subcategory of $\hat{\mathcal{C}}$. On the other hand we also have isomorphisms

$$\mathbf{Alg}(\hat{\Omega})(\Omega^\infty X, \Omega^\infty Y) \ \cong\ \hat{\mathcal{C}}(X, \Omega^\infty Y) \ \cong\ \varinjlim_m \mathcal{C}(X, \Omega^m Y)$$

which will be of more use to us. Write $L_\Omega(\mathcal{C}) \hookrightarrow \hat{\mathcal{C}}$ for the essential image of Ω^∞.

Theorem 4. *$\Omega^\infty : \mathcal{C} \to L_\Omega(\mathcal{C})$ is the localisation of $\mathcal{C}$ at θ.*

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor such that every $F(\theta_X)$ is an isomorphism. Extend F to a functor $\hat{F} : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ and consider the composition $\hat{F}\Omega^\infty$. By construction we have

$$\hat{F}\Omega^\infty X \ \cong\ \varinjlim \big(FX \to F\Omega X \to F\Omega^2 X \to \cdots\big)$$

but by assumption, every map in this colimit is an isomorphism, and so we see that $\hat{F}\Omega^\infty \cong F$. In other words, F factors through the essential image of Ω^∞. We need to check that the factoring map $\hat{F}$ is unique. So let $G : L_\Omega(\mathcal{C}) \to \mathcal{D}$ be any functor such that $G\Omega^\infty = F$. Pick $X \in L_\Omega(\mathcal{C})$. Since $L_\Omega(\mathcal{C})$ is defined to be the essential image of Ω^∞, there must be $X' \in \mathcal{C}$ such that $X \cong \Omega^\infty X'$, and hence $G(X) = F(X') = \hat{F}(X)$. Let

$$G_{\Omega^\infty X, \Omega^\infty Y} :\ L_\Omega(\mathcal{C})(\Omega^\infty X, \Omega^\infty Y) \longrightarrow \mathcal{D}(G\Omega^\infty X, G\Omega^\infty Y)$$

be the component maps of G. Replacing $L_\Omega(\mathcal{C})(\Omega^\infty X, \Omega^\infty Y)$ by a colimit as above, we see that $G_{\Omega^\infty X, \Omega^\infty Y}$ is an inverse limit of maps of the form

$$\phi_m :\ \mathcal{C}(X, \Omega^m Y) \longrightarrow \mathcal{D}(G\Omega^\infty X, G\Omega^\infty Y).$$

We have a commutative diagram in $\mathcal{V}$ (cf. the proof of [Sei08, Lemma 1.1])

$$\begin{array}{ccc} \mathcal{C}(X, \Omega^m Y) & \xrightarrow{\quad \phi_m \quad} & \mathcal{D}(G\Omega^\infty X, G\Omega^\infty Y) \\ & {\searrow}^{G\Omega^\infty_{X,\Omega^m Y}} \qquad {\swarrow}_{\psi} & \\ & \mathcal{D}(G\Omega^\infty X, G\Omega^\infty \Omega^m Y) & \end{array}$$

where ψ is induced by the canonical morphism $Y \to \Omega^m Y$. Because Ω^∞ inverts θ, it follows that ψ is an isomorphism. In particular, $G_{\Omega^\infty X, \Omega^\infty Y}$ is the inverse limit of the system of maps $G\Omega^\infty_{X,\Omega^m Y} = \hat{F}\Omega^\infty_{X,\Omega^m Y}$. Running the same argument for $\hat{F}$ shows that G must be naturally isomorphic to $\hat{F}$. □

3. Examples

Here we let k be a field; all categories will be linear over k.

Example 5. Let $\mathcal{C}$ be a k-linear category and $T : F \to \mathrm{id}$ a well-copointed[3] endofunctor on $\mathcal{C}$. Running our constructions in $\mathcal{C}^{\mathrm{op}}$ yields a localisation $L_F(\mathcal{C}^{\mathrm{op}})$ that agrees

[3] i.e. T^{op} is well-pointed; Seidel uses the term **ambidextrous** [Sei08].

with Seidel's construction [Sei08]. In particular, if $\mathcal{C}$ is a pretriangulated dg category and F is a dg functor, then $L_F(\mathcal{C}^{\mathrm{op}})$ can be identified as the dg quotient of $\mathcal{C}^{\mathrm{op}}$ by the pretriangulated subcategory spanned by those objects that are annihilated by some power of F [Sei08, Lemma 1.3].

