REFLEXIVE DG CATEGORIES IN ALGEBRA AND TOPOLOGY

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Abstract. Reflexive dg categories were introduced by Kuznetsov and Shinder to abstract the duality between bounded and perfect derived categories. In particular this duality relates their Hochschild cohomologies, autoequivalence groups, and semiorthogonal decompositions. We establish reflexivity in a variety of settings including affine schemes, simple-minded collections, chain and cochain dg algebras of topological spaces, Ginzburg dg algebras, and Fukaya categories of cotangent bundles and surfaces as well as the closely related class of graded gentle algebras. Our proofs are based on the interplay of reflexivity with gluings, derived completions, and Koszul duality. In particular we show that for certain (co)connective dg algebras, reflexivity is equivalent to derived completeness.

Contents

1. Introduction

Duality theorems are abundant in algebra and geometry. Reflexivity - recently introduced in [KS25] and with a precursor in [BZNP17] - generalises the duality between bounded and perfect derived categories seen in [Bal11, BZNP17, Che21]. Given a dg category $\mathcal{C}$ over a field k, its **perfectly valued dg category** $\mathcal{D}_{\mathrm{fd}}(\mathcal{C})$ consists of the dg modules over $\mathcal{C}$ which take values in the category of perfect complexes over k. For finite dimensional algebras and proper schemes, this construction permutes their perfect and bounded derived categories. The functor $\mathcal{C} \mapsto \mathcal{D}_{\mathrm{fd}}(\mathcal{C})$ is contravariant, and a dg category $\mathcal{C}$ is called **reflexive** if the natural functor

$$\mathcal{C} \to \mathcal{D}_{\mathrm{fd}}\big(\mathcal{D}_{\mathrm{fd}}(\mathcal{C})\big)$$

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is a Morita equivalence. In [KS25] it was shown that for a reflexive dg category $\mathcal{C}$, there is an isomorphism between the triangulated autoequivalence groups of $\mathcal{D}^{\mathrm{perf}}(\mathcal{C})$ and of $\mathcal{D}_{\mathrm{fd}}(\mathcal{C})$ and a bijection between semiorthogonal decompositions. In [Goo24], reflexive dg categories are shown to be exactly the reflexive objects in the Morita homotopy category. As a consequence, the Hochschild cohomologies and derived Picard groups of $\mathcal{D}^{\mathrm{perf}}(\mathcal{C})$ and of $\mathcal{D}_{\mathrm{fd}}(\mathcal{C})$ are shown to coincide. Note that the analogous statement for Hochschild *homology* is false [Goo24, Example 5.6]. The main families of dg categories currently known to be reflexive are

- the (perfect or bounded) derived categories of proper schemes;
- proper connective dg algebras;
- homologically smooth and proper dg categories;
- Fukaya categories of Milnor fibres of certain hypersurface singularities.

The purpose of this article is to expand this list. Important is that we work almost always over *arbitrary* fields k, unlike the earlier [BZNP17] and [KS25] who work over characteristic zero and perfect fields respectively.

Affine schemes.
We characterise all reflexive noetherian affine schemes:

Theorem A (Theorem 4.0.4)**.** *Let R be a commutative noetherian k-algebra. Then $\mathcal{D}^{\mathrm{perf}}(R)$ is reflexive if and only if R is a finite product of complete local k-algebras, each of which has a residue field which is a finite extension of k.*

Simple-minded collections and silting objects.
Simple-minded collections and silting objects in triangulated categories abstract simple modules and projective generators for finite dimensional algebras [KY14]. Algebraic triangulated categories admitting simple-minded collections or silting objects are Morita equivalent to certain (co)connective dg algebras. The theorem below is a simplification of our results:

Theorem B (cf. Theorem 5.3.2, Theorem 5.4.1)**.** *Let A be a* ***locally proper*** *dg algebra, i.e. $H^i(A)$ is finite dimensional for all $i \in \mathbb{Z}$.*

(1) *If A is connective and $H^0(A)$ is local then A is reflexive.*
(2) *If A is strictly coconnective, $A^1 \cong 0$, and $H^0(A) \cong A^0$ is a commutative semisimple k-algebra, then A is reflexive.*

We also give a version for smooth coconnective dg algebras (Corollary 5.4.5). As corollaries of our more general theorems, we give some reflexivity theorems for simple-minded collections (Example 3.3.3 and Example 5.4.3), silting objects (Corollary 3.4.2), and relative singularity categories (Example 5.3.5).

Topological spaces. Associated to a topological space X are two natural dg-k-algebras: the algebra of cochains $C^\bullet(X,k)$ and the algebra of chains on loops $C_\bullet(\Omega X,k)$. The following is a simplified version of the results of Section 7.2:

Theorem C. *Let X be a path connected topological space and k a field. Suppose that $H_i(\Omega X,k)$ is finite dimensional for all $i \geq 0$. Then both $C^\bullet(X,k)$ and $C_\bullet(\Omega X,k)$ are reflexive as long as either of the following conditions hold:*

(1) *X is simply connected.*
(2) *$\pi_1(X)$ is a finite p-group, where $p = \mathrm{char}(k) > 0$.*

Moreover, there are equivalences

$$\mathcal{D}_{\mathrm{fd}}(C^\bullet X) \simeq \mathcal{D}^{\mathrm{perf}}(C_\bullet \Omega X) \qquad \mathcal{D}^{\mathrm{perf}}(C^\bullet X) \simeq \mathcal{D}_{\mathrm{fd}}(C_\bullet \Omega X).$$

We remark that the assumptions of the theorem are satisfied by finite simply connected CW complexes (in fact, $C^\bullet(X,k)$ is reflexive whenever $\oplus_{i\in\mathbb{N}} H_i(X,k)$ is finite dimensional) and classifying spaces of finite p-groups (where we moreover have $C_\bullet \Omega BG \simeq kG$). We give further examples from string topology, rational homotopy theory, p-complete groups, and symplectic geometry. Note that $\mathcal{D}_{\mathrm{fd}}(C_\bullet \Omega X)$ can be interpreted as the category of ∞-local systems on X with finite fibres.

Ginzburg dg algebras and Calabi–Yau completions.

Ginzburg dg algebras are a class of Calabi–Yau dg algebras first introduced in [Gin06] and constructed from a quiver with superpotential. Under certain assumptions, *all* complete exact Calabi–Yau dg algebras are Ginzburg dg algebras [VdB15]. In [Kel09], Ginzburg dg algebras were interpreted as deformed 3-Calabi–Yau completions. We prove the following (cf. Proposition 6.1.4, Proposition 6.2.5):

Theorem D. *Let Q be a finite quiver and k a field of characteristic zero.*

(1) *For all $n \geq 2$, the (completed, undeformed) n-Calabi–Yau completion $\hat{\Pi}_n(Q)$ is reflexive.*
(2) *Let W be a superpotential on Q such that the cycles appearing in W have length at least 3. Then the completed Ginzburg dg algebra $\hat{\Gamma}(Q,W)$ associated to the pair (Q,W) is reflexive.*

Fukaya categories of surfaces and graded gentle algebras.

We study the reflexivity of topological Fukaya categories of surfaces in the sense of [HKK17]. These categories admit formal generators whose endomorphism dg algebras are graded versions of **gentle algebras** [AS87]. The gentle algebras which arise in this way are smooth, but not necessarily proper. The Koszul dual perspective, that of *finite dimensional* graded gentle algebras, has been studied by many authors [OPS18, Opp19, LP20, APS23].

Theorem E. *Let Σ be a graded marked surface and let* $\mathrm{Fuk}(\Sigma)$ *denote its partially wrapped Fukaya category. If Σ contains at least one boundary arc and contains no boundary component which is marked entirely and which has vanishing winding number, then* $\mathrm{Fuk}(\Sigma)$ *is reflexive.*

We believe that this result is sharp (cf. Example 9.4.6). Theorem E is a consequence of the following statement:

Theorem F. *Let A be a finite dimensional graded gentle algebra. Then A is reflexive and $\mathcal{D}_{\mathrm{fd}}(A)$ is generated as a thick subcategory by the simple A-modules, i.e.* $\mathbf{thick}_A(A/\operatorname{rad}(A)) = \mathcal{D}_{\mathrm{fd}}(A)$.

In the newest update of [OPS18], the authors describe a geometric model for the thick closure of $A/\operatorname{rad}(A)$. Since the grading of A can be arbitrary (in particular nonconnective), no previous techniques were able to show that this geometric model does indeed describe the whole category $\mathcal{D}_{\mathrm{fd}}(A)$. The proof of Theorem F is heavily based on the following result, which describes the behaviour of reflexivity under semiorthogonal gluing:

Theorem G. *Let $\mathcal{T}$ be a proper dg category. If $\mathcal{D}^{\mathrm{perf}}(\mathcal{T})$ admits a semiorthogonal decomposition into reflexive dg categories, then $\mathcal{T}$ is reflexive.*

The theorem holds true under the more general assumption that $\mathcal{T}$ is **semireflexive** (cf. Theorem 8.0.3). To prove Theorem F, one exploits that $\mathcal{D}^{\mathrm{perf}}(A)$ admits a semiorthogonal decomposition into perfect derived categories of simpler gentle algebras whose reflexivity follows easily from our other criteria.

Methodology. The key technical insight of this paper can be summed up by the following slogan:

Reflexivity = well-behaved $\mathcal{D}_{\mathrm{fd}}$-generators + derived completeness

If A is a dg algebra and M an A-module, we put $A^{!!}_M := \mathbb{R}\mathrm{End}_{\mathbb{R}\mathrm{End}_A(M)}(M)$ and regard $A^{!!}_M$ as a derived completion of A along M. Indeed, in algebro-geometric settings this often recovers the adic completion of A at an ideal [DGI06]. To make the above slogan precise, we prove the following 'two-out-of-three' theorem:

Theorem H (Theorem 2.3.8)**.** *Let A be a dg algebra and M a thick generator for $\mathcal{D}_{\mathrm{fd}}(A)$. If any two of the following hold then so does the third:*
(1) *A is reflexive.*
(2) *M is a thick generator for $\mathcal{D}_{\mathrm{fd}}(\mathbb{R}\mathrm{End}_A(M))$.*
(3) *The derived completion map $A \to A^{!!}_M$ is a quasi-isomorphism.*

It is hence important for us to identify thick generators for $\mathcal{D}_{\mathrm{fd}}(A)$; indeed, Lemma 4.0.2 is the key input to the proof of Theorem A. For connective dg algebras one can do this using the standard t-structure, and for coconnective dg algebras with semisimple H^0 one can do this using the co-t-structures of [KN13]. This allows us to prove the following precursor to Theorem B:

Theorem I (Theorem 3.3.1 and Proposition 3.4.1)**.** *Let A be a dg algebra with $H^0(A)$ finite dimensional. Suppose that either of the following conditions hold:*
- *A is connective.*
- *A is coconnective and $H^0(A)$ is semisimple.*

Then A is reflexive if and only if it is derived complete at $H^0(A)/\operatorname{rad} H^0(A)$.

Now we have dealt with the existence of good generators, we need a method to check when an algebra is derived complete. To do this we pass through the well-known relationship between derived completion and Koszul duality. Indeed, Theorem B above follows by combining Theorem I with derived completion results of the kind established in [Boo22] using Koszul duality. This approach to reflexivity via Koszul duality already has an antecedent in [LU22]. We also develop a notion of reflexivity for dg coalgebras, and show that a dg coalgebra C is reflexive precisely when its Koszul dual dg algebra ΩC is (Proposition 5.5.4). The interaction between reflexivity of C and reflexivity of its linear dual C^* is subtle, but their interaction is key to our proof of Theorem C via the Koszul duality between the dg algebra of chains on loops and the dg coalgebra of chains [RZ18]. Koszul duality is moreover crucial to the proof of Theorem D, as one needs to identify Ginzburg dg algebras as Koszul duals of certain cyclic A_∞-algebras, as in [Seg08, Kel09].

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1.1. **Notation and conventions.** Throughout, k will denote a fixed base field. A **complex** is a cochain complex of k-vector spaces. We will usually use cohomological indexing for complexes; one can convert between homological and cohomological indexing by the formula $M_i = M^{-i}$. The shift of a complex M will be denoted by $M[1]$, so that $M[1]^i \cong M^{i+1}$. The category of complexes is closed symmetric monoidal, with product given by the usual tensor product of complexes.

A **dg category** is a category enriched in complexes. The derived category of right modules over a dg category $\mathcal{A}$ will be denoted by $\mathcal{D}(\mathcal{A})$; it is a pretriangulated dg category. An $\mathcal{A}$-module M is **proper** or **perfectly valued** if $H^*(M(a))$ is a finite dimensional graded vector space for each $a \in \mathcal{A}$. The derived category of perfectly valued modules over $\mathcal{A}$ will be denoted $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$; it is a pretriangulated dg category. A **perfect** module is a compact object of $\mathcal{D}(\mathcal{A})$; these form a pretriangulated dg category which we will denote by $\mathcal{D}^{\mathrm{perf}}(\mathcal{A})$. A dg category $\mathcal{A}$ is **proper** if $H^*(\mathcal{A}(a,b))$ is finite dimensional for all $a, b \in \mathcal{A}$. This is equivalent to each representable $\mathcal{A}(a,-)$ being a proper module, or $\mathcal{A}$ itself being a proper $\mathcal{A}$-bimodule.

A **dg algebra** is a dg category with one object, i.e. a complex with a compatible multiplication. A dg algebra A is **connective** if $H^i(A) \cong 0$ for $i > 0$ and **coconnective** if $H^i(A) \cong 0$ for $i < 0$. A dg algebra is **finite dimensional** if its underlying complex is finite dimensional. A finite dimensional dg algebra is clearly proper. A **module** over a dg algebra is a complex M with an action map $A \otimes M \to M$. We will sometimes refer to modules as **dg modules** for emphasis. All modules are by default right modules.

2. Preliminaries

In this section we recall the definition of reflexivity, before showing that reflexivity is closely linked to the notion of derived completeness.

2.1. **Reflexivity.**

Definition 2.1.1. A dg category $\mathcal{A}$ is **reflexive** (resp. **semireflexive**) if the natural map

$$\mathrm{ev}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{op}}); \quad a \mapsto (M \mapsto M(a))$$

is a Morita equivalence (resp. quasi-fully-faithful).

Remark 2.1.2. Equivalently, one can use the related **coevaluation map**

$$\mathrm{coev}_{\mathcal{A}} : \mathcal{A} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A}))$$

or its opposite $\mathrm{coev}_{\mathcal{A}^{\mathrm{op}}}$ in the above definition, as per [KS25, Lemma 3.10].

Example 2.1.3.

(1) A smooth proper dg category $\mathcal{A}$ satisfies $\mathcal{D}_{\mathrm{fd}}(\mathcal{A}) \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{A})$ and hence is reflexive (one inclusion is clear and the other is well-known, see e.g. [KS25]). This also follows from the fact that they are the dualisable objects in the closed symmetric monoidal category Hmo of dg categories localised at Morita equivalences. In particular, if $\mathcal{X}$ is a smooth proper DM stack over a field of characteristic zero, then $\mathcal{D}^{\mathrm{perf}}(\mathcal{X})$ is reflexive [BLS16].

(2) Proper connective dg algebras are reflexive. When k is perfect this was shown in [KS25] and in general this appears in [GRS24].
(3) If X is a proper scheme over k then both $\mathcal{D}^{\text{perf}}(X)$ and $\mathcal{D}^b_{\text{coh}}(X)$ are reflexive. In characteristic zero this appears in [BZNP17] (in fact, a relative version for algebraic spaces is given). In [KS25] this was proved for projective schemes over perfect fields. For all fields this appears in [GRS24].
(4) In [GRS24], Azumaya algebras over proper schemes over any field were shown to be reflexive.
(5) In [Goo24] the power series ring $k[\![t]\!]$ was shown to be reflexive.
(6) The polynomial ring $k[t]$ is not reflexive; this follows from Theorem 4.0.4 below.
(7) Let $f : \mathbb{C}^n \to \mathbb{C}$ be a weighted homogeneous polynomial with an isolated critical point, and let V be its Milnor fibre. Associated to V are its Fukaya category $\mathcal{F} := \mathcal{F}(V)$ and its wrapped Fukaya category $\mathcal{W} := \mathcal{W}(V)$. If a mild numerical condition is satisfied, then Lekili–Ueda show that both $\mathcal{F}$ and $\mathcal{W}$ are reflexive [LU22, Theorem 6.11]. In fact, they show that $\mathcal{D}_{\text{fd}}(\mathcal{F}) \simeq \mathcal{W}$ and $\mathcal{D}_{\text{fd}}(\mathcal{W}) \simeq \mathcal{F}$. Similar examples not fitting into the above framework are given in [Li24].
(8) Non-proper non-examples are easy to come by: e.g. there are no finite dimensional modules over the Weyl algebra, or algebras of graded Laurent polynomials, and so they cannot be reflexive.
(9) Proper dg categories are semireflexive [KS25].

Remark 2.1.4. A key feature of a reflexive dg category $\mathcal{A}$ is that there is some common information between $\mathcal{D}_{\text{fd}}(\mathcal{A})$ and $\mathcal{D}^{\text{perf}}(\mathcal{A})$:

(1) For any dg category $\mathcal{A}$, $\mathcal{D}_{\text{fd}}(\mathcal{A})$ is determined by $\mathcal{D}^{\text{perf}}(\mathcal{A})$; reflexivity guarantees the converse.
(2) In [KS25], it was shown that for $\mathcal{A}$ reflexive there is a bijection between semiorthogonal decompositions of $\mathcal{D}_{\text{fd}}(\mathcal{A})$ and of $\mathcal{D}^{\text{perf}}(\mathcal{A})$, and an isomorphism between the triangulated autoequivalence groups of these categories.
(3) It follows immediately from the results of [Goo24] that if $\mathcal{A}$ is reflexive, then there is an isomorphism between the derived Picard groups of $\mathcal{D}_{\text{fd}}(\mathcal{A})$ and $\mathcal{D}^{\text{perf}}(\mathcal{A})$. It was also shown in *op. cit.* that $\mathcal{D}_{\text{fd}}(\mathcal{A})$ and $\mathcal{D}^{\text{perf}}(\mathcal{A})$ have the same Hochschild cohomology.

Remark 2.1.5.

(1) $\mathcal{A}$ is reflexive if and only if $\mathcal{A}^{\text{op}}$ is [KS25].
(2) Since $\mathcal{D}^{\text{perf}}(\mathcal{A})$ is a Morita fibrant replacement of $\mathcal{A}$, and a Morita equivalence between pretriangulated idempotent complete dg categories is a quasi-equivalence, a dg category is reflexive if and only if the natural map

$$\mathcal{D}^{\text{perf}}(\mathcal{A})^{\text{op}} \to \mathcal{D}_{\text{fd}}(\mathcal{D}_{\text{fd}}(\mathcal{A})^{\text{op}}); \quad M \mapsto \mathbb{R}\text{Hom}_{\mathcal{A}}(M, -)$$

is a quasi-equivalence. From this point of view, reflexivity is seen to be a representability property for functors defined on $\mathcal{D}_{\text{fd}}(\mathcal{A})$.
(3) Let Hmo denote the Morita homotopy category of dg categories. It is a closed symmetric monoidal category, with monoidal structure induced by the derived tensor product of dg categories and internal hom induced by the internal hom of dg categories [Rod12]. In [Goo24] it was shown that the reflexive dg categories are precisely the reflexive objects in Hmo. Loosely,

this is because we have

$$\mathbb{R}\mathrm{Hom}_{\mathrm{Hmo}}(\mathcal{A}, k) \simeq \mathbb{R}\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{A}, \mathcal{D}^{\mathrm{perf}}(k)) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{A}^{op})$$

so that the Morita dual of a dg category $\mathcal{A}$ coincides with $\mathcal{D}_{\mathrm{fd}}(\mathcal{A}^{op})$.

2.2. **$\mathcal{D}_{\mathrm{fd}}$-generators.** We are interested in dg categories of the form $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. In many situations of interest these come with natural generators. For example, for finite dimensional algebras one takes the sum of all the simple modules, and for commutative local augmented k-algebras one takes k itself. A special case of [Nee21, Theorem 0.15] shows that $\mathcal{D}_{\mathrm{fd}}(X)$ admits a generator whenever X is a proper scheme. In this section we record some simple properties of these generators and give some examples.

Definition 2.2.1. Let $\mathcal{A}$ be a dg category and $S \in \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ a perfectly valued $\mathcal{A}$-module. Say that S is a **$\mathcal{D}_{\mathrm{fd}}$-generator** if we have an equality $\mathbf{thick}_{\mathcal{D}(\mathcal{A})}(S) = \mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ of subcategories of $\mathcal{D}(A)$.

When $\mathcal{D}(\mathcal{A})$ admits a t-structure, we can reduce the question of existence of a $\mathcal{D}_{\mathrm{fd}}$-generator for $\mathcal{A}$ to a question about a $\mathcal{D}_{\mathrm{fd}}$-generator for the heart:

Proposition 2.2.2. *Let A be a connective dg algebra. Then $\mathcal{D}_{\mathrm{fd}}(A)$ is generated by the finite dimensional simple $H^0(A)$-modules.*

Proof. Since A is connective, the cohomology of any A-module can be viewed as an $H^0(A)$-module via restriction along $A \to H^0(A)$. If $X \in \mathcal{D}_{\mathrm{fd}}(A)$ then $X \in \mathbf{thick}(H^*(X))$ using the standard t-structure. Each $H^i(X)$ is finite dimensional, and in particular a finite length $H^0(A)$-module, and so admits a finite filtration whose factors are finite dimensional simple $H^0(A)$-modules. It follows that each $H^i(A)$ is contained in the thick subcategory of $\mathcal{D}(H^0(A))$ generated by the finite dimensional simples. Now $H^*(X)$ is in the image of the restriction functor $\mathcal{D}_{\mathrm{fd}}(H^0(A)) \to \mathcal{D}_{\mathrm{fd}}(A)$ and so $H^*(X)$ is in the thick subcategory of A generated by the finite dimensional simple $H^0(A)$-modules. □

Corollary 2.2.3. *Let A be a connective dg algebra such that $H^0(A)$ is finite dimensional. Let $S = H^0(A)/\mathrm{rad}H^0(A)$ be the maximal semisimple quotient of $H^0(A)$, and regard S as an A-module. Then S is a $\mathcal{D}_{\mathrm{fd}}$-generator for A.*

Proof. Up to multiplicity, S is the direct sum of the simple $H^0(A)$-modules. Hence S generates the same thick subcategory as the simple $H^0(A)$-modules do. □

Remark 2.2.4. There is an analogous theorem for coconnective dg algebras with semisimple H^0 due to Keller and Nicolás [KN13]. The proof makes use of weight structures, which we will return to in Section 3.

2.3. **Derived completion.** When A admits a $\mathcal{D}_{\mathrm{fd}}$-generator S, standard tilting theorems imply that $\mathcal{D}_{\mathrm{fd}}(A)$ is Morita equivalent to the endomorphism dg algebra $\mathbb{R}\mathrm{End}_A(S)$. In this section we explore the properties of this construction from the viewpoint of reflexivity. We pay particular attention to the two-fold application of this construction, known as the **derived double centraliser** or the **derived completion** [DGI06, Efi10]. We will show that in favourable circumstances, being derived complete with respect to a $\mathcal{D}_{\mathrm{fd}}$-generator is equivalent to being reflexive.

Definition 2.3.1. Let A be a dg algebra and M an A-module. We define a new dg algebra $A^!_M \coloneqq \mathbb{R}\mathrm{End}_A(M)^{\mathrm{op}}$. We refer to $A^!_M$ as the **centraliser of A relative to M**.

We think of $A^!_M$ as an M-relative dual of A - indeed, when M is a $\mathcal{D}_{\mathrm{fd}}$-generator it is the Morita dual of A, and when M is the base field it is (up to an opposite) the Koszul dual of A. We will explore this latter perspective further in Section 5.

Clearly M is itself a right $A^!_M$-module, and so we may form the double dual $A^{!!}_M \coloneqq (A^!_M)^!_M$. We call this the **derived completion of** A **along** M. Observe that there is a functor

$$\mathbb{R}\mathrm{Hom}_A(-, M)\colon \mathcal{D}(A)^{\mathrm{op}} \to \mathcal{D}(A^!_M)$$

which sends A to M, which induces a map of dg algebras

$$A = \mathbb{R}\mathrm{Hom}_A(A, A) \to \mathbb{R}\mathrm{Hom}_{A^!_M}(M, M)^{\mathrm{op}} = A^{!!}_M.$$

Definition 2.3.2. We say that a dg algebra A is **derived complete with respect to** $M \in \mathcal{D}(A)$ if the natural map $A \to A^{!!}_M$ is a quasi-isomorphism. In [DGI06] this is called **dc-completeness**.

Lemma 2.3.3. *If A is derived complete with respect to M then $A^!_M$ is derived complete with respect to M.*

Proof. Consider the composition

$$\mathcal{D}(A^!_M) \xrightarrow{\mathbb{R}\mathrm{Hom}(-,M)} \mathcal{D}(A^{!!}_M)^{\mathrm{op}} \xrightarrow{\simeq} \mathcal{D}(A)^{\mathrm{op}}$$

The second functor is the equivalence induced by restricting along $A \to A^{!!}_M$. The long composite is a coproduct-preserving functor which is fully faithful, since it is fully faithful when restricted to the generator M: one has by hypothesis a natural quasi-isomorphism $\mathbb{R}\mathrm{End}_{A^!_M}(M)^{\mathrm{op}} =: A^{!!}_M \simeq A$. Therefore the first map is fully faithful, and so the induced map $A^!_M \to (A^!_M)^{!!}_M$ is an equivalence. □

Remark 2.3.4.

(1) It was shown in [DGI06, Proposition 4.20] that if $(R, \mathfrak{m}, K)$ is a commutative noetherian local ring, then the map $R \to R^{!!}_K$ coincides with the $\mathfrak{m}$-adic completion map $R \to \hat{R}_{\mathfrak{m}}$. More generally, the same holds for any regular quotient of R. Efimov generalised this to a non-affine version [Efi10].

(2) If $\mathbf{thick}_A(M) = \mathbf{thick}_A(M')$, then clearly $A^!_M$ and $A^!_{M'}$ are Morita equivalent. By [Efi10, Proposition 3.2], in this situation $A^{!!}_M$ and $A^{!!}_{M'}$ are also Morita equivalent.

(3) By [Efi10, Proposition 3.4], derived completion respects Morita equivalences: if A and B are Morita equivalent dg algebras, with M an A-module and N the corresponding B-module, then $A^{!!}_M$ and $B^{!!}_N$ are Morita equivalent.

Lemma 2.3.5. *Let A be a dg algebra and M an A-module. If $A \in \mathbf{thick}_A(M)$ then A is derived complete with respect to M.*

Proof. We have an equivalence

$$\mathbf{thick}_A(M) \xrightarrow{\mathbb{R}\mathrm{Hom}_A(-,M)} \mathcal{D}^{\mathrm{perf}}(A^!_M)^{\mathrm{op}}$$

Indeed the functor is fully faithful restricted to the generator M by definition of $A^!_M$. Then since $\mathcal{D}^{\mathrm{perf}}(A^!_M) = \mathbf{thick}(A^!_M)$, it is essentially surjective. Since $A \in \mathbf{thick}_A(M)$ it restricts to an equivalence

$$\mathcal{D}^{\mathrm{perf}}(A) \xrightarrow{\mathbb{R}\mathrm{Hom}_A(-,M)} \mathbf{thick}_{A^!_M}(M)^{\mathrm{op}}$$

Indeed it is fully faithful as it is the restriction of a fully faithful functor. It is essentially surjective since A maps to M. Therefore, by definition the map $A \to A^{!!}_M$ is a quasi-isomorphism. □

Let A be a finite dimensional dg algebra. Orlov introduced the concept of a **dg radical** J_- of A [Orl20]. It follows from the results of [Orl20] that every module over a finite dimensional dg algebra A whose underlying chain complex is finite dimensional is in the thick subcategory generated by A/J_-.

Corollary 2.3.6.

(1) *Let A be a proper dg algebra and M a $\mathcal{D}_{\mathrm{fd}}$-generator of A. Then A is derived complete with respect to M.*
(2) *Let A be a finite dimensional dg algebra. Then A is derived complete with respect to A/J_-.*

Proof. If A is proper, then $A \in \mathcal{D}_{\mathrm{fd}}(A) = \mathbf{thick}_A(M)$ and if A is finite dimensional then $A \in \mathbf{thick}(A/J_-)$. Both claims now follow from Lemma 2.3.5. □

Remark 2.3.7. In the general setting, the existence of a $\mathcal{D}_{\mathrm{fd}}$-generator for finite dimensional dg algebras is subtle. For example, Efimov constructed an example of a formal coconnective finite dimensional dg algebra A, and a module over it which has finite dimensional cohomology but which is not quasi-isomorphic to a finite dimensional module [Efi20].

We can now link reflexivity to derived completeness:

Theorem 2.3.8 ('two-out-of-three' theorem)**.** *Let A be a dg algebra and M a $\mathcal{D}_{\mathrm{fd}}$-generator for A. If any two of the following hold then so does the third:*

(1) *A is reflexive.*
(2) *M is a $\mathcal{D}_{\mathrm{fd}}$-generator for $A^!_M$.*
(3) *A is derived complete at M.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{perf}}(A)^{\mathrm{op}} & \xrightarrow{\quad \mathrm{ev} \quad} & \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{op}}) \\ \downarrow & & \downarrow \simeq \\ \mathcal{D}^{\mathrm{perf}}(A^{!!}_M)^{\mathrm{op}} \simeq \mathbf{thick}_{A^!_M}(M) & \hookrightarrow & \mathcal{D}_{\mathrm{fd}}(A^!_M) \simeq \mathcal{D}_{\mathrm{fd}}(\mathbf{thick}_A(M)^{\mathrm{op}}) \end{array}$$

The assumption that M is a $\mathcal{D}_{\mathrm{fd}}$-generator for A implies that the right-hand vertical map is an isomorphism. Condition (1) is equivalent to the upper horizontal map being an equivalence, condition (2) is equivalent to the left-hand vertical map being an equivalence, and condition (3) is equivalent to the lower horizontal map being an equivalence. □

From the proof of Theorem 2.3.8, one can immediately deduce the following:

Corollary 2.3.9. *Let A be a dg algebra. If A is derived complete with respect to a $\mathcal{D}_{\mathrm{fd}}$-generator, then A is semireflexive.*

Remark 2.3.10. In [Efi10], the completion of a dg category $\mathcal{A}$ along any subcategory of $\mathcal{D}(\mathcal{A})$ is defined. In particular, one can complete $\mathcal{A}$ along $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$, and this completion comes with a natural map $\mathcal{A} \to \hat{\mathcal{A}}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$. Although this does not agree with the natural morphism to $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(A)$, c.f. Remark 4.0.9, one can formulate a result similar to Theorem 2.3.8 where $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(A)$ is replaced by $\hat{\mathcal{A}}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{A})}$.

Proposition 2.3.11. *If A is a proper dg algebra and M is a $\mathcal{D}_{\mathrm{fd}}$-generator for A, then A is reflexive if and only if M is a $\mathcal{D}_{\mathrm{fd}}$-generator for $A^!_M$. In this case $A^!_M$ is also reflexive.*

Proof. The equivalence follows from Theorem 2.3.8 combined with Corollary 2.3.6. If A is reflexive then so is $\mathcal{D}_{\mathrm{fd}}(A)$, which is Morita equivalent to $A^!_M$, up to an opposite. □

Lemma 2.3.12. *Let A be a dg algebra and $M \in \mathcal{D}_{\mathrm{fd}}(A)$. Suppose that A is derived complete with respect to M. Then:*

1. *If $\mathbf{thick}_A(M) = \mathcal{D}_{\mathrm{fd}}(A)$ and $\mathbf{thick}_{A^!_M}(M) = \mathcal{D}_{\mathrm{fd}}(A^!_M)$, then A and $A^!_M$ are reflexive.*
2. *The following are equivalent:*
   (a) *A is reflexive and $\mathbf{thick}_A(M) = \mathcal{D}_{\mathrm{fd}}(A)$.*
   (b) *$A^!_M$ is reflexive and $\mathbf{thick}_{A^!_M}(M) = \mathcal{D}_{\mathrm{fd}}(A^!_M)$.*

Proof. If the generation conditions in (1) hold, then since A is derived complete, Theorem 2.3.8 implies that A is reflexive. Hence $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^!_M)$ is also reflexive. For (2), if $\mathbf{thick}_A(M) = \mathcal{D}_{\mathrm{fd}}(A)$ and A is reflexive, then Theorem 2.3.8 implies that $\mathbf{thick}_{A^!_M}(M) = \mathcal{D}_{\mathrm{fd}}(A^!_M)$. Also $\mathcal{D}_{\mathrm{fd}}(A) = \mathcal{D}^{\mathrm{perf}}(A^!_M)$ is reflexive. Conversely, suppose that $A^!_M$ is reflexive and $\mathbf{thick}_{A^!_M}(M) = \mathcal{D}_{\mathrm{fd}}(A^!_M)$. Then $\mathcal{D}_{\mathrm{fd}}(A^!_M) = \mathbf{thick}_{A^!}(M) \simeq \mathcal{D}^{\mathrm{perf}}((A^{!!}_M)^{\mathrm{op}}) \simeq \mathcal{D}^{\mathrm{perf}}(A^{\mathrm{op}})$ is reflexive. Hence A is reflexive. Since A is derived complete, so is $A^!_M$. Then Theorem 2.3.8 applied to $A^!_M$ implies that M generates $\mathcal{D}_{\mathrm{fd}}(A^{!!}_M)$. Since A is derived complete, restriction induces an equivalence $\mathcal{D}_{\mathrm{fd}}(A^{!!}_M) \simeq \mathcal{D}_{\mathrm{fd}}(A)$, and so M generates $\mathcal{D}_{\mathrm{fd}}(A)$, as required. □

2.4. **Restricted reflexivity.** Let $\mathcal{A}$ be a dg category and $\mathcal{B}$ a pretriangulated dg subcategory of $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$. Applying the $\mathcal{D}_{\mathrm{fd}}$ functor to the inclusion $\mathcal{B}^{\mathrm{op}} \hookrightarrow \mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{op}}$ gives a map $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{op}}) \to \mathcal{D}_{\mathrm{fd}}(\mathcal{B}^{\mathrm{op}})$. Composition with $\mathrm{ev}_{\mathcal{A}}$ hence yields a map

$$\mathrm{ev}_{\mathcal{A},\mathcal{B}} \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(\mathcal{B}^{\mathrm{op}})$$

which we call the **restricted evaluation functor**. We say that $\mathcal{A}$ is **$\mathcal{B}$-restricted reflexive** when $\mathrm{ev}_{\mathcal{A},\mathcal{B}}$ is a quasi-equivalence. When $\mathcal{B} = \mathbf{thick}_A(M)$ for some $M \in \mathcal{D}_{\mathrm{fd}}(A)$, then we replace $\mathcal{B}$ by M in the above notation.