Example 6. Let $\mathcal{C}$ be a dg-k-category and $\theta : \mathrm{id} \to \Omega$ a well-pointed dg endofunctor on $\mathcal{C}$. Then $L_\Omega(\mathcal{C})$ is precisely the localisation $\mathcal{S}\mathcal{C}$ constructed by Chen and Wang [CW24, §6][4]. Hence if $\mathcal{C}$ is pretriangulated then $L_\Omega(\mathcal{C})$ is a model for the dg quotient $\mathcal{C}/\mathbf{thick}\,(\mathrm{cone}(\theta_{\mathrm{X}}) : \mathrm{X} \in \mathcal{C})$ by [CW24, Theorem 6.4]. Note that $L_\Omega(\mathcal{C})$ is a strictification of Keller's ind-categorical description of the dg quotient [Kel99]. Indeed, if $\mathcal{D}$ is a pretriangulated dg subcategory of $\mathcal{C}$ then the dg quotient $\mathcal{C}/\mathcal{D}$ can be described as the subcategory of $\hat{\mathcal{C}}$ on those ind-objects X right orthogonal to $\mathcal{D}$ and which fit into an exact triangle $c \to X \to Y \to$ with $c \in \mathcal{C}$ and $Y \in \hat{\mathcal{D}}$, as made clear in [Dri04, 4.9]. This provides a high-level viewpoint on some computations of stable Ext made by the author in [Boo21, Theorem 6.4.6].

Example 7. Let $\mathcal{A}$ be a dg-k-category and $F : \mathcal{A} \to \mathcal{A}$ a dg endofunctor. Define a new dg category $\mathcal{A}_F$ with the same objects as $\mathcal{A}$, and hom-complexes given by $\mathcal{A}_F(X,Y) \coloneqq \oplus_n \mathcal{A}(F^n X, Y)$. The composition of $F^i X \to Y$ and $F^j Y \to Z$ is given by $F^{i+j} X \to F^i Y \to Z$. The resulting endofunctor F of $\mathcal{A}_F$ is well-pointed, by the natural transformation with components $\mathrm{id}_{FX} \in \mathcal{A}_F(X, FX)$; this is in fact the universal way to make F well-pointed. Then $L_F(\mathcal{A}_F)$ is Keller's dg orbit category [Kel05, 5.1]. Note that $L_F(\mathcal{A}_F)$ need not be pretriangulated, even if $\mathcal{A}$ was.

4. Spectra

As above, all categories, functors, etc. remain enriched over $\mathcal{V}$. Let $\mathcal{C}$ be a category and Ω an endofunctor.[5] A **spectrum** is a sequence $X_0, X_1, X_2, \ldots$ of objects in $\mathcal{C}$ with morphisms $\sigma_n : X_n \to \Omega X_{n+1}$. A spectrum is an **$\Omega$-spectrum** when the morphisms σ_n are all isomorphisms.[6] There is an evident category $\mathrm{Sp}_\Omega(\mathcal{C})$ of spectra together with a full subcategory $\underline{\mathrm{Sp}}_\Omega(\mathcal{C})$ of Ω-spectra. Since limits in $\mathcal{V}$-**Cat** are computed pointwise, there is an equivalence of categories

$$\underline{\mathrm{Sp}}_\Omega(\mathcal{C}) \cong \varprojlim \left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right)$$

and when $\mathcal{V} = \mathbf{Set}$ then $\mathrm{Sp}_\Omega(\mathcal{C})$ can also be obtained as the analogous 2-limit taken in **Cat**.[7] Observe that the map $X \mapsto X_n$ which assigns a spectrum its n^{th} level can be regarded as a functor $\mathrm{Sp}_\Omega(\mathcal{C}) \to \mathcal{C}$.