Example 2.4.1. Clearly $\mathcal{A}$ is reflexive precisely when it is $\mathcal{D}_{\mathrm{fd}}(A)$-restricted reflexive.

Proposition 2.4.2. *Let $\mathcal{A}$ be a dg category and $\mathcal{B} \subseteq \mathcal{D}_{\mathrm{fd}}(A)$ a reflexive subcategory. If any two of the following three conditions hold, then so does the third:*

1. *The inclusion $\mathcal{B} \hookrightarrow \mathcal{D}_{\mathrm{fd}}(A)$ is a quasi-equivalence.*
2. *$\mathcal{A}$ is $\mathcal{B}$-restricted reflexive.*
3. *$\mathcal{A}$ is reflexive.*

Before we begin the proof, observe that when $\mathcal{B} = \mathbf{thick}_{\mathcal{D}(\mathcal{A})}(M)$, then condition (1) says that M is a $\mathcal{D}_{\mathrm{fd}}$-generator for $\mathcal{A}$.

Proof. We have already observed in Example 2.4.1 that if (1) holds, then (2) is equivalent to (3), even without the reflexivity hypothesis on $\mathcal{B}$. We need only show that (2) and (3) together imply (1). If (2) and (3) hold, then $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ is also reflexive and moreover the natural map $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})^{\mathrm{op}}) \to \mathcal{D}_{\mathrm{fd}}(\mathcal{B}^{\mathrm{op}})$ is necessarily a quasi-equivalence. We can conclude that (1) holds by applying $\mathcal{D}_{\mathrm{fd}}$ to this quasi-equivalence and using that both $\mathcal{D}_{\mathrm{fd}}(\mathcal{A})$ and $\mathcal{B}$ are reflexive. □

In general the relationship between reflexivity and restricted reflexivity seems unclear; neither implies the other.

Remark 2.4.3. If $\mathcal{A}$ is a proper dg category and $\mathcal{B} \subseteq \mathcal{D}_{\mathrm{fd}}(A)$ contains the image of $\mathcal{A}$ under the Yoneda embedding, then one can adapt the arguments of [KS25] to show that $\mathrm{ev}_{\mathcal{A},\mathcal{B}}$ is quasi-fully faithful (one should call this property **$\mathcal{B}$-restricted semireflexivity**).

3. Connective and coconnective dg algebras

Coconnective dg algebras with semisimple H^0 appear naturally as algebras of cochains on topological spaces, as derived endomorphism algebras of semisimple modules, and, more generally, as derived endomorphisms of simple-minded collections in the sense of [KY14]. In this section, we use the weight structures constructed by Keller and Nicolás [KN13] to investigate when such dg algebras are reflexive. A similar argument also gives a criterion for reflexivity of connective dg algebras with finite dimensional H^0, which appear when considering silting objects.

3.1. **Weight Structures.** A weight structure (or a co-t-structure) on a triangulated category generalises the properties of the brutal truncation functors, just like t-structures generalise the properties of the good truncation functors. Weight structures were introduced independently by Pauksztello [Pau08] and Bondarko [Bon10]. It was shown in [KN13] that the derived category of a coconnective dg algebra with semisimple H^0 admits a particularly well-behaved weight structure. We begin by recalling these results.

Definition 3.1.1. A **weight structure** (or **co-t-structure**) $(\mathcal{T}^{w>0}, \mathcal{T}^{w\leq 0})$ on a triangulated category $\mathcal{T}$ with shift functor [1] is a pair of additive subcategories closed under summands that satisfy the following conditions:

(1) $\mathcal{T}^{w>0}$ is closed under $[-1]$ and $\mathcal{T}^{w\leq 0}$ is closed under [1].
(2) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{T}^{w>0}, \mathcal{T}^{w\leq 0}) \cong 0$.
(3) For every object $X \in \mathcal{T}$, there is a truncation triangle

$$\sigma_{>0}X \to X \to \sigma_{\leq 0}X \to$$

with $\sigma_{>0}X \in \mathcal{T}^{w>0}$ and $\sigma_{\leq 0}X \in \mathcal{T}^{w\leq 0}$.

Example 3.1.2. The typical example of a weight structure is the one on the homotopy category of an additive category $\mathcal{K}(\mathcal{C})$ given by $(\mathcal{K}(\mathcal{C})^{w>0}, \mathcal{K}(\mathcal{C})^{w\leq 0})$ where $\mathcal{K}(\mathcal{C})^{w>0}$ consists of complexes isomorphic to those concentrated in positive degrees and $\mathcal{K}(\mathcal{C})^{w\leq 0}$ consists of those concentrated in non-positive degrees.

Theorem 3.1.3 ([KN13, Corollary 5.1])**.** *Let A be a coconnective dg algebra with $H^0(A)$ semisimple. Then:*

(1) *There is a weight structure $(\mathcal{D}(A)^{w>0}, \mathcal{D}(A)^{w\leq 0})$ on $\mathcal{D}(A)$ given by*

$$\mathcal{D}(A)^{w>0} = \{X \in \mathcal{D}(A) \mid H^i(X) = 0 \text{ for } i \leq 0\}$$
$$\mathcal{D}(A)^{w\leq 0} = \{X \in \mathcal{D}(A) \mid H^i(X) = 0 \text{ for } i > 0\}.$$

(2) *For every $X \in \mathcal{D}(A)$ there is a truncation triangle $\sigma_{>0}X \to X \to \sigma_{\leq 0}X \to$ such that the map $\sigma_{>0}X \to X$ induces an isomorphism on H^i for $i \leq 0$ and the map $X \to \sigma_{\leq 0}X$ induces an isomorphism on H^i for $i > 0$.*

Remark 3.1.4. If A is strictly coconnective, i.e. $A^i = 0$ for $i < 0$, then we may take $\sigma_{\leq 0}A$ to be the brutal truncation A^0.

The following result of Keller and Nicolás provides a $\mathcal{D}_{\mathrm{fd}}$-generator for our algebras of interest.

Proposition 3.1.5 ([KN13, 5.6.1 a)])**.** *Let A be a coconnective dg algebra such that $H^0(A)$ is a finite dimensional semisimple k-algebra. Then $\sigma_{\leq 0}A$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for A.*

For a coconnective dg algebra A with finite dimensional semisimple H^0 we set

$$A^! \coloneqq A^!_{\sigma_{\leq 0}A} = \mathbb{R}\mathrm{Hom}_A(\sigma_{\leq 0}A, \sigma_{\leq 0}A)^{\mathrm{op}}$$

We note that by [KN13, Lemma 6.2], $A^!$ is a connective dg algebra. By Proposition 3.1.5, there is a quasi-equivalence $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^!)^{\mathrm{op}}$. For the remainder of this section, we will say that A is **derived complete** to mean that A is derived complete at $\sigma_{\leq 0}A$.

3.2. **The simple-projective bijection.** The aim of this subsection is to relate the simple $H^0(A)$ modules and the simple $H^0(A^!)$ modules. We begin with a version of the simple-projective bijection which follows from the results of [KN13].

Theorem 3.2.1. *Let A be a coconnective dg algebra with $H^0(A)$ a finite dimensional semisimple algebra. Then there is a bijection between the indecomposable summands of $H^0(A^!)$ and the isomorphism classes of simple $H^0(A)$-modules.*

Proof. Put $S \coloneqq \sigma_{\leq 0}A$. Any indecomposable summand of $H^0(A^!)$ is of the form $P = e_P H^0(A^!)$ for some primitive idempotent e_P of $H^0(A^!) = \mathrm{Hom}_{\mathcal{D}(A)}(S,S)^{\mathrm{op}}$. This idempotent splits in $\mathcal{D}(A)$ and produces an indecomposable summand $\overline{\phi(P)}$ of S and so a summand $\phi(P)$ of $H^0(S) \simeq H^0(A)$. Suppose that $\phi(P)$ is decomposable, so that $\phi(P) \simeq \bigoplus S_i$ for some simple summands S_i of $H^0(A)$. By [KN13, Lemma 5.5], there are indecomposable summands $\tilde{A}_i$ of A in $\mathcal{D}(A)$ lifting each of the S_i. By Lemma 5.4 *op. cit.* the isomorphism $\bigoplus S_i \to \phi(P)$ can be lifted to a map $f : \bigoplus A_i \to \overline{\phi(P)}$ which is an isomorphism on H^0. Since $\phi(P) \in \mathcal{D}(A)^{w\leq 0}$, the map f factors as $\tilde{f} : \sigma_{\leq 0}(\bigoplus A_i) \to \overline{\phi(P)}$ and $H^0(\tilde{f})$ is an isomorphism. But then $\tilde{f}$ is a quasi-isomorphism and so $\overline{\phi(P)} \simeq \bigoplus \sigma_{\leq 0}A_i$ using Lemma 4.2 *op. cit.* . This contradicts the indecomposablility of $\overline{\phi(P)}$. Therefore $\phi(P)$ is an indecomposable summand of $H^0(A)$ and hence a simple $H^0(A)$ module.

Given a simple $H^0(A)$ module, Corollary 5.7 *op. cit.* and its proof show that it can be lifted to a summand of S. The same argument as above shows that it must be indecomposable and so it corresponds to a primitive idempotent in $H^0(A^!)$.

Suppose that $\phi(P') \simeq \phi(P)$. Then by the uniqueness of Corollary 5.7 *op. cit.* , we have that $\overline{\phi(P)} \simeq \overline{\phi(P')}$. Therefore there are isomorphisms

$$e_P H^0(A^!) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(S, \overline{\phi(P)}) \simeq \mathrm{Hom}_{\mathcal{D}(A)}(S, \overline{\phi(P')}) \simeq e_{P'} H^0(A^!)$$

where the first isomorphism follows from the bijection between idempotents and summands for $H^0(A^!)$. □

Recall that a ring R is **semiperfect** if it has a complete set of orthogonal idempotents e_i such that each $e_i R e_i$ is local. Local rings and Artinian rings are semiperfect. For semiperfect rings there is a bijection between indecomposable projectives and simples given by taking projective covers. Theorem 3.2.1 allows us to relate the simple $H^0(A)$-modules to the projective $H^0(A^!)$ modules. We will show now that $H^0(A^!)$ is semiperfect, and so we can relate the simple $H^0(A)$-modules to the simple $H^0(A^!)$-modules.

Remark 3.2.2. If we assume that $H^0(A)$ is a product of division algebras, then by Lemma 3.2.5 and Corollary 24.12 in [Lam01], every indecomposable projective is a summand of $H^0(A^!)$. So in this setting the statement of Theorem 3.2.1 can simplified to replace indecomposable summands of $H^0(A^!)$ with indecomposable projective $H^0(A^!)$-modules. In fact, up to Morita equivalence, the following lemma shows that we may always assume this.

Lemma 3.2.3. *Let A be a coconnective dg algebra such that $H^0(A)$ is a semisimple algebra. Then there is a dg algebra A' such that $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A')$, and $H^0(A')$ is a product of division algebras.*

Proof. Suppose $H^0(A) \simeq \bigoplus_{i=1,\dots,n,j=1,\dots,k_i} S_{i,j}$ is a decomposition of $H^0(A)$ into simple right modules such that $S_{i,j} \simeq S_{i,j'}$ for $1 \le j,j' \le k_i$ for all i, and $S_{i,j} \not\simeq S_{i',j'}$ if $i \neq i'$. Then $H^0(A) \simeq M_{k_1}(D_1) \times \cdots \times M_{k_n}(D_n)$ as algebras, where each $D_i \simeq \mathrm{End}_{H^0(A)}(S_{i,1})$ is a division algebra. By [KN13, Lemma 5.5], there is an A-linear quasi-isomorphism $A \simeq \bigoplus A_{i,j}$ where $H^0(A_{i,j}) \simeq S_{i,j}$. For each i and each $1 \le j,j' \le k_i$ there is an isomorphism $S_{i,j} \to S_{i,j'}$ which lifts to a map $f : A_{i,j} \to A_{i,j'}$ by Lemma 5.4 *op. cit.* , and moreover by naturality of Lemma 5.4 *op. cit.* it follows that f is a quasi-isomorphism. So if $A' := \mathbb{R}\mathrm{Hom}_A(\bigoplus_i A_i, \bigoplus_i A_i)$ where $A_i := A_{i,1}$ then it follows that $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A')$. Furthermore we have that $H^0(A') \simeq D_1 \times \cdots \times D_n$, as required. □

Lemma 3.2.4. *Let A be a coconnective dg algebra with semisimple $H^0(A)$. Then $H^0(\sigma_{\le 0}A)$ is a semisimple $H^0(A^!)$-module.*

Proof. Set $S = \sigma_{\le 0}A$. The functor H^0 induces a map of k-algebras

$$H^0(A^!)^{\mathrm{op}} \cong \mathrm{End}_{\mathcal{D}(A)}(S) \to \mathrm{End}_{H^0(A)}(H^0(S)) \cong \mathrm{End}_{H^0(A)}(H^0(A)) \cong H^0(A)$$

using the isomorphism $H^0(A) \cong H^0(S)$. Since $H^0(A)$ is a semisimple k-algebra, it is a semisimple $H^0(A^!)$-module, and moreover the isomorphism $H^0(A) \cong H^0(S)$ is $H^0(A)$-linear. It remains to check that this is an isomorphism of $H^0(A^!)$-modules where $H^0(A)$ is viewed as an $H^0(A^!)$ module via restriction and $H^0(S)$ is viewed as an $H^0(A^!)$ module using the fact that $H^0(A^!)^{\mathrm{op}}$ is the endomorphism ring of S in $\mathcal{D}(A)$. To check this, note that the map $H^0(A^!)^{\mathrm{op}} \to H^0(A)$ can be identified with the map

$$H^0(A^!)^{\mathrm{op}} \cong \mathrm{Hom}_{\mathcal{D}(A)}(S,S) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{D}(A)}(A,S) \xrightarrow{f_*^{-1}} \mathrm{Hom}_{\mathcal{D}(A)}(A,A) \simeq H^0(A)$$

where $f : A \to S$ and where f_* agrees with the isomorphism $H^0(f)\colon H^0(A) \to H^0(S)$ across the identification $\mathrm{Hom}_{\mathcal{D}(A)}(A,-) \simeq H^0(-)$. One can then check directly that the actions agree. □

Lemma 3.2.5. *Let A be a coconnective dg algebra such that $H^0(A)$ is a product of division algebras. Then $H^0(A^!)$ is semiperfect.*

Proof. Suppose $H^0(A) = D_1 \times \cdots \times D_n$ is a product of division algebras. By [KN13, Lemma 5.5], A splits into a direct sum of indecomposables $A \simeq \bigoplus A_i$ such that $H^0(A_i) \simeq D_i$. It follows that $S := \sigma_{\le 0}A \simeq \bigoplus S_i$ where we put $S_i := \sigma_{\le 0}A_i$. Therefore $H^0(A^!)^{\mathrm{op}} \simeq \mathrm{Hom}_{\mathcal{D}(A)}(\bigoplus S_i, \bigoplus S_i)$. The maps $e_i : S_i \to S \to S_i$ clearly form a complete set of orthogonal idempotents so it remains to show that each $e_iH^0(A)^{\mathrm{op}}e_i = \mathrm{Hom}_{\mathcal{D}(A^!)}(S_i, S_i)$ is local. Consider the map of algebras from the proof of Lemma 3.2.4,

$$H^0(A^!)^{\mathrm{op}} \to H^0(A)$$

induced by taking H^0. Clearly it restricts to the subalgebras

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(A)}(S_i, S_i) \to \mathrm{Hom}_{H^0(A)}(H^0(S_i), H^0(S_i)) &\simeq \mathrm{Hom}_{H^0(A)}(H^0(A_i), H^0(A_i)) \\ &\simeq D_i\end{aligned}$$

We note that this map reflects units. Indeed if $f : S_i \to S_i$ is such that $H^0(f)$ is an isomorphism, then since the cohomology of S_i only lives in degree zero, f is a quasi-isomorphism. Since D_i is a division algebra this implies that the kernel of the above map is exactly the set of non-units. Therefore the non-units form an ideal and $\mathrm{Hom}_{\mathcal{D}(A)}(S_i, S_i)$ is local. □

Proposition 3.2.6. *Let A be a coconnective dg algebra with $H^0(A)$ a product of division algebras. Then there is an isomorphism $H^0(A^!)/\operatorname{rad} H^0(A^!) \simeq H^0(A)^{\mathrm{op}}$. Therefore the simple $H^0(A^!)$ modules are exactly the indecomposable summands of $H^0(\sigma_{\leq 0}A)$.*

Proof. Let $J := \operatorname{rad} H^0(A^!)$. Since $H^0(A^!)$ is semiperfect by Lemma 3.2.5, it follows that $H^0(A^!)/J$ is the maximal semisimple quotient of $H^0(A^!)$. Hence the map $H^0(A^!)^{\mathrm{op}} \twoheadrightarrow H^0(A)$ from the proof of Lemma 3.2.4 factors as a map $H^0(A^!)^{\mathrm{op}}/J \twoheadrightarrow H^0(A)$. Since $H^0(A^!)$ is semiperfect, $H^0(A^!)/J$ is isomorphic to a product of N matrix rings over division algebras. Here N is the number of isomorphism classes of simples, which equals the number of isomorphism classes of indecomposable projectives. As it is semisimple, the only quotient rings of $H^0(A^!)/J$ are products of its connected components, and so $H^0(A)^{op}$ must be some product of connected components of $H^0(A^!)/J$. However by Theorem 3.2.1, the number of connected components of $H^0(A)^{op}$ and of $H^0(A)/J$ are equal. It follows that the map $H^0(A^!)^{\mathrm{op}}/J \twoheadrightarrow H^0(A)$ is an isomorphism. Since $H^0(\sigma_{\leq 0}A) \simeq H^0(A)$ as $H^0(A^!)^{\mathrm{op}}$-modules the second claim also follows. □

The proof of Lemma 3.2.5 also shows the following:

Corollary 3.2.7. *If A is a coconnective dg algebra with $H^0(A)$ a division algebra, then $H^0(A^!)$ is local.*

If a commutative noetherian ring is complete at a maximal ideal, then it must be local. We conclude a similar result along these lines:

Corollary 3.2.8. *Let B be a connective dg algebra augmented over k. If B is derived complete at k, then $H^0(B)$ is local.*

Proof. If B is connective then $B^! := \mathbb{R}\mathrm{Hom}_B(k, k)^{\mathrm{op}}$ is coconnective as follows from the the standard t-structure. Furthermore

$$H^0(B^!) = \mathrm{Hom}_{\mathcal{D}(B)}(k, k)^{\mathrm{op}} \simeq \mathrm{Hom}_{H^0(B)}(k, k)^{\mathrm{op}} \simeq k$$

and so by Corollary 3.2.7 we see that $H^0(B) \simeq H^0(B^{!!})$ is local. □

Remark 3.2.9. The results of this section should be compared to the extensive literature relating simple-minded collections, silting objects, t-structures, and weight structures. See for example [AN09, KN13, KY14, Bon23, Fus23]. A simple-minded collection in an algebraic triangulated category is the same information as a coconnective dg algebra with H^0 a product of division algebras, and a silting object is the same as a connective dg algebra.