There is a shift endofunctor S of $\mathrm{Sp}_\Omega(\mathcal{C})$ given on sequences by $(SX)_i = X_{i+1}$. The Ω functor extends to an endofunctor of $\mathrm{Sp}_\Omega(\mathcal{C})$, and one can easily check that $\Omega S = S\Omega$. There is a natural transformation $\sigma : \mathrm{id} \to \Omega S$ defined on sequences by $\sigma_n : X_n \to \Omega X_{n+1} = \Omega S(X_n)$, making ΩS into a well-pointed endofunctor.[8]

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[4]The motivating example of [CW24] is the case when $\mathcal{C}$ is the **Yoneda dg category** of an algebra A and Ω is the **noncommutative differential forms** functor; the localisation $\mathcal{S}\mathcal{C}$ is then a model for the dg singularity category of A.

[5]When $\mathcal{V} = \mathbf{Ab}$ then this is precisely the notion of **looped category** from [Bel00].

[6]One sometimes calls the first kind of object a **prespectrum** and the other simply a **spectrum**.

[7]Presumably a similar statement holds for general $\mathcal{V}$, possibly with some additional assumptions.

[8]The argument showing that ΩS is well-pointed is precisely the argument which shows that σ is a well-defined natural transformation.

Let $\mathcal{L}$ denote the localisation $L_{\Omega S}(\mathrm{Sp}_\Omega \mathcal{C})$, which is a a subcategory of the category of ind-spectra $\widehat{\mathrm{Sp}_\Omega}(\mathcal{C})$. This category comes equipped with a localisation functor $\Omega^\infty S^\infty \coloneqq (\Omega S)^\infty : \mathrm{Sp}_\Omega(\mathcal{C}) \to \mathcal{L}$. Since Ω and S commute, so do $\hat{\Omega}$ and $\hat{S}$, and hence they are mutually inverse functors on $\mathcal{L}$.

Observe that there is a natural fully faithful functor $\iota : \widehat{\mathrm{Sp}_\Omega}(\mathcal{C}) \to \mathrm{Sp}_{\hat{\Omega}}(\hat{\mathcal{C}})$ defined as follows. If $X : J \to \mathrm{Sp}_\Omega(\mathcal{C})$ is an ind-spectrum, then $(\iota X)_n$ is the ind-object $J \xrightarrow{X} \mathrm{Sp}_\Omega(\mathcal{C}) \xrightarrow{(-)_n} \mathcal{C}$. The connecting maps are obtained analogously.[9]

We refer to the composition $\iota\Omega^\infty S^\infty$ as the **spectrification** functor; by construction its image lies in the subcategory $\underline{\mathrm{Sp}}_{\hat{\Omega}}(\hat{\mathcal{C}})$. One can easily compute that if X is a spectrum, we have $(\iota\Omega^\infty S^\infty X)_n \cong \varinjlim \big(X_n \to \Omega X_{n+1} \to \Omega^2 X_{n+2} \to \cdots\big)$, where we take the filtered colimit in $\hat{\mathcal{C}}$. The structure maps are induced from those of X.

Example 8. When $\mathcal{C}$ has filtered colimits, the composition

$$\varinjlim \circ\, \iota\Omega^\infty S^\infty : \mathrm{Sp}_\Omega(\mathcal{C}) \to \underline{\mathrm{Sp}}_\Omega(\mathcal{C})$$

is (an enriched version of) the classical spectrification appearing in e.g. [LMSM86].

Example 9. Suppose that the endofunctor Ω was actually well-pointed, by a natural transformation θ. This yields a functor $\Theta : \mathcal{C} \to \mathrm{Sp}_\Omega(\mathcal{C})$ defined by $\Theta(X)_n = X$, with the structure maps $\sigma_n : X \to \Omega X$ given by θ. Then the spectrification of $\Theta(X)$ has at all levels the localisation $\Omega^\infty(X)$.

Example 10. Suppose that the endofunctor Ω admits a left adjoint Σ. This yields a functor $\Sigma^\infty : \mathcal{C} \to \mathrm{Sp}_\Omega(\mathcal{C})$ by putting $\Sigma^\infty(X)_n = \Sigma^n X$. The structure map $\Sigma^n X \to \Omega\Sigma^{n+1} X$ is the adjunct of the identity map on Σ^{n+1}. Note that by composition with Ω^n this yields maps $\Omega^n\Sigma^n X \to \Omega^{n+1}\Sigma^{n+1} X$. Put

$$\Omega^\infty\Sigma^\infty X \coloneqq \varinjlim \big(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots\big)$$

where again we take the filtered colimit in $\hat{\mathcal{C}}$. This construction is topologically known as the **free infinite loop space on** X. One can check that the n^{th} level of the spectrification of $\Sigma^\infty X$ is precisely $\Omega^\infty\Sigma^\infty(\Sigma^n X)$, which recovers the classical topological fact that $\Omega^\infty\Sigma^\infty X$ is the zeroth level of the spectrification of $\Sigma^\infty X$.