3.3. Reflexivity of coconnective dg algebras with semisimple H^0.

Theorem 3.3.1. *Let A be a coconnective dg algebra with $H^0(A)$ a finite dimensional semisimple k-algebra. Then A is derived complete if and only if it is reflexive.*

Proof. Lemma 3.2.3 ensures the existence of a coconnective dg algebra A', Morita equivalent to A, such that $H^0(A')$ is a product of division algebras. In particular A is reflexive if and only if A' is. Moreover, across the induced equivalence $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}_{\mathrm{fd}}(A')$, the proof of *op. cit.* shows that $\sigma_{\leq 0}A$ corresponds to $\sigma_{\leq 0}A'$, and hence A is derived complete if and only if A' is. So replacing A by A' we may assume that $H^0(A)$ is a product of division algebras. Set $S := \sigma_{\leq 0}A$. By Proposition 3.1.5, S is a $\mathcal{D}_{\mathrm{fd}}$-generator for A. So by Theorem 2.3.8, it is enough to show that S is a $\mathcal{D}_{\mathrm{fd}}$-generator for $A^!$. By [KN13, Lemma 6.2], $A^!$ is connective so by Proposition 2.2.2 the simple $H^0(A^!)$-modules generate $\mathcal{D}_{\mathrm{fd}}(A^!)$. By Proposition 3.2.6, the simple $H^0(A^!)$-modules are summands of $H^0(S)$. Therefore $S \simeq H^0(S) \in \mathcal{D}(A^!)$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for $A^!$. □

Corollary 3.3.2. *Every proper coconnective dg algebra with semisimple H^0 is reflexive.*

Proof. Follows from Corollary 2.3.6 and Theorem 3.3.1. □

Example 3.3.3. The following dg categories are reflexive by Corollary 3.3.2:

(1) If $\mathcal{T}$ is a proper pretriangulated dg category admitting a simple-minded collection in the sense of [KY14], then $\mathcal{T}$ is reflexive.
(2) If $\mathcal{A}$ is a small Hom-finite k-linear abelian category and $S_1, \ldots, S_n$ are are a collection of simple objects such that $\mathrm{Ext}^*_{\mathcal{A}}(\bigoplus S_i, \bigoplus S_i)$ is bounded, then the thick subcategory of $\mathcal{D}^b(\mathcal{A})$ they generate is reflexive. (e.g. the S_i are all of finite projective dimension or all of finite injective dimension.)
(3) Let A be a finite dimensional algebra which admits a grading with $A/\operatorname{rad}(A)$ in degree zero and $\operatorname{rad} A$ in positive degrees. Then A can be viewed as a formal coconnective dg algebra with semisimple H^0, which is hence reflexive. This includes exterior algebras, truncated polynomial rings, and many algebras given by path algebras of quivers with relations.
(4) If X is a finite CW complex then the dg algebra of cochains on X is a proper coconnective dg algebra with semisimple H^0, and hence is reflexive.

Remark 3.3.4. In Section 5, we use Koszul duality to give more general conditions for derived completeness which can be applied to non-proper examples.

3.4. Reflexivity for connective dg algebras.

Similar, though more straightforward, techniques allow us to prove that certain connective dg algebras are reflexive precisely when they are derived complete:

Proposition 3.4.1. *Let A be a connective dg algebra with finite dimensional $H^0(A)$. Then A is reflexive if and only if it is derived complete at $H^0(A)/\operatorname{rad} H^0(A)$.*

Proof. For brevity put $R := H^0(A)/\operatorname{rad} H^0(A)$, so that R is a finite dimensional semisimple k-algebra. By Corollary 2.2.3, R is a $\mathcal{D}_{\mathrm{fd}}$-generator for A. We wish to apply Theorem 2.3.8, for which it will suffice to show that R is a $\mathcal{D}_{\mathrm{fd}}$-generator for

$A^! := \mathbb{R}\mathrm{Hom}_A(R,R)^{op}$. The standard t-structure implies that $A^!$ is coconnective, and furthermore we have isomorphisms

$$H^0(A^!) \cong \mathrm{Hom}_{\mathcal{D}(A)}(R,R)^{op} \cong \mathrm{Hom}_{H^0(A)}(R,R)^{op} = \mathrm{Hom}_R(R,R)^{op} \cong R^{op}$$

which is a semisimple finite dimensional k-algebra. Since $\sigma_{\leq 0}A^!$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for $A^!$ by Proposition 3.1.5, it is enough to show that we have an $A^!$-linear quasi-isomorphism $\sigma_{\leq 0}A^! \simeq R$. But we already have an $H^0(A^!)$-linear isomorphism $H^0(\sigma_{\leq 0}A^!) \cong R$, and by a similar argument to the proof of Theorem 3.2.1, this can be lifted to a quasi-isomorphism of $A^!$-modules $\sigma_{\leq 0}A^! \simeq R$, as required. □

Corollary 3.4.2. *Let $\mathcal{T}$ be a hom-finite algebraic triangulated category admitting a silting object X such that $\mathrm{Ext}^i_{\mathcal{T}}(X,X)$ vanishes for $i \ll 0$. Then $\mathcal{T}$ is reflexive.*

Proof. Put $A := \mathbb{R}\mathrm{End}_{\mathcal{T}}(X)$, so that $\mathcal{T}$ is Morita equivalent to A. Since X was silting, A is connective. By assumption A is bounded below, and since $\mathcal{T}$ was hom-finite A is hence proper. So since A is connective and proper, we may without loss of generality assume that A is finite dimensional (cf. [GRS24, Appendix]). Then A is derived complete by Corollary 2.3.6 and hence reflexive by Proposition 3.4.1. □

4. Commutative rings

In this section we show that a commutative noetherian k-algebra is reflexive if and only if it is a finite product of complete local k-algebras, each with residue field finite over k. If R is a commutative k-algebra and $\mathfrak{m}$ a maximal ideal, we let $k(\mathfrak{m}) := R/\mathfrak{m}$ denote the residue field of R at $\mathfrak{m}$. We let $\mathrm{Fin}(R) \subseteq \mathrm{Spec}(R)$ denote the set of those maximal ideals $\mathfrak{m} \subseteq R$ such that $k(\mathfrak{m})$ is a finite extension of k.

Example 4.0.1. When R is finite type over k, then $\mathrm{Fin}(R)$ is simply $\mathrm{MaxSpec}(R)$, the set of maximal ideals of R.

Lemma 4.0.2. *Let R be a commutative k-algebra. There is a quasi-equivalence*

$$\mathcal{D}_{\mathrm{fd}}(R) \simeq \bigoplus_{\mathfrak{m} \in \mathrm{Fin}(R)} \mathbf{thick}_R(k(\mathfrak{m}))$$

where the right hand side denotes the orthogonal sum of triangulated categories.

Proof. We first claim that $\mathcal{D}_{\mathrm{fd}}(R) = \mathbf{thick}_R\{k(\mathfrak{m})\colon \mathfrak{m} \in \mathrm{Fin}(R)\}$. If $k(\mathfrak{m})$ is a finite extension of k, then it is certainly contained in $\mathcal{D}_{\mathrm{fd}}(R)$, and so the right hand side is contained in the left. On the other hand, if $M \in \mathcal{D}_{\mathrm{fd}}(R)$, then $M \in \mathbf{thick}_R(H^*(M))$ by an induction argument using the standard t-structure, and so it is enough to show that every finite dimensional R-module is contained in the right hand side. Every finite dimensional R-module M is finite length, and so admits a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

with each $M_i/M_{i-1} \simeq k(\mathfrak{m}_i)$ for some maximal ideals $\mathfrak{m}_i$. Since M is finite dimensional over k, each of the submodules M_i are finite dimensional over k, and hence each $\mathfrak{m}_i$ is contained in $\mathrm{Fin}(R)$. Hence M is contained in $\mathbf{thick}_R\{k(\mathfrak{m})\colon \mathfrak{m} \in \mathrm{Fin}(R)\}$ as desired. Finally we just need to show that $\mathbf{thick}_R\{k(\mathfrak{m})\colon \mathfrak{m} \in \mathrm{Fin}(R)\}$ splits as the orthogonal sum $\bigoplus_{\mathfrak{m}\in\mathrm{Fin}(R)} \mathbf{thick}_R(k(\mathfrak{m}))$. But this follows from a standard localisation argument: if x is a closed point of $\mathrm{Spec}(R)$ and M, N two R-modules then one has $\mathrm{Ext}^i_R(M,N)_x \cong \mathrm{Ext}^i_{R_x}(M_x,N_x)$ since localisation is flat. In particular, if M and N are supported at two distinct closed points, then $\mathrm{Ext}^i_R(M,N)$ must vanish, as its support is empty. □

Lemma 4.0.3. *Let R be a non-zero reflexive connected commutative k-algebra. Then exactly one of the residue fields of R is a finite extension of k.*

Proof. Using the splitting of Lemma 4.0.2 we obtain a quasi-equivalence

$$\mathcal{D}_{\mathrm{fd}}(R)^{\mathrm{op}} \simeq \bigoplus_{\mathfrak{m}\in \mathrm{Fin}(R)} \mathcal{D}^{\mathrm{perf}}(\mathbb{R}\mathrm{End}_R(k(\mathfrak{m}))).$$

If $\mathrm{Fin}(R)$ has more than two elements, then $\mathcal{D}_{\mathrm{fd}}(R)^{\mathrm{op}}$ admits a non-trivial orthogonal decomposition, cf. Definition 8.0.1. As in [KS25, Lemma 3.7] it follows that $\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(R)^{\mathrm{op}})$ admits an orthogonal decomposition. It is non-trivial since each $\mathcal{D}_{\mathrm{fd}}(\mathbb{R}\mathrm{End}_R(k(\mathfrak{m})))$ is nonzero; for example it contains $k(\mathfrak{m})$. Since R was reflexive by assumption, it follows that $\mathcal{D}^{\mathrm{perf}}(R) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(R)^{\mathrm{op}})$ admits a non-trivial orthogonal decomposition. It follows that R is a decomposable R-module, and hence R must be disconnected, which is a contradiction. If none of the residue fields of R are finite extensions of k, then $\mathcal{D}_{\mathrm{fd}}(R) \simeq 0$ and R cannot be reflexive. □

Theorem 4.0.4. *Let R be a commutative noetherian k-algebra. Then R is reflexive if and only if it is a finite product of complete local k-algebras whose residue fields are finite extensions of k.*

Proof. Since R is noetherian, it is a finite product of connected k-algebras, and R is reflexive if and only if each of its connected components is (since $\mathcal{D}^{\mathrm{perf}}(R)$ splits as the product of the perfect derived categories of the connected components of R). So without loss of generality we may assume that R is connected. We will prove the forwards implication of the theorem; the proof of the backwards implication is similar. If R is reflexive, then Lemma 4.0.3 implies that R possesses a unique maximal ideal $\mathfrak{m}$ such that its residue field $K := R/\mathfrak{m}$ is a finite extension of k. By Lemma 4.0.2, K is a $\mathcal{D}_{\mathrm{fd}}$-generator for R. Note that $R^!_K = \mathbb{R}\mathrm{Hom}_R(K,K)^{\mathrm{op}}$ is a coconnective dg algebra with $H^0(R^!_K) = K$, which can be computed using the minimal projective resolution for K. As a consequence, we may also assume that $(R^!_K)^0 = K$. By Proposition 3.1.5 it follows that $\sigma_{\leq 0}R^!_K = (R^!_K)^0$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for $R^!_K$. Hence K is a $\mathcal{D}_{\mathrm{fd}}$-generator for $R^!_K$. Since R is reflexive, it now follows from Theorem 2.3.8 that the natural map $R \to R^{!!}_K$ is an isomorphism. By Remark 2.3.4(1), this is the case if and only if R is $\mathfrak{m}$-adically complete. In particular, R must be local and $\mathfrak{m}$ its unique maximal ideal. □

Remark 4.0.5. Let R be a non-regular complete local k-algebra of Krull dimension at least one. Then R is reflexive by Theorem 4.0.4, but neither R nor $R^!_K$ are proper. This gives to our knowledge the first example of a reflexive dg category such that neither R nor $\mathcal{D}_{\mathrm{fd}}(R)$ are proper.

Example 4.0.6. Suppose that R is finite type over k and reflexive. Then R is a finite product of Artinian local k-algebras: by Example 4.0.1 and Theorem 4.0.4, it is a finite product of complete local k-algebras. But a complete k-algebra is finite type exactly when it is Artinian.

Remark 4.0.7. The field extension condition cannot be dropped: if K is an infinite field extension of k then certainly K is complete local, but not a reflexive k-algebra. However, via the Cohen structure theorem, the assumption on the residue field K can be dropped in one direction of Theorem 4.0.4 by changing the base field. Indeed, let R be a commutative complete local noetherian k-algebra with residue field K.

Then the Cohen structure theorem tells us that R is a complete local K-algebra, and hence by Theorem 4.0.4 R is a reflexive K-algebra.

Remark 4.0.8. Let $(R, \mathfrak{m}, K)$ be a local noetherian k-algebra. If $\mathrm{Kos}(\mathfrak{m})$ denotes the Koszul complex of $\mathfrak{m}$, then it follows from the main result of [PSY14] that R is complete local if and only if it is derived complete with respect to $\mathrm{Kos}(\mathfrak{m})$. In particular, if K is finite over k, then R is reflexive if and only if it is $\mathrm{Kos}(\mathfrak{m})$-derived complete. We remark that in this setting, $\mathrm{Kos}(\mathfrak{m})$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for R precisely when R is regular: in one direction, simply use that $\mathrm{Kos}(\mathfrak{m}) \simeq K$ when R is regular. In the other direction, note that $\mathrm{Kos}(\mathfrak{m})$ is perfect, and hence if it is a $\mathcal{D}_{\mathrm{fd}}$-generator then K is perfect, which implies that R is regular.

Remark 4.0.9. We note a comparison to the derived completion along $\mathcal{D}_{\mathrm{fd}}$ in the sense of [Efi10]. If R is a commutative noetherian ring, then $\hat{R}_{\mathcal{D}_{\mathrm{fd}}(R)}$ is the one-object dg category $\prod_{\mathfrak{m}\in\mathrm{Fin}(R)} \hat{R}_{\mathfrak{m}}$, whereas $\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(R)$ need not have a single compact generator. Observe that the map $R \to \hat{R}_{\mathcal{D}_{\mathrm{fd}}(R)}$ is precisely the map of dg algebras $\mathrm{End}_{\mathcal{D}^{\mathrm{perf}}(R)}(R) \to \mathrm{End}_{\mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(R))}(\mathrm{ev}R)$ induced by the evaluation functor. We note also that $\hat{R}_{\mathcal{D}_{\mathrm{fd}}(R)}$ is the pseudocompact completion of R.

5. Reflexivity via Koszul duality

In this section we use Koszul duality to understand reflexivity. The key point for us will be that the Koszul double dual of an augmented dg algebra computes its derived completion along k, a viewpoint which was first taken in [DGI06]. We begin by recalling facts about dg coalgebras and the bar and cobar constructions. We then use known derived completeness results to show that large classes of (co)connective dg algebras are reflexive. We then define what it means for a dg coalgebra to be reflexive, and compare this notion to the one for algebras across both linear and Koszul duality.

5.1. **Coalgebras, comodules, coderived categories.** We begin by quickly recalling some standard facts about dg coalgebras and their coderived categories. For comprehensive references on this section and the next, see e.g. [Lef03, Pos11, LV12]. Throughout this section, we let R be a commutative finite dimensional semisimple k-algebra (i.e. a finite product of finite field extensions of k). If M is a dg R-module then we denote by $M^\vee := \mathrm{Hom}_R(M, R)$ its R-linear dual.

A **dg-R-coalgebra** is a comonoid in the symmetric monoidal category of dg-R-modules: explicitly it is a complex of R-modules C with a coassociative comultiplication $\Delta\colon C \to C \otimes_R C$ and a counit $\eta\colon C \to R$. The condition that Δ is a chain map translates into the condition that d is a coderivation for Δ. A dg coalgebra is **coaugmented** if η admits a section which is a morphism of R-coalgebras. In this case, the coaugmentation coideal $\bar{C} := \ker\eta$ becomes a noncounital dg coalgebra under the reduced comultiplication $\bar{\Delta}$. Say that a coaugmented dg coalgebra C is **conilpotent** if for all $c \in \bar{C}$ there exists $N \in \mathbb{N}$ such that $\bar{\Delta}^N(c) = 0$.

If C is a dg-R-coalgebra then its R-linear dual $C^\vee$ is a dg-R-algebra under the operations $\eta^\vee$ and $\Delta^\vee$. In fact, $C^\vee$ is a **pseudocompact** dg-R-algebra, meaning a topological dg algebra obtained as an inverse limit of discrete finite dimensional dg algebras, equipped with the inverse limit topology. If C is conilpotent then $C^\vee$ is **pronilpotent**, meaning that the finite dimensional algebras occurring in the inverse limit are all nilpotent extensions of R. The R-linear dual functor gives an equivalence of categories between dg-R-coalgebras and pseudocompact dg-R-algebras, and between conilpotent dg-R-coalgebras and pronilpotent dg-R-algebras.

If C is a dg-R-coalgebra, a (right) **dg-C-comodule** is a dg-R-module V together with a coaction map $\rho\colon V \to V \otimes C$ such that $(\mathrm{id}_V \otimes \Delta)\rho = (\rho \otimes \mathrm{id}_C)\rho$. A **$C$-colinear map** between two dg-C-comodules U, V is an R-linear map $U \to V$ which is compatible with the coactions. These define an abelian dg category $C{-}\mathbf{Comod}$ of dg-C-comodules. We let $\mathrm{Hot}(C) \coloneqq H^0(C{-}\mathbf{Comod})$ denote the corresponding homotopy category; it is a triangulated category. The subcategory $\mathbf{CoAcy}(C)$ of **coacyclic** C-comodules is then defined to be the smallest localising subcategory of $\mathrm{Hot}(C)$ containing the totalisations of exact triples of C-comodules. The **coderived category** of C is the Verdier quotient $\mathcal{D}^{\mathrm{co}}(C) \coloneqq \mathrm{Hot}(C)/\mathbf{CoAcy}(C)$. By [Pos11, §5.5] the triangulated category $\mathcal{D}^{\mathrm{co}}(C)$ is compactly generated by the full subcategory $\mathbf{fd}(C) \hookrightarrow \mathcal{D}^{\mathrm{co}}(C)$ on those comodules weakly equivalent to finite dimensional comodules. When C is conilpotent over R there is a natural equivalence $\mathbf{fd}(C) \simeq \mathbf{thick}_{\mathcal{D}^{\mathrm{co}}(C)}(R)$.

One can also recover $\mathcal{D}^{\mathrm{co}}(C)$ as the homotopy category of a model structure on the category of C-comodules; the cofibrations are the injections and the weak equivalences are the maps with coacyclic cone. In particular every comodule is cofibrant. Via taking dg quotients one can also enhance $\mathcal{D}^{\mathrm{co}}(C)$ to a pretriangulated dg category, and we will frequently regard it as such.

A weak equivalence between comodules is a quasi-isomorphism, but the converse is not true. In particular, there is a quotient map $\mathcal{D}^{\mathrm{co}}(C) \to \mathcal{D}(C{-}\mathbf{Comod})$ which is full and essentially surjective, but not faithful (fullness follows from Brown representability). Recalling that the R-linear dual $C^\vee$ is a pseudocompact dg algebra, the linear dual functor gives a contravariant equivalence between $C{-}\mathbf{Comod}$ and the category $C^\vee{-}\mathbf{pcMod}$ of pseudocompact $C^\vee$-modules. Thus, we obtain a natural quasi-equivalence of pretriangulated dg categories between $\mathcal{D}(C{-}\mathbf{Comod})^{\mathrm{op}}$ and $\mathcal{D}(C^\vee{-}\mathbf{pcMod})$. The forgetful functor from pseudocompact $C^\vee$-modules to all $C^\vee$-modules induces a functor $\mathcal{D}(C^\vee{-}\mathbf{pcMod}) \to \mathcal{D}(C^\vee)$ which is neither essentially surjective nor full. Hence by combining the above functors we obtain an R-linear dual functor $\mathcal{D}^{\mathrm{co}}(C)^{\mathrm{op}} \to \mathcal{D}(C^\vee)$ which sends a comodule N to its R-linear dual $N^\vee$. In general this functor need not be full, faithful, or essentially surjective.

5.2. **Bar and cobar constructions.** Let A be a dg-R-algebra. Recall that A is **augmented** if there is an R-algebra map $A \to R$ splitting the unit. In this case, the augmentation ideal $\bar{A} \coloneqq \ker(A \to R)$ becomes a nonunital dg algebra.

If V is a dg-R-module, its **tensor coalgebra** is the dg-R-coalgebra given by $T^c_R(V) \coloneqq R \oplus V \oplus (V \otimes_R V) \oplus \cdots$ with comultiplication given by the deconcatenation coproduct, which sends a tensor $v_1 \otimes \cdots \otimes v_n$ to the sum $\sum_i (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n)$. It is not hard to see that $T^c_R V$ is a conilpotent dg-R-coalgebra. In fact, $T^c_R(V)$ is the cofree conilpotent coalgebra on V, in the sense that T^c_R is the right adjoint to the forgetful functor from conilpotent dg-R-coalgebras to dg-R-modules.

If A is an augmented dg-R-algebra, then its **bar construction** is the conilpotent dg coalgebra $B_R A$ whose underlying graded coalgebra is $T^c_R(\bar{A}[1])$. The differential combines the usual differential on the tensor coalgebra with the multiplication on A. We denote the R-linear dual of $B_R A$ by $B^\vee_R A$; it is a pseudocompact dg-R-algebra.

Example 5.2.1. If A is the square-zero extension $R[\epsilon]/\epsilon^2$, with ϵ in degree $1-n$, then $B^\vee_R A$ is the pseudocompact algebra $R[\![x]\!]$ with x placed in degree n. More generally, the square-zero extension $R \oplus V$ has dual bar construction given by the free pseudocompact dg-R-algebra on the pseudocompact dg-R-module $V^\vee[-1]$.

There is a dual construction, the cobar construction, which sends conilpotent coalgebras to algebras. In brief, let C be a conilpotent dg-R-coalgebra. The **cobar construction** of C is the dg algebra ΩC whose underlying graded algebra is $T(\bar{C}[-1])$, the tensor algebra on the shifted augmentation ideal of C. The differential combines the natural internal differential on the tensor algebra with the comultiplication on C.

Theorem 5.2.2 (Koszul duality, cf. [Pos11])**.**
(1) *Ω and B form an adjunction $\Omega\colon \mathbf{dgCog}_R^{\mathrm{conil}} \longleftrightarrow \mathbf{dgAlg}_R^{\mathrm{aug}} \colon B$.*
(2) *Let A be an augmented dg-R-algebra. The counit $\Omega BA \to A$ is a quasi-isomorphism of algebras.*
(3) *Let C be a conilpotent dg-R-coalgebra. The unit $C \to B\Omega C$ is a weak equivalence of coalgebras (i.e. is sent to a quasi-isomorphism by Ω).*

Remark 5.2.3. In fact, Positselski proves that there exist model structures on the categories of augmented dg algebras and conilpotent dg coalgebras making $\Omega \dashv B$ into a Quillen equivalence. A weak equivalence of algebras is precisely a quasi-isomorphism. The weak equivalences of coalgebras are created by Ω; every weak equivalence is a quasi-isomorphism but the converse is not true.

Following [BCL25] we will refer to any pair (C, A) consisting of a conilpotent dg-R-coalgebra C and an augmented dg-R-algebra A such that $\Omega C \simeq A$ as a **Koszul duality pair**. The prototypical examples of Koszul duality pairs are pairs of the form $(C, \Omega C)$ and (BA, A), and up to pairwise weak equivalence every pair is of this form.

Theorem 5.2.4 (Module-comodule Koszul duality, cf. [Pos11])**.** *If (C, A) is a Koszul duality pair then there is a quasi-equivalence of pretriangulated dg categories $\mathcal{D}^{\mathrm{co}}(C) \simeq \mathcal{D}(A)$ which sends R to A and C to R.*

Module-comodule Koszul duality can be used to show that the dual bar construction computes derived endomorphisms. Let C be a conilpotent dg coalgebra and M, N two C-comodules. We write $\mathbb{R}\mathrm{Hom}_C(M, N) \in \mathcal{D}(k)$ for the derived mapping space between M and N computed in the dg category $\mathcal{D}^{\mathrm{co}}(C)$. This can be computed as the complex of C-colinear morphisms $\mathrm{Hom}_C(M, \tilde{N})$, where $\tilde{N}$ is a fibrant replacement of N (note that since every comodule is cofibrant, we do not need to replace M).

Proposition 5.2.5. *Let A be an augmented dg-R-algebra. Then there is a dg-R-algebra quasi-isomorphism $B_R^\vee A \simeq \mathbb{R}\mathrm{End}_A(R)$.*

Proof. Put $C \coloneqq B_R A$. By module-comodule Koszul duality, there are quasi-isomorphisms of dg-R-algebras

$$\mathbb{R}\mathrm{End}_A(R) \simeq \mathbb{R}\mathrm{End}_{\mathcal{D}(A)}(R) \simeq \mathbb{R}\mathrm{End}_{\mathcal{D}^{\mathrm{co}}(C)}(C)$$

and since C is an injective C-comodule, it is fibrant, and we can compute its derived endomorphisms as $\mathbb{R}\mathrm{End}_{\mathcal{D}^{\mathrm{co}}(C)}(C) \simeq \mathrm{End}_C(C) \cong C^\vee$, as desired. □

In particular, there is a dg-R-algebra quasi-isomorphism $B_R^\vee A \simeq (A^!_R)^{\mathrm{op}}$, which we will implicitly use going forward.

5.3. **Reflexivity for connective algebras.**

Definition 5.3.1. Say that a dg algebra A is **locally proper** if each $H^i(A)$ is a finite dimensional vector space.

If A is a connective dg algebra, there is a dg algebra map $A \to H^0(A)$.

Theorem 5.3.2. *Let A be a connective locally proper dg-k-algebra. Suppose that the finite dimensional k-algebra $R \coloneqq H^0(A)/\mathrm{rad}H^0(A)$ is commutative, and that the natural map $A \twoheadrightarrow H^0(A) \twoheadrightarrow R$ of dg algebras admits a splitting. Then A is reflexive.*

Proof. Clearly A is an augmented dg-R-algebra with R commutative semisimple. We wish to apply Proposition 3.4.1, for which we need to know that the natural map $A \to B^\vee_R B^\vee_R A$ is a quasi-isomorphism. But this is [Boo22, 4.2.8] - note that although the theorem is stated for $R = k$ only, the proof adapts, as already observed in [Boo22, §8.1]. □

Corollary 5.3.3. *Let A be a connective locally proper dg algebra. Suppose that $H^0(A)/\mathrm{rad}H^0(A) \cong k$. Then A is reflexive.*

Remark 5.3.4 (Obstructions to augmentations). When R is separable over k, then $H^0(A)$ becomes an augmented R-algebra by the Wedderburn–Malcev theorem. In order to lift the map $R \to H^0(A)$ to a dg algebra map $R \to A$, we then simply need to lift it to a map $R \to A^0$. Standard obstruction theory methods show that the existence of such a lift is classified by the k-invariant, which is valued in the R-linear Hochschild cohomology $HH^*_R(H^0(A), \tau_{<0}A)$ associated to the two-stage Postnikov tower $A \to H^0(A)$ of dg-R-algebras.

Example 5.3.5 (Relative singularity categories). Let R be a commutative Gorenstein algebra over an algebraically closed field of characteristic zero. Suppose that $A = \mathrm{End}_R(R \oplus M)$ is a noncommutative resolution of R in the sense of [Boo21], with M a basic MCM R-module. Let $e = \mathrm{id}_R \in A$ be the obvious idempotent, so that A/AeA is the stable endomorphism algebra $\underline{\mathrm{End}}_R(M)$. Since M was basic, A/AeA is a nilpotent extension of a finite dimensional k-algebra of the form $k \times \cdots \times k$. Then [Boo21] gives a Morita equivalence between the derived quotient $A/^{\mathbb{L}}AeA$ and the relative singularity category $\Delta_R(A) \coloneqq \mathcal{D}^{\mathrm{perf}}(A)/\mathbf{thick}(eA)$. When R is a complete local hypersurface singularity, then $A/^{\mathbb{L}}AeA$ satisfies the conditions of Theorem 5.3.2 and is hence reflexive. It follows that the dg category $\Delta_R(A)$ is also reflexive.

5.4. **Reflexivity for coconnective algebras.**

If A is a strictly coconnective dg algebra, note that A is an $H^0(A)$-algebra.

Theorem 5.4.1. *Let A be a strictly coconnective locally proper dg algebra. Assume that the following conditions hold:*

1. *$H^0(A)$ is a commutative semisimple k-algebra.*
2. *$H^1(A)$ vanishes.*
3. *A admits an augmentation as a dg-$H^0(A)$-algebra.*

Then A is reflexive.

Proof. By Theorem 3.3.1, we need only show that A is derived complete with respect to $R \coloneqq H^0(A)$. To do this, we prove a coconnective version of [Boo22, 4.2.8].

First observe that A is an augmented R-algebra. Let $\tilde{A}$ be an A_∞-minimal model for A; it is a strictly coconnective A_∞-R-algebra with $\tilde{A}^0 \cong R$ and $\tilde{A}^1 \cong 0$, such that each $\tilde{A}^i$ is finite dimensional. Put $E \coloneqq B_R^\vee \tilde{A}$, where B_R now means the A_∞-bar construction. Since the A_∞-bar construction is quasi-isomorphism invariant, we have $E \simeq B_R^\vee A$. By the connectivity and finiteness conditions on $\tilde{A}$, we see that E is a connective dg algebra with each E^i finite dimensional and $E^0 \cong R$. It follows that BE is cohomologically raylike in the sense of [Boo22]. Hence, [Boo22, 4.1.11] tells us that there is a natural quasi-isomorphism $A_R^{!!} \simeq \hat{\Omega}(E^\vee)$ between the derived completion of A and the completed cobar construction on $E^\vee$. As in the proof of [Boo22, 4.2.8], since $E^\vee \simeq BA$, to prove that A is derived complete it now suffices to prove that the natural completion map $c\colon \Omega(E^\vee) \to \hat{\Omega}(E^\vee)$ is a quasi-isomorphism. The analogous proof in [Boo22, 4.2.6] is nontrivial, but in our setting this is clear, since $\Omega(E^\vee)$ is the tensor algebra on a complex concentrated in degrees ≥ 2. □

Remark 5.4.2. If A is a strictly coconnective locally proper dg algebra with $A^1 \cong 0$, then A satisfies both (2) and (3) of Theorem 5.4.1. Note that we also have $H^0(A) \cong A^0$ under these assumptions.

Example 5.4.3 (Simple-minded collections). Let k be an algebraically closed field. Let Λ be an ungraded finite dimensional k-algebra and $\mathcal{T} \subseteq \mathcal{D}^b(\mathbf{mod}\Lambda)$ a triangulated subcategory admitting a **rigid** simple-minded collection $\{S_1, \dots, S_n\}$ (rigidity means that $\operatorname{Ext}^1_\Lambda(S_i, S_j)$ vanishes for all i, j). Let A be the derived endomorphism algebra of $S \coloneqq \oplus_i S_i$, so that $\mathcal{T}$ and A are Morita equivalent. Clearly A is locally proper. As in Lemma 3.2.3, we may assume that $H^0(A)$ is a product of division algebras, and - since k is algebraically closed - a finite product of copies of k. Moreover we may assume that A is strictly coconnective via the use of minimal resolutions. By the rigidity hypothesis, $H^1(A)$ vanishes, so that by an application of Theorem 5.4.1 we see that A, and hence $\mathcal{T}$, is reflexive as long as the natural map $H^0A \to A$ admits a retract. This is the case if, for example, the standard t-structure on $\mathcal{D}^b(\mathbf{mod}\Lambda)$ restricts to a t-structure on $\mathcal{T}$ adjacent to the co-t-structure of [KN13], since in this case the desired retract $A \to H^0A$ is the truncation map.