Remark 11. For the purposes of algebraic topology, especially constructing a symmetric monoidal smash product of spectra, the above approach is known to be completely inadequate [Lew91]. One either needs to use model categories of highly structured spectra, as in e.g. [MMSS01], or use ∞-categories from the beginning, as in [Lur17]. We note that similar constructions to that of this section in a homotopy-invariant setting have already been given in [Hel97, §8].

5. Stabilisation, cospectra, and comparisons

Once again we work in the enriched setting. Let $\mathcal{C}$ be a category and Ω an endofunctor of $\mathcal{C}$. Following Heller [Hel68, §1], we define a new category $\mathcal{S}_\Omega\mathcal{C}$, the **stabilisation** of $\mathcal{C}$, as follows. The objects are the pairs (c, i) with $c \in \mathcal{C}$ and $i \in \mathbb{Z}$. The morphisms are defined to be

$$\mathcal{S}_\Omega\mathcal{C}((c,i),(d,j)) \coloneqq \varinjlim_k \mathcal{C}(\Omega^{k+i}c, \Omega^{k+j}d)$$

[9]More abstractly, a spectrum is a certain kind of pro-object, and the natural comparison functor indpro$\mathcal{C} \to$ proind$\mathcal{C}$ gives the map from ind-spectra to spectra in ind-objects.

with composition inherited from $\mathcal{C}$. For brevity we will write $[-,-]$ for the hom-objects in $\mathcal{S}_\Omega\mathcal{C}$; with this notation we clearly have $[(c,i),(d,j)] \simeq [(c,i+l),(d,j+l)]$ for all $l \in \mathbb{Z}$. The functor Ω extends to the stabilisation by putting $\Omega(c,i) \coloneqq (\Omega c, i)$, and one can easily verify via the Yoneda lemma that there is a natural isomorphism $\Omega(c,i) \cong (c,i+1)$. In particular, Ω is an autoequivalence of $\mathcal{S}_\Omega\mathcal{C}$, with inverse $(c,i) \mapsto (c,i-1)$. There is an obvious functor $\mathcal{C} \to \mathcal{S}_\Omega\mathcal{C}$ sending c to $(c,0)$, which is universal with respect to stabilising Ω [Hel68, Proposition 1.1].

Observe that there is a natural comparison map $\Phi : \underline{\mathrm{Sp}}_\Omega(\mathcal{C}) \to \mathcal{S}_\Omega\mathcal{C}$ defined by sending a spectrum X to the pair $(X_0, 0) \cong (X_i, i)$.

Proposition 12. *Suppose that $\theta : \mathrm{id} \to \Omega$ is a well-pointed endofunctor on a locally finitely presentable*[10] *category $\mathcal{C}$. We denote by $\Omega^\infty : \mathcal{C} \to \mathcal{C}$ the corresponding localisation functor, with image $L_\Omega\mathcal{C} \hookrightarrow \mathcal{C}$.*

1. *The localisation $L_\Omega\mathcal{C}$ is a coreflective subcategory of $\mathcal{S}_\Omega\mathcal{C}$, with coreflection given by the functor η which sends (d,i) to $(\Omega^\infty(d),0) \cong (\Omega^\infty(d),n)$.*
2. *The localisation $L_\Omega\mathcal{C}$ is a coreflective subcategory of $\underline{\mathrm{Sp}}_\Omega(\mathcal{C})$, with coreflection given by the functor ε which sends a spectrum X to the constant spectrum on $\Omega^\infty(X_0)$ (with structure maps as in Example 9).*
3. *There is a natural comparison map $\Psi : \mathcal{S}_\Omega\mathcal{C} \to \underline{\mathrm{Sp}}_\Omega(\mathcal{C})$ which sends (c,i) to the constant spectrum on $\Omega^\infty(c)$.*
4. *There are natural isomorphisms $\Phi\Psi \cong \eta$ and $\Psi\Phi \cong \varepsilon$.*
5. *The following are equivalent:*
   - *Φ is an equivalence, with inverse Ψ.*
   - *Both $\underline{\mathrm{Sp}}_\Omega(\mathcal{C})$ and $\mathcal{S}_\Omega\mathcal{C}$ are naturally equivalent to $L_\Omega\mathcal{C}$.*

Proof. For (1), the inclusion functor is the composition $L_\Omega\mathcal{C} \hookrightarrow \mathcal{C} \to \mathcal{S}_\Omega\mathcal{C}$; this is fully faithful since $\Omega^k\Omega^\infty \cong \Omega^\infty$ as functors on $\mathcal{C}$. For the coreflection, we compute

$$[(\Omega^\infty c, 0),(d,i)] \cong \varinjlim_k \mathcal{C}(\Omega^\infty c, \Omega^{k+i}d) \cong \varinjlim_k \mathcal{C}(\Omega^\infty c, \Omega^k d) \cong L_\Omega\mathcal{C}(\Omega^\infty c, \Omega^\infty d)$$

where in the last step we use the natural isomorphism $\Omega^\infty\Omega^\infty \cong \Omega^\infty$. The proof of (2) is similar; here the inclusion functor is the composition $L_\Omega\mathcal{C} \hookrightarrow \mathcal{C} \xrightarrow{\Theta} \underline{\mathrm{Sp}}_\Omega(\mathcal{C})$ where Θ is the functor of Example 9. For (3), since $\underline{\mathrm{Sp}}_\Omega(\mathcal{C})$ stabilises Ω, the universal property of the stabilisation ensures the existence of Ψ and the proof of [Hel68, Proposition 1.1] yields the desired description. Claim (4) is a simple computation and claim (5) follows easily. □

Remark 13. When $\mathcal{V} = \mathbf{Set}$, one can regard $\mathcal{S}_\Omega\mathcal{C}$ as the colimit of the diagram $\mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \cdots$, which one could call the category of **Ω-cospectra**.[11,12] If J denotes the doubly-infinite diagram $\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \cdots$ then we obtain a natural comparison map $\underline{\mathrm{Sp}}_\Omega(\mathcal{C}) \cong \varprojlim J \longrightarrow \varinjlim J \cong \mathcal{S}_\Omega\mathcal{C}$ which agrees with the comparison map Φ defined above. Hence, in this setting, Φ is an equivalence precisely when Ω **has eventual image duality** in the sense of [Lei24]. For more on the duality between spectra and cospectra, see [Gra95, §4].

[10] One can remove this assumption by replacing $\mathcal{C}$ by $\hat{\mathcal{C}}$; for readability we refrain from doing this.

[11] More generally, this holds when $\mathcal{V}$ is a presheaf category (e.g. **sSet**), since colimits in $\mathcal{V}$ are computed pointwise. In general, colimits in $\mathcal{V}$-**Cat** can be computed as in [Wol74].

[12] To obtain the cospectra of [Lim59], one should instead take the corresponding 2-colimit. Presumably one can then adapt the arguments of the previous section to construct a cospectrification functor which replaces a cospectrum by an Ω-cospectrum. Note that [AI22] refers to the higher-categorical version of cospectra as **telescopes**.

Remark 14. Suppose that $\theta : \mathrm{id} \to \Omega$ is a well-pointed endofunctor on $\mathcal{C}$. Although both $\underline{\mathrm{Sp}}_{\Omega}(\mathcal{C})$ and $\mathcal{S}_{\Omega}\mathcal{C}$ satisfy a universal property with respect to stabilising Ω, neither construction need actually invert the map θ.

Remark 15. For certain left triangulated categories $(\mathcal{C}, \Omega)$, the stabilisation $\mathcal{S}_{\Omega}\mathcal{C}$ can be realised as a generalised singularity category [Bel00, Theorem 3.8], cf. [Buc21, KV87, CW25]. Dually, for certain right triangulated categories, the costabilisation $\underline{\mathrm{Sp}}_{\Omega}(\mathcal{C})$ has a similar interpretation [Bel00, Theorem 3.11], cf. [Gra95].

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