The following result is roughly the Koszul dual of Theorem 5.3.2:

Proposition 5.4.4. *Let R be a commutative semisimple finite dimensional k-algebra, and let A be an augmented dg-R-algebra such that $\mathbb{R}\mathrm{End}_A(R)$ is connective and proper. Then A is derived complete over R.*

Proof. Since $\mathbb{R}\mathrm{End}_A(R)$ is connective and proper, there is a finite dimensional dg algebra E and a quasi-isomorphism $E \simeq \mathbb{R}\mathrm{End}_A(R)$ (see e.g. [GRS24, Appendix]). Hence the dg coalgebras BA and $E^\vee$ are quasi-isomorphic. But by assumption they are both coconnective, so by [Lef03, Proposition 1.3.5.1.e] they are actually weakly equivalent. As in [Boo22, Proposition 4.1.7] we see that the derived completion map $A \to A_R^{!!}$ is quasi-isomorphic to the completion map $\Omega(E^\vee) \to \hat{\Omega}(E^\vee)$. So it suffices to check that $\Omega(E^\vee)$ is complete with respect to its maximal ideal $\mathfrak{m}$. But this is clear since $E^\vee$ was coconnective and finite dimensional: for a fixed I there exists an N such that for all $i \leq I$ and all $n \geq N$ the map $\Omega(E^\vee) \to \Omega(E^\vee)/\mathfrak{m}^n$ is an isomorphism on H^i. □

Corollary 5.4.5. *Let A be a smooth coconnective dg algebra with $H^0(A)$ a commutative finite dimensional semisimple k-algebra. Suppose that the algebra map $H^0(A) \to A$ admits a retract. Then A is reflexive.*

Proof. Putting $R \coloneqq H^0(A)$, it is clear that A is an augmented dg-R-algebra. By Theorem 3.3.1 it suffices to check that A is derived complete. By an application of Proposition 5.4.4, it suffices to check that $A^! \coloneqq \mathbb{R}\mathrm{End}_A(R)$ is connective and proper. Connectivity follows from [KN13, Lemma 6.2] and properness follows from the smoothness assumption on A, cf. [BCL25, Proposition 5.25]. □

Remark 5.4.6. In the situation of Corollary 5.4.5, we obtain an isomorphism of derived Picard groups $\mathrm{DPic}(A) \cong \mathrm{DPic}(A^!)$. This generalises the main result of [MYH19], which assumes that $H^0(A) \cong k$, and in addition that $A^!$ is concentrated in degree zero.

5.5. **Reflexivity for coalgebras.** Recall that we write $\mathbb{R}\mathrm{Hom}_C(-,-)$ for the derived mapping space functor of $\mathcal{D}^{\mathrm{co}}(C)$. The following definition appears in [BCL25] and is a homotopical version of Takeuchi's definition of a quasi-finite comodule [Tak77].

Definition 5.5.1. Let C be a dg coalgebra. Say that a C-comodule M is **homotopy quasi-finite** if, for all $X \in \mathbf{fd}(C)$, the complex $\mathbb{R}\mathrm{Hom}_C(X, M)$ is proper. We abbreviate homotopy quasi-finite by **hqf** and denote the full subcategory of $\mathcal{D}^{\mathrm{co}}(C)$ on the hqf comodules by $\mathbf{hqf}(C)$.

Remark 5.5.2. The category **hqf** is Morita invariant, in the sense that if there is a quasi-equivalence $\mathcal{D}^{\mathrm{co}}(C) \simeq \mathcal{D}^{\mathrm{co}}(C')$ then $\mathbf{hqf}(C) \simeq \mathbf{hqf}(C')$. This is because $\mathbf{fd}(C)$ has a Morita invariant description as the compact objects in $\mathcal{D}^{\mathrm{co}}(C)$.

Proposition 5.5.3. *Let (C, A) be a Koszul duality pair. Module-comodule Koszul duality induces the following quasi-equivalences of pretriangulated dg categories:*

(1) $\mathcal{D}^{\mathrm{perf}}(A) \simeq \mathbf{fd}(C)$.
(2) $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathbf{hqf}(C)$.

Proof. This appears in [BCL25] but we give a self-contained proof. The quasi-equivalence of (1) is well known and follows from restricting the Koszul duality equivalence $\mathcal{D}(A) \simeq \mathcal{D}^{\mathrm{co}}(C)$ to compact objects. To show that (2) holds, let M be an A-module and let N be its corresponding C-comodule. Then we have

$$\begin{aligned} M \in \mathcal{D}_{\mathrm{fd}}(A) &\iff \mathbb{R}\mathrm{Hom}_A(\mathcal{D}^{\mathrm{perf}}(A), M) \in \mathcal{D}_{\mathrm{fd}}(k) \\ &\iff \mathbb{R}\mathrm{Hom}_C(\mathbf{fd}(C), N) \in \mathcal{D}_{\mathrm{fd}}(k) && \text{by (1)} \\ &\iff N \in \mathbf{hqf}(C) && \text{by definition.} \end{aligned}$$ □

Note that there is a natural functor $\mathrm{ev}_C \colon \mathbf{fd}(C)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(\mathbf{hqf}(C)^{\mathrm{op}})$ that sends a finite dimensional C-comodule X to the $\mathbf{hqf}(C)$-module $N \mapsto \mathbb{R}\mathrm{Hom}_C(X, N)$. Say that a dg coalgebra C is **reflexive** if ev_C is a quasi-equivalence.

Proposition 5.5.4. *Let (C, A) be a Koszul duality pair. Then A is reflexive if and only if C is reflexive.*

Proof. The diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{perf}}(A)^{\mathrm{op}} & \xrightarrow{\simeq} & \mathbf{fd}(C)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathrm{ev}_A} & & \downarrow{\scriptstyle \mathrm{ev}_C} \\ \mathcal{D}_{\mathrm{fd}}(\mathcal{D}_{\mathrm{fd}}(A)^{\mathrm{op}}) & \xrightarrow{\simeq} & \mathcal{D}_{\mathrm{fd}}(\mathbf{hqf}(C)^{\mathrm{op}}) \end{array}$$

commutes, which implies that ev_A is a quasi-equivalence precisely when ev_C is. □

5.6. **Reflexivity and linear duality.** Let R be a finite dimensional commutative semisimple k-algebra and let C be a dg-R-coalgebra. We wish to study the relationship between reflexivity of C and reflexivity of its R-linear dual $C^\vee$.

Let A be a dg algebra and M a proper A-module. Say that M **admits a finite dimensional model** if there is a finite dimensional A-module M' and an A-linear quasi-isomorphism $M \simeq M'$.

Proposition 5.6.1. *Let (C, A) be a Koszul duality pair. Then:*

(1) *Sending $R \mapsto C^\vee$ induces a quasi-equivalence $\mathbf{thick}_A(R) \xrightarrow{\simeq} \mathcal{D}^{\mathrm{perf}}(C^\vee)^{\mathrm{op}}$.*
(2) *Across the quasi-equivalence of (1), the restricted evaluation map $\mathrm{ev}_{A,R}$ corresponds to the linear duality map $\mathbf{fd}(C)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(C^\vee)$.*
(3) *The following are equivalent:*
   (a) *A is R-restricted reflexive.*
   (b) *The linear duality map $\mathbf{fd}(C)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(C^\vee)$ is a quasi-equivalence.*
(4) *The linear duality map $\mathbf{fd}(C)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(C^\vee)$ is quasi-essentially surjective precisely when every proper $C^\vee$-module admits a finite dimensional model.*

Proof. Claim (1) is clear since $C^\vee \simeq \mathbb{R}\mathrm{End}_A(R)$. To prove (2), for brevity put $\mathcal{R} \coloneqq \mathbf{thick}_{\mathcal{D}(A)}(R)^{\mathrm{op}}$. Applying $\mathcal{D}_{\mathrm{fd}}$ to the equivalence of (1) hence yields an equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{R}) \simeq \mathcal{D}_{\mathrm{fd}}(C^\vee)$. To check that the diagram

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{perf}}(A)^{\mathrm{op}} & \xrightarrow{\simeq} & \mathbf{fd}(C)^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathrm{ev}_{A,R}} & & \downarrow{\scriptstyle (-)^\vee} \\ \mathcal{D}_{\mathrm{fd}}(\mathcal{R}) & \xrightarrow{\simeq} & \mathcal{D}_{\mathrm{fd}}(\mathcal{D}^{\mathrm{perf}}(C^\vee)) \end{array}$$

commutes, it is enough to check it on the generator A of $\mathcal{D}^{\mathrm{perf}}(A)$ together with its endomorphisms. But both compositions send the object A to the $\mathcal{D}^{\mathrm{perf}}(C^\vee)$-module which sends $C^\vee$ to R, and an element $a \in A$ to its action on R. The equivalence of (3a) and (3b) is immediate from the proof of (2). Finally, (4) follows from the fact that the essential image of $\mathbf{fd}(C)^{\mathrm{op}} \to \mathcal{D}_{\mathrm{fd}}(C^\vee)$ consists exactly of those modules with finite dimensional models. □

If C is a conilpotent dg-R-coalgebra, observe that there is a natural completion map $\Omega C \to B_R(C^\vee)^\vee$. Following [Boo22, Proposition 4.1.7], this completion map is a quasi-isomorphism when C is finite dimensional in each degree and either connective or 2-coconnective (i.e. connected with $C^0 \cong k$ and $C^1 \cong 0$).

Corollary 5.6.2. *Let (C, A) be a Koszul duality pair such that R is a $\mathcal{D}_{\mathrm{fd}}$-generator for A. If C is reflexive then $C^\vee$ is reflexive. The converse is true as long as R is a $\mathcal{D}_{\mathrm{fd}}$-generator for $C^\vee$ and the completion map $A \to B_R(C^\vee)^\vee$ is a quasi-isomorphism.*

Proof. If C is reflexive then so is ΩC. Hence $\mathcal{D}_{\mathrm{fd}}(\Omega C)$ is also reflexive. But by Proposition 5.6.1(1), $\mathcal{D}_{\mathrm{fd}}(\Omega C)$ is Morita equivalent (up to an opposite) to $C^\vee$, and hence $C^\vee$ is reflexive. For the converse, if $C^\vee$ is reflexive then by Proposition 5.6.1(1) again, $\mathcal{D}_{\mathrm{fd}}(\Omega C)$ is reflexive. Since R is a $\mathcal{D}_{\mathrm{fd}}$-generator for $C^\vee$, there is a Morita equivalence between $(C^\vee)^!_R$ and $\mathcal{D}_{\mathrm{fd}}(\Omega C)$, and hence $(C^\vee)^!_R$ is reflexive. But $(C^\vee)^!_R$ is the opposite of $B_R(C^\vee)^\vee$, which by assumption is ΩC. Hence ΩC is reflexive, and hence C is reflexive, as desired. □

Remark 5.6.3. In the converse situation of Corollary 5.6.2, one can show that the linear duality map $\mathbf{fd}(C) \to \mathcal{D}_{\mathrm{fd}}(C^\vee)^{\mathrm{op}}$ is a quasi-equivalence. In particular, every proper $C^\vee$-module admits a finite dimensional model.

6. Ginzburg dg algebras

We use our results on Koszul duality to show that completed Ginzburg dg algebras [Gin06] and completed undeformed Calabi–Yau completions [Kel09] are reflexive. A similar treatment appears in [Kel09, Appendix] and [HLW23].

6.1. **Calabi–Yau completions.** Let Q be a finite quiver with vertex set Q_0 and arrow set Q_1. Write kQ_0 for the semisimple algebra on the vertex idempotents. The path algebra of Q is then $kQ \coloneqq T_{kQ_0}(kQ_1)$. We denote composition in the path algebra from left to right, so that ab means 'follow arrow a then arrow b'. If M is a kQ_0-module we write $M^\vee$ for its kQ_0-linear dual. In particular, the space of arrows kQ_1 is a kQ_0-bimodule; its dual is the bimodule of 'opposite arrows', often written $\overline{kQ_1} \coloneqq kQ_1^\vee$. If $a : u \to v$ is an arrow we denote its corresponding opposite arrow by $a^\vee : v \to u$.

Pairing an arrow with its dual gives natural kQ_0-linear pairings $kQ_1 \otimes_{kQ_0} kQ_1^\vee \to kQ_0$ and $kQ_1^\vee \otimes_{kQ_0} kQ_1 \to kQ_0$ which we denote by $\langle -,- \rangle$. Concretely, if a, b are arrows, then $\langle a, b^\vee \rangle$ is zero unless $a = b$, in which case it is $\mathrm{tail}(a) = \mathrm{head}(b^\vee)$. The pairing $\langle a^\vee, b \rangle$ behaves similarly.

Fix an integer n and let R_n be the graded kQ_0-module

$$R_n \coloneqq kQ_0 \oplus kQ_1[-1] \oplus kQ_1^\vee[1-n] \oplus kQ_0[-n].$$

We equip R_n with a multiplication making it into a graded kQ_0-algebra. Writing a basis element of R_n in the form $(x, a, b^\vee, y)$, the multiplication is given by

$$(x, a, b^\vee, y)(u, r, s^\vee, v) = (xu, xr + au, xs^\vee + b^\vee u, \langle a, s^\vee \rangle + \langle b^\vee, r \rangle + xv + yu)$$

It is easy to verify that this multiplication makes R_n into a graded kQ_0-algebra.

Remark 6.1.1. Let S be the graded kQ_0-algebra given by the square-zero extension $kQ \oplus kQ_1[-1]$. In other words, S is the quotient of the path algebra kQ by the square of the arrow ideal. The grading is given by placing the arrows in degree one. Let M be the graded S-bimodule $S^\vee[-n]$, with action given by the pairing $\langle -,- \rangle$. Then R_n is the trivial extension algebra $S \oplus M$. This is a basic example of a construction known as cyclic completion [Seg08].

Let $\Pi_n(Q)$ be Keller's (undeformed) Calabi–Yau completion of kQ, in the sense of [Kel09], and let $\hat{\Pi}_n(Q)$ be the completion of $\Pi_n(Q)$ at the arrow ideal. Note that both of these are augmented dg-kQ_0-algebras. In what follows, if A is an augmented dg algebra we write $A^! \coloneqq B(A)^\vee$. In addition if A is finite dimensional we write $A^\dagger \coloneqq \Omega(A^\vee)$, following the notation of [HLW23].

Lemma 6.1.2. *For a finite quiver Q, there are dg-kQ_0-algebra quasi-isomorphisms*
1. $R_n^\dagger \simeq \Pi_n(Q)$.
2. $R_n^! \simeq \hat{\Pi}_n(Q)$.
3. $\hat{\Pi}_n(Q)^! \simeq R_n$.

Proof. By the definition of the cobar construction, the underlying graded algebra of $R_n^\dagger$ is freely generated over kQ_0 by the arrows kQ_1 in degree zero, the dual arrows $kQ_1^\vee$ in degree $2-n$, and loops z_i, one at each vertex i, placed in degree $1-n$. The differential satisfies $da = 0, da^\vee = 0$, and $dz_i = \sum_a e_i[a, a^\vee]e_i$ where e_i is the idempotent at vertex i. Claim (1) then follows from the description of $\Pi_n(Q)$ as a Ginzburg dg category given in [Kel09, Theorem 6.3]. Since $R_n^! \simeq B(R_n)^\vee$, Claim (2) follows from Claim (1) by an application of [Boo22, Proposition 4.1.7]. Observe that the dg radical of the graded algebra R_n is the ideal $kQ_1[-1] \oplus kQ_1^\vee[1-n] \oplus kQ_0[-n]$. Hence by Corollary 2.3.6 we see that R_n is derived complete along kQ_0. Claim (3) now follows from Claim (2) using the quasi-isomorphism $R_n \simeq R_n^{!!}$. □

Remark 6.1.3. Note that the relations $dz_i = \sum_a e_i[a, a^\vee]e_i$ impose the preprojective relations on the cohomology of $\Pi_n(Q)$. In particular, $\Pi_2(Q)$ is the dg preprojective algebra of Q. When Q has no cycles and is not of ADE type, then $\Pi_2(Q)$ is a resolution of the classical preprojective algebra of Q [Her16].

Proposition 6.1.4. *For $n \geq 2$, both R_n and $\hat{\Pi}_n(Q)$ are reflexive.*

Proof. Since $n \geq 2$, the algebra R_n is coconnective and we have $H^0 R_n \cong kQ_0$, which is a finite dimensional semisimple k-algebra. Moreover R_n is derived complete at kQ_0 by Lemma 6.1.2. Hence by Theorem 3.3.1, we see that both R_n and $R_n^!$ are reflexive, as required. □

Remark 6.1.5. When Q has no cycles, the natural map $\Pi_n(Q) \to \hat{\Pi}_n(Q)$ is an isomorphism (in particular a quasi-isomorphism!), and hence for $n \geq 2$ we see that $\Pi_n(Q)$ is reflexive. When Q is a tree, $n = 2$, and $\mathrm{char}(k) \neq 2$ this was proved in [EL17], who used it to compute the symplectic cohomology of an associated Liouville manifold obtained by plumbing copies of T^*S^2 along Q.

Remark 6.1.6. When the quiver Q itself is graded, then $\Pi_n(Q)$ inherits an extra grading, often known as an **Adams grading**. In good situations, one can use this to prove that $\Pi_n(Q)$ is formal, or complete at the arrow ideal.

Remark 6.1.7. For $n \leq 0$, the algebra R_n is neither connective nor coconnective. However, R_1 is coconnective, with $H^0(R_1) \cong R_1^0$ a square-zero extension of kQ_0. In particular, one should be able to generalise Proposition 6.1.4 to the $n = 1$ case; ideally this would follow from a similar analysis along the lines of Section 3 where one relaxes the condition that H^0 be semisimple to the condition that it be Artinian. It is possible that one can use the compactly generated co-t-structures of [Pau08] to do this (specifically, the co-t-structure generated by R_1 itself together with all of its negative shifts).

6.2. **Completed Ginzburg algebras.** In the $n = 3$ case we now want to turn on a superpotential W, to obtain reflexivity results for deformed 3-Calabi–Yau completions. These were first studied by Ginzburg [Gin06] and are hence known as Ginzburg dg algebras. In this section, we restrict k to be a characteristic zero field.

Recall that a **(completed) superpotential** W on a quiver Q is an element of the completed cocentre $\widehat{kQ}/[\widehat{kQ},\widehat{kQ}]$. In simpler terms, a superpotential is a possibly infinite linear combination of cycles, with only a finite number of cycles of any given length occurring. We call a superpotential **finite** if it has only finitely many terms. Let W be a superpotential and a an arrow of Q. We define the **cyclic derivative** of W with respect to a to be the sum $\partial_a W \coloneqq \sum_{W=uav} vu \in \widehat{kQ}$.

The **Ginzburg dg algebra** $\Gamma(Q,W)$ associated to a quiver with finite superpotential (Q,W) is defined as follows. The underlying graded algebra is the same as that of $\Omega(R_3^\vee)$. The differential is defined by $da = 0$, $da^\vee = \partial_a W$, and $dz_i = \sum_a e_i[a,a^\vee]e_i$. In other words, it is the same as $\Pi_3(Q)$ but the differentials of the opposite arrows are deformed by the superpotential W. We have $H^0(\Gamma(Q,W)) \cong kQ/(\partial_a W)_{a\in Q_1}$, the **Jacobi algebra** $\mathrm{Jac}(W)$ associated to the superpotential. Similarly, one can define a **completed Ginzburg dg algebra** $\hat{\Gamma}(Q,W)$ from a quiver with superpotential, and $H^0(\hat{\Gamma}(Q,W))$ is the **completed Jacobi algebra** $\widehat{\mathrm{Jac}}(W)$.

Remark 6.2.1. There is a natural completion map $\Gamma(Q,W) \to \hat{\Gamma}(Q,W)$. When (Q,W) is **Jacobi-finite**, i.e. $\mathrm{Jac}(W)$ is a finite dimensional algebra, then this map is a quasi-isomorphism; one can show this in a similar manner to the proof of [Boo22, Theorem 4.3.6].

We will use the notation $W = W^{\geq m}$ to mean that all of the cycles appearing in W have length $\geq m$. This ensures that each term in the cyclic derivatives of W has length at least $m-1$.

Proposition 6.2.2 (Van den Bergh)**.** *Let Q be a quiver and $W = W^{\geq 2}$ a superpotential on Q. Then there exists the structure of an A_∞-kQ_0-algebra R_3^W on the graded vector space R_3 and a quasi-isomorphism $B(R_3^W)^\vee \simeq \hat{\Gamma}(Q,W)$. If $W = W^{\geq 3}$ then R_3^W is minimal (i.e. the differential vanishes). If $W = W^{\geq 4}$ then the underlying graded algebra of R_3^W agrees with the previously defined algebra structure on R_3. If $W = 0$ then $R_3^W = R_3$.*

Proof. This is [Kel09, A.15]. Abstract existence of the A_∞ structure simply follows from the fact that the Ginzburg algebra is a dg algebra. To actually construct the A_∞ products m_r, the idea is that to obtain R_3^W from R_3, one need only add terms of the form $m_r(a_1,\ldots,a_r) = \pm b^\vee/r$ corresponding to length $r+1$ cycles in W whose cyclic derivative with respect to b is a cyclic permutation of $a_1\cdots a_r$. Note that here we are using that k has characteristic zero. In particular, if W has no 2-cycles then we do not modify the differential of R_3, and if W has no 3-cycles then we do not modify the multiplication. □

Remark 6.2.3. Cyclic invariance of W ensures that the above constructed A_∞ structure is actually a cyclic A_∞ structure in the sense of [KS09]; the relevant inner product on R_3^W is the one described above. A concrete description of the above constructed m_r for the two-loop one-vertex quiver is given in [BW24] in terms of necklace polynomials.

Remark 6.2.4. If Q is a quiver and $\lambda = (\lambda_i)_{i\in Q_0}$ is a set of weights on Q, then the **deformed dg preprojective algebra** $\Pi_2(Q,\lambda)$ is defined similarly to $\Pi_2(Q)$, but where we now modify dz_i by a $\lambda_i e_i$ term [Kel09, KY18]. The relevant modification of R_2 now requires a curvature term, which our methods cannot handle.

Note that R_3^W is A_∞-quasi-isomorphic to a dg algebra, namely the dg algebra $\Omega B(R_3^W)$. We will abusively say that R_3^W is reflexive to mean that this latter dg algebra is reflexive.

Proposition 6.2.5. *Let Q be a finite quiver and $W = W^{\geq 3}$ a superpotential on Q. Then both R_3^W and $\hat{\Gamma}(Q, W)$ are reflexive.*

Proof. This is similar to the proof of Proposition 6.1.4. Put $A \coloneqq \Omega B(R_3^W)$. Then A is a coconnective dg algebra with finite dimensional semisimple H^0. Moreover A is a proper dg algebra and hence derived complete. Since $A^! \simeq \hat{\Gamma}(Q, W)$, an application of Theorem 3.3.1 proves the desired statement. □

Remark 6.2.6. The proof of Proposition 6.2.5 shows that R_3^W is derived complete, and so we obtain an A_∞-quasi-isomorphism $R_3^W \simeq \mathbb{R}\mathrm{End}_{\hat{\Gamma}(Q,W)}(kQ_0)$. When $W = W^{\geq 4}$, we see that R_3^W is the graded algebra $R_3 \cong \mathrm{Ext}^*_{\hat{\Pi}_3(Q)}(kQ_0, kQ_0)$ equipped with higher A_∞ multiplications. The CY property for $\hat{\Pi}_3(Q)$ yields a description of R_3 in terms of $\mathrm{Ext}^*_{kQ}(kQ_0, kQ_0)$ which reduces to the cyclic completion of Remark 6.1.1. More generally, one should view R_3^W as a 'deformed cyclic completion' of $\mathrm{Ext}^*_{kQ}(kQ_0, kQ_0)$.

7. Chains and cochains on topological spaces

In this section we prove that in a wide variety of situations, the dg (co)algebra of (co)chains on a topological space is reflexive.

7.1. **∞-local systems and Koszul duality.** Here we broadly follow the approach of [BD19, Section 5.1]. Fix a field k and let X be a topological space. We define a dg category $\mathcal{C}(X)$ as follows. First view X as a Kan complex via the singular simplicial set functor. Apply the homotopy coherent rigidification functor $\mathfrak{C}$ to the ∞-groupoid X to obtain a simplicially enriched category $\mathfrak{C}X$. Linearise the mapping spaces to obtain a category $k\mathfrak{C}X$ enriched in simplicial k-vector spaces. Finally, apply the normalised chains functor N of the Dold–Kan correspondence to each mapping space to obtain a dg category $\mathcal{C}(X) \coloneqq N(k\mathfrak{C}X)$. Note that the $Nk\mathfrak{C}$ functor is the left adjoint of the dg nerve functor, so that the dg nerve of $\mathcal{C}X$ is equivalent to X. We have a quasi-equivalence between the derived category $\mathcal{D}(\mathcal{C}X)$ and the category of ∞-local systems of dg k-vector spaces on X.

Remark 7.1.1. The objects of $\mathcal{C}(X)$ are in bijection with the points of X, and the mapping spaces are - up to quasi-isomorphism - given by taking k-linear chains on the corresponding path space.

We let $C_\bullet(X, k)$ denote the dg coalgebra of k-chains on X, and we let $C^\bullet(X, k)$ denote the dg algebra of k-chains on X. We will typically abuse notation and simply denote them as $C_\bullet X$ and $C^\bullet X$, leaving the base field implicit. Observe that $C^\bullet X$ is the k-linear dual of $C_\bullet X$. Fixing a point $x \in X$, let $\Omega_x X$ be the space of x-based Moore loops in X. The 'concatenation of loops' operation on $\Omega_x X$ gives the connective dg coalgebra $C_\bullet \Omega_x X$ the structure of a dg algebra.

Remark 7.1.2. If $G_x X$ denotes the x-based Kan loop group of the singular simplicial set of X, then $C_\bullet(G_x X)$ is a dg Hopf algebra, and there is a quasi-isomorphism $C_\bullet(G_x X) \simeq C_\bullet(\Omega_x X)$ of dg algebras [GJ09].

If X is path connected then, suppressing the point x from the notation, by [BD19, Remark 5.2] we have a quasi-equivalence of dg categories

$$\mathcal{C}(X) \simeq C_\bullet \Omega X$$

and hence to study $\mathcal{C}(X)$ we may as well study the dg algebra $A \coloneqq C_\bullet \Omega X$. Observe that we have an isomorphism $H^0(A) \cong k\pi_1(X)$, and in particular A admits a natural augmentation over k.

Remark 7.1.3. Although there is a natural map $C_\bullet \Omega X \to k\pi_1(X)$, it generally does not admit a section. Even when $k\pi_1(X)$ is a commutative semisimple k-algebra (for example, when k has characteristic zero and $\pi_1(X)$ is finite abelian), the dg algebra $C_\bullet \Omega X$ is rarely a $k\pi_1(X)$-algebra unless X is simply connected.

Theorem 7.1.4 ([RZ18, CHL21]). *Let X be a path connected topological space. Then there is a dg algebra quasi-isomorphism $\Omega(C_\bullet X) \simeq C_\bullet(\Omega X)$.*

Corollary 7.1.5. *Let X be a path connected topological space. Then there is a quasi-isomorphism $(C_\bullet \Omega X)^! \simeq C^\bullet X$.*

Remark 7.1.6. When X is simply connected, Theorem 7.1.4 reduces to a much earlier theorem of Adams, who introduced the cobar construction Ω for precisely this reason [Ada57].

Note that $C^\bullet X$ is coconnective, and has $H^0(C^\bullet X) \simeq k$ if X is path connected.

Example 7.1.7. Take $X = \mathbb{CP}^\infty$, which is a $K(\mathbb{Z}, 2)$, so that ΩX is S^1. Hence, the dg algebra $A = C_\bullet \Omega X$ is (quasi-isomorphic to) the square-zero extension $k[\epsilon]/\epsilon^2$ with ϵ in homological degree one. Hence the Koszul dual is $A^! = k[t]$ with t in cohomological degree two. This is indeed the cohomology ring of X (and hence proves that $C^\bullet X$ is formal).

Example 7.1.8. The following essentially appears as [CHL21, Example 4.3]. Put $X = S^1$ so that ΩX is the discrete space $\mathbb{Z}$. So $A \coloneqq C_\bullet \Omega X \simeq k[t, t^{-1}]$ is concentrated in degree zero (the augmentation is given by $t \mapsto 1$). We know that $\Omega C_\bullet X \simeq A$. On the other hand, $C^\bullet X \simeq k[\epsilon]/\epsilon^2$, with ϵ in cohomological degree one. It follows that $C_\bullet X$ is *quasi-isomorphic* to the dg coalgebra C given by the linear dual of $k[\epsilon]/\epsilon^2$. However, they are *not weakly equivalent*, since their cobar constructions disagree: $\Omega C_\bullet X \simeq A$ while $\Omega C \simeq k[t]$. In fact, C is the coalgebra of cochains on the non-grouplike simplicial set $\Delta^1/\partial\Delta^1$, and the fact that A is the localisation of C at t is a general phenomenon [CHL21, Corollary 4.4].

Remark 7.1.9. If X is simply connected, then the quasi-isomorphism type of the coalgebra $C_\bullet X$ determines its weak equivalence type. If X is not simply connected, this fails, as Example 7.1.8 demonstrates.

Lemma 7.1.10. *Let X be a path connected topological space. Then $\mathcal{D}_{\mathrm{fd}}(C^\bullet X)$ is Morita equivalent to $C_\bullet(\Omega X)^{!!}$.*

Proof. By Corollary 7.1.5 we have a quasi-isomorphism $C_\bullet(\Omega X)^! \simeq C^\bullet X$. By Proposition 3.1.5, k is a $\mathcal{D}_{\mathrm{fd}}$-generator for $C^\bullet X$, so that $\mathcal{D}_{\mathrm{fd}}(C^\bullet X)$ is Morita equivalent to $C^\bullet(X)^!$. Combining these results we obtain the desired statement. □

7.2. **Reflexivity results.** Say that X is **k-finite type** if each $H_i(X,k)$ is finite dimensional over k. By the Universal Coefficient Theorem, this is equivalent to $C^\bullet X$ being locally proper. Say that X is **k-finite** if the graded vector space $H_\bullet(X,k)$ is finite dimensional; this is equivalent to $C^\bullet X$ being proper. A finite CW complex is clearly k-finite. Observe also that if X is a homotopy retract of a k-finite (type) space, then X itself is k-finite (type). If X is a k-finite type topological space, then it necessarily has finitely many path components.

Lemma 7.2.1. *Let X be a simply connected k-finite topological space. Then ΩX is a k-finite type space.*

Proof. It suffices to check that $H^q(\Omega X,k)$ is finite dimensional for every q. This is a standard argument using the cohomological Serre spectral sequence associated to the path fibration $\Omega X \to PX \to X$, where $PX \simeq *$ is the path space of X. This spectral sequence has E_2 page $H^p(X, H^q(\Omega X,k)) \cong H^p(X,k) \otimes_k H^q(\Omega X,k)$ and converges to $H^{p+q}(PX,k)$. Since X was k-finite, the spectral sequence degenerates after finitely many pages. Since X was simply connected, we have $H^0(\Omega X) \cong k$, and so the $q=0$ column of the E_2 page consists of a finite number of finite dimensional vector spaces (namely, the $H^p(X,k)$). Since PX is contractible - and in particular has vanishing H^1 - we see that the $q=1$ column of the E_2 page must also consist of a finite number of finite dimensional vector spaces. Continuing inductively we see that all entries on the E_2 page are finite dimensional, as desired. □

Proposition 7.2.2. *Let X be a k-finite topological space. Then $C^\bullet X$ is reflexive. If X is simply connected, then $C_\bullet \Omega X$ is reflexive.*

Proof. The first claim follows from Corollary 3.3.2, since $H^0(X)$ is semisimple. For the second claim, note that $H_0(\Omega X) \simeq k$ if X is simply connected. In particular, $C_\bullet \Omega X$ is a connective augmented k-algebra, which by Lemma 7.2.1 is locally proper. Hence it is reflexive by Theorem 5.3.2. □

Example 7.2.3. Take $X = S^1$ from Example 7.1.8, so that we have quasi-isomorphisms $C_\bullet \Omega X \simeq k[t,t^{-1}]$ and $C^\bullet X \simeq k[\epsilon]/\epsilon^2$. We compute $(C^\bullet X)^! \simeq k[\![t]\!]$, which is the completion of $k[t,t^{-1}]$ at the maximal ideal $(t-1)$. In particular $C_\bullet \Omega X$ is not derived complete. This example shows that the simply connected hypothesis in Proposition 7.2.2 cannot be dropped.

Say that a topological space X is **$k\pi_1$-local** if $k\pi_1(X)$ is a finite dimensional local k-algebra (i.e. a nilpotent extension of k). This property will be key for us, since it implies that k is a $\mathcal{D}_{\mathrm{fd}}$-generator for $C_\bullet \Omega X$.

Lemma 7.2.4. *Let X be a topological space and k a field. Then X is $k\pi_1$-local if and only if either of the following conditions is satisfied:*

1. *X is simply connected.*
2. *k has characteristic p and $\pi_1(X)$ is a finite p-group.*

Proof. This follows from the main theorem of [Ren71]. □

Theorem 7.2.5. *Let X be a path connected $k\pi_1$-local topological space.*

1. *There is a natural equivalence $\mathcal{D}_{\mathrm{fd}}(C_\bullet \Omega X) \simeq \mathcal{D}^{\mathrm{perf}}(C^\bullet X)$.*
2. *If $C_\bullet X$ is reflexive then so is $C^\bullet X$.*
3. *If ΩX is k-finite type, then both $C_\bullet X$ and $C^\bullet X$ are reflexive.*

(4) *If $C_\bullet X$ is reflexive then there is a natural equivalence*

$$\mathcal{D}_{\mathrm{fd}}(C^\bullet X) \simeq \mathcal{D}^{\mathrm{perf}}(C_\bullet \Omega X).$$

Proof. Since X is $k\pi_1$-local, k is a $\mathcal{D}_{\mathrm{fd}}$-generator for $C_\bullet \Omega X$ by Corollary 2.2.3. So putting $A \coloneqq C_\bullet \Omega X$, we have $\mathcal{D}_{\mathrm{fd}}(A) \simeq \mathcal{D}^{\mathrm{perf}}(A^!) \simeq \mathcal{D}^{\mathrm{perf}}(C^\bullet X)$ by Corollary 7.1.5, which proves claim (1). Claim (2) follows from Corollary 5.6.2. To prove claim (3), by an application of (2) we need only check that $C_\bullet X$ is reflexive. But this is the same as checking that $\Omega C_\bullet X \simeq C_\bullet \Omega X$ is reflexive, and this follows from Theorem 5.3.2. Claim (4) follows from (1). □

Remark 7.2.6. The category $\mathcal{D}_{\mathrm{fd}}(C_\bullet \Omega X)$ appearing in part (1) of Theorem 7.2.5 is the dg category of ∞-local systems of finite dimensional k-vector spaces on X.

Example 7.2.7. Part (3) of Theorem 7.2.5 says that for a pair (X, k) which is of Eilenberg–Moore type in the sense of [DGI06], both $C_\bullet(X, k)$ and $C^\bullet(X, k)$ are reflexive.

Example 7.2.8 (Adams–Hilton models). Let X be a finite CW complex with one 0-cell and no 1-cells. Then ΩX is a k-finite type space, via the use of Adams–Hilton models [AH56]. Since X is simply connected, we can conclude that both $C_\bullet X$ and $C^\bullet X$ are reflexive.

Example 7.2.9 (Rational homotopy theory). Suppose that k has characteristic zero and let X be a simply connected k-finite type space. Then $C_\bullet \Omega X$ is the universal enveloping algebra of the Whitehead Lie algebra $W(X) \coloneqq \pi_*(\Omega X) \otimes_{\mathbb{Z}} k$ [FHT01]. If $W(X)$ is a k-finite type Lie algebra, the PBW theorem tells us that ΩX is k-finite type. In particular, we can conclude that both $C_\bullet X$ and $C^\bullet X$ are reflexive.

Example 7.2.10 (Classifying spaces). Let G be a finite p-group and $X = BG$ its classifying space. Then we have $\Omega X \simeq G$, which certainly is of k-finite type for any k. So if k has characteristic p, then this gives a new proof that the finite dimensional algebra $C_\bullet \Omega X \simeq kG$ is reflexive. We can also conclude that both $C_\bullet BG$ and $C^\bullet BG$ are reflexive.

Example 7.2.11 (p-compact groups). This is a generalisation of the previous example. Let k be a field of characteristic p and let (X, BX, e) be a p-compact group in the sense of [DW94]: X is k-finite, BX is pointed and p-complete, and $e : X \to \Omega BX$ is a homotopy equivalence. Then $C_\bullet \Omega X$ and $C^\bullet X$ are reflexive by Proposition 7.2.2. Moreover, $C_\bullet \Omega BX \simeq C_\bullet X$ is a proper connective dg algebra and hence reflexive. If BX is $k\pi_1$-local, then by Theorem 7.2.5(3) we see that $C^\bullet BX$ is also reflexive. Note that we have $\pi_1(BX) \simeq \pi_0 X$.

Example 7.2.12 (String topology). Let M be a compact connected manifold, so that there is an S^1-invariant quasi-isomorphism $\mathrm{HH}_\bullet(C_\bullet \Omega M) \simeq C_\bullet(\mathcal{L}M)$, where $\mathrm{HH}_\bullet$ denotes the Hochschild homology complex and $\mathcal{L}M$ denotes the free loop space of M [Jon87]. In particular, if M is $k\pi_1$-local then Proposition 7.2.2, Theorem 7.2.5(4), and the Morita invariance of Hochschild homology gives us a natural quasi-isomorphism

$$\mathrm{HH}_\bullet(\mathcal{D}_{\mathrm{fd}}(C^\bullet M)) \simeq C_\bullet(\mathcal{L}M).$$

Moreover, by e.g. [Goo24] and the Morita invariance of Hochschild cohomology [Kel03] we also obtain a natural quasi-isomorphism

$$\mathrm{HH}^\bullet(\mathcal{D}_{\mathrm{fd}}(C^\bullet M)) \simeq \mathrm{HH}^\bullet(C_\bullet \Omega M)$$

and when X is in addition a d-dimensional Poincaré duality space it follows from e.g. [BCL25, Example 9.38] that we have a quasi-isomorphism

$$\mathrm{HH}^\bullet(\mathcal{D}_{\mathrm{fd}}(C^\bullet M)) \simeq C_\bullet(\mathcal{L}M)[-d].$$

In characteristic zero and when M is simply connected, these are quasi-isomorphisms of BV-algebras [TZ07].

The following is a slight generalisation of Theorem 7.2.5 in the simply connected setting (recall that if $\pi_1(X) \cong 0$ then $H^1(X) \cong 0$ by the Hurewicz and Universal Coefficient Theorems). The proof is a straightforward application of Theorem 5.4.1.

Proposition 7.2.13. *Let X be a path connected k-finite type topological space such that $H^1(X)$ vanishes. Then the dg algebra $C^\bullet X$ is reflexive.*

Example 7.2.14 (Fukaya categories of cotangent bundles). Let M be a compact path connected smooth manifold. In this setting, $C_\bullet \Omega M$ is Morita equivalent to $\mathcal{W}(T^*M)$, the derived wrapped Fukaya category of the symplectic manifold T^*M. If M is $k\pi_1$-local, then using Theorem 7.2.5 we obtain an equivalence $\mathcal{D}_{\mathrm{fd}}(\mathcal{W}(T^*M)) \simeq \mathcal{D}^{\mathrm{perf}}(C^\bullet M)$. When M is in addition simply connected, then $C^\bullet M$ is Morita equivalent to $\mathcal{F}(T^*M)$, the compact Fukaya category of T^*M, and we hence obtain equivalences

$$\mathcal{D}^{\mathrm{perf}}(\mathcal{W}(T^*M)) \simeq \mathcal{D}_{\mathrm{fd}}(\mathcal{F}(T^*M)) \quad \text{and} \quad \mathcal{D}_{\mathrm{fd}}(\mathcal{W}(T^*M)) \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(T^*M)).$$

Versions of the above equivalences for more general simply connected symplectic manifolds were given in [EL23] and versions for Milnor fibres were given in [LU22]. We will see more about Fukaya categories in Section 9.

8. Gluing reflexive dg categories

In this section we note that one can glue (semi)reflexive dg categories along semiorthogonal decompositions. Semiorthogonal decompositions, introduced in [BK89], are a fundamental tool in derived noncommutative algebraic geometry, as they allow one to decompose invariants and to isolate singular behaviour. A survey of their uses can be found in [Kuz14].

Definition 8.0.1. Let $\mathcal{T}$ be a triangulated category. A **semiorthogonal decomposition** of $\mathcal{T}$ is a pair of thick subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathcal{T}$ satisfying the following properties:

(1) The smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{T}$ itself.
(2) $\mathrm{Hom}_{\mathcal{T}}(b, a) = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

In this situation we write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$.

If $\mathcal{T}$ is a pretriangulated dg category, we say that a semiorthogonal decomposition of $\mathcal{T}$ is a semiorthogonal decomposition of the triangulated category $H^0(\mathcal{T})$.

Remark 8.0.2. Semiorthogonal decompositions were studied in the context of reflexivity in [KS25]. If $\mathcal{T}$ is a pretriangulated dg category with a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ then there is a semiorthogonal decomposition $\mathcal{D}_{\mathrm{fd}}(\mathcal{T}) = \langle \mathcal{D}_{\mathrm{fd}}\mathcal{B}, \mathcal{D}_{\mathrm{fd}}\mathcal{A} \rangle$ [KS25, Lemma 3.7]. Moreover, if $\mathcal{T}$ is reflexive then so are $\mathcal{A}$ and $\mathcal{B}$; this follows from naturality of the evaluation functor. Alternatively, this can be seen using the monoidal characterisation of [Goo24], using the fact that reflexive objects are closed under retracts.

Theorem 8.0.3. *Let $\mathcal{T}$ be a semireflexive dg category that admits a semiorthogonal decomposition $\mathcal{D}^{\mathrm{perf}}(\mathcal{T}) = \langle \mathcal{A}, \mathcal{B} \rangle$. If $\mathcal{A}$ and $\mathcal{B}$ are reflexive then so is $\mathcal{T}$.*

Proof. In this proof, to avoid opposite categories appearing we use the coevaluation functor instead of the evaluation functor. Recall that $\mathrm{coev}_{\mathcal{A}}$ is defined in [KS25] as the composition of $\mathrm{ev}_{\mathcal{A}}$ with the linear dual functor $(-)^*$. Moreover, (semi)reflexivity can be checked using the coevaluation functor in exactly the same manner as the evaluation functor, cf. [KS25, Lemma 3.10].

Without loss of generality we may assume that $\mathcal{T}$ is pretriangulated and idempotent complete, so that $\mathcal{T} = \mathcal{D}^{\mathrm{perf}}(\mathcal{T})$. Applying [KS25, Lemma 3.7] twice, we obtain a semiorthogonal decomposition

$$\mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T}) = \langle \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{A}), \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{B}) \rangle$$

It follows from the proof of [KS25, Lemma 3.7] that the semiorthogonal decompositions are compatible with the (co)evaluation functors, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{coev}_{\mathcal{A}}} & \mathcal{D}_{\mathrm{fd}}\left(\mathcal{D}_{\mathrm{fd}}(\mathcal{A})\right) \\ \downarrow{\scriptstyle i_{\mathcal{A}}} & & \downarrow{\scriptstyle i'_{\mathcal{A}}} \\ \mathcal{T} & \xrightarrow{\mathrm{coev}_{\mathcal{T}}} & \mathcal{D}_{\mathrm{fd}}\left(\mathcal{D}_{\mathrm{fd}}(\mathcal{T})\right). \end{array}$$

Here $i_{\mathcal{A}}$ and $i'_{\mathcal{A}}$ denote the inclusions. Note that these functors have left adjoints, which we denote by $\pi_{\mathcal{A}}$ and $\pi'_{\mathcal{A}}$ respectively. By assumption $\mathcal{T}$ is semireflexive (i.e. $\mathrm{coev}_{\mathcal{T}}$ is fully faithful) and so to prove that it is reflexive we need only check that $\mathrm{coev}_{\mathcal{T}}$ is essentially surjective. Take $M \in \mathcal{D}_{\mathrm{fd}}\mathcal{D}_{\mathrm{fd}}(\mathcal{T})$. Then there is an exact triangle

$$i'_{\mathcal{B}}\pi'_{\mathcal{B}}M \to M \to i'_{\mathcal{A}}\pi'_{\mathcal{A}}M \to$$

Since $\mathcal{A}$ is reflexive we have $\pi'_{\mathcal{A}}M \simeq \mathrm{coev}_{\mathcal{A}}(a)$ for some $a \in \mathcal{A}$. So we have

$$i'_{\mathcal{A}}\pi'_{\mathcal{A}}M \simeq i'_{\mathcal{A}}\,\mathrm{coev}_{\mathcal{A}}(a) \simeq \mathrm{coev}_{\mathcal{T}}\, i_{\mathcal{A}}(a)$$

Similarly, $i'_{\mathcal{B}}\pi'_{\mathcal{B}}M \simeq \mathrm{coev}_{\mathcal{T}}\, i_{\mathcal{B}}(b)$ for some $b \in \mathcal{B}$, and since $\mathcal{T}$ is semireflexive the morphism

$$\mathrm{coev}_{\mathcal{T}}\, i_{\mathcal{A}}(a)[-1] \to \mathrm{coev}_{\mathcal{T}}\, i_{\mathcal{B}}(b)$$

in the triangle above can be lifted to some $f\colon i_{\mathcal{A}}(a)[-1] \to i_{\mathcal{B}}(b)$. Therefore it follows that $\mathrm{coev}_{\mathcal{T}}(\mathrm{cone}(f)) \simeq M$, as required. □

Remark 8.0.4. Examples of semireflexive dg categories include proper dg categories (Example 2.1.3) and dg algebras which are derived complete with respect to a $\mathcal{D}_{\mathrm{fd}}$-generator (Corollary 2.3.9).

Remark 8.0.5. If $\mathcal{A}$, $\mathcal{B}$ are two dg categories and M is a $\mathcal{B}$-$\mathcal{A}$-bimodule, then one can glue $\mathcal{A}$ and $\mathcal{B}$ along M to produce a new dg category $\mathcal{A} \xrightarrow{M} \mathcal{B}$. The objects of the gluing are $\mathrm{Ob}(\mathcal{A}) \sqcup \mathrm{Ob}(\mathcal{B})$, and the morphisms are given by upper-triangular matrices. Even if both $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated, then usually $\mathcal{A} \xrightarrow{M} \mathcal{B}$ will fail to be; in this situation we let $\mathcal{A} \vdash_M \mathcal{B} := \mathcal{D}^{\mathrm{perf}}(\mathcal{A} \xrightarrow{M} \mathcal{B})$. Orlov [Orl16] shows that $\mathcal{A} \vdash_M \mathcal{B}$ admits a semiorthogonal decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$, and moreover if $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ is any pretriangulated dg category then there is a bimodule M and a quasi-equivalence $\mathcal{T} \simeq \mathcal{A} \vdash_M \mathcal{B}$. Note that $\mathcal{A} \vdash_M \mathcal{B}$ is reflexive exactly when the gluing $\mathcal{A} \xrightarrow{M} \mathcal{B}$ is reflexive.

9. Graded gentle algebras

We apply the techniques of this paper to study reflexivity for partially wrapped Fukaya categories of surfaces in the sense of [HKK17] and the related class of graded gentle algebras.

9.1. **Topological notions.** By a **marked surface** we mean a pair $(\Sigma, \mathcal{M})$, where Σ is a compact, oriented smooth surface with boundary $\partial\Sigma \neq \emptyset$ and where $\mathcal{M} \subseteq \partial\Sigma$ is a compact subset which intersects each boundary component non-trivially and such that each connected component of $\mathcal{M}$ has non-trivial interior. We often omit the set $\mathcal{M}$ from the notation and refer to a marked surface as simply Σ. Boundary components $B \subseteq \partial\Sigma$ such that $B \subseteq \mathcal{M}$ are called **fully marked** while all other connected components of $\mathcal{M}$, namely those homeomorphic to $[0,1]$, are called **marked intervals**. As consequence of this definition, $\mathcal{M}$ is the union of the fully marked components and the marked intervals. By a **boundary segment** we mean the closure in Σ of a connected component of $\partial\Sigma \setminus \mathcal{M}$.

A **simple arc** on a graded surface $(\Sigma, \mathcal{M})$ is an embedded path $\gamma\colon [0,1] \hookrightarrow \Sigma$ such that $\gamma^{-1}(\mathcal{M}) = \{0,1\}$. Such an arc is said to be **finite** if both its end points lie in marked intervals. By an **isotopy** between simple arcs γ_0 and γ_1 we mean a map $H\colon [0,1] \times [0,1] \to \Sigma$ such that for all $t \in [0,1]$, $H_t \coloneqq H|_{\{t\}\times[0,1]}$ is a simple arc and such that $H^{-1}(\mathcal{M}) = [0,1] \times \{0,1\}$ as well as $H_0 = \gamma_0$ and $H_1 = \gamma_1$.

An **arc system** Γ is a collection of pairwise disjoint and pairwise non-isotopic simple arcs on Σ. It follows that Γ is necessarily a finite set. An arc system Γ is **full** if its complement

$$\Sigma \setminus \Gamma \coloneqq \Sigma \setminus \big(\bigcup_{\gamma\in\Gamma} \gamma([0,1])\big)$$

is homeomorphic to a disjoint union of disks, the boundary of each of which contains at most one boundary segment. Likewise, Γ is **finitely-full** if its complement is homeomorphic to a disjoint union of disks as above as well as half-open cylinders $C = [0,1)\times S^1$ such that $\partial C = \{0\}\times S^1$ corresponds to a fully marked component. A full (resp. finitely-full) arc system is **formal** if the boundary of every disk component of $\Sigma \setminus \Gamma$ contains a (and hence exactly one) boundary segment.

Of course, full formal arc systems can only exist on surfaces with at least one marked interval but in fact, this is also a sufficient condition.

Proposition 9.1.1 ([HKK17])**.** *If Σ contains at least one marked interval, then Σ admits a full, formal arc system.*

We note that the same is true for finitely-full, formal arc systems under the same assumptions; the proof is similar.

A **flow** from an arc $\gamma_1 \in \Gamma$ to a (possibly identical) arc $\gamma_2 \in \Gamma$ is the homotopy class of a path f inside $\mathcal{M}$ which follows the natural orientation of $\partial\Sigma$ and which starts on a start or end point of γ_1 and which ends on a start or end point of γ_2. A flow f is **constant** if it agrees with the homotopy class of a constant path. Flows can be composed as paths and a flow is **irreducible** if it cannot be decomposed further into non-constant flows between arcs of Γ. As such, every flow between arcs of Γ is a finite composition of irreducible ones in a unique way.

9.2. **Partially wrapped Fukaya categories and gentle algebras.** By a **graded marked surface** we mean a triple $(\Sigma, \mathcal{M}, \eta)$ consisting of a graded surface $(\Sigma, \mathcal{M})$ and a **line field** η on Σ, that is, a section $\eta\colon \Sigma \to \mathbb{P}(T\Sigma)$ of the projectivised tangent bundle. As with $\mathcal{M}$, the line field is frequently omitted from our notation, so that the graded marked surface above is simply referred to by Σ. The presence of a line field allows one to endow every simple arc on Σ with the extra structure of a grading, cf. [HKK17]. The set of all gradings on every simple arc is (non-canonically) in bijection with $\mathbb{Z}$ and given two graded simple arcs γ_1, γ_2 in an arc system, any flow f from γ_1 to γ_2 naturally inherits a degree $|f| \in \mathbb{Z}$ which is compatible with the composition of flows, cf. [HKK17, (3.17)]. We denote by $\mathrm{Flow}(\gamma_1, \gamma_2)$ the set of flows from γ_1 to γ_2.

Definition 9.2.1 ([HKK17]). Let Σ be a graded marked surface with at least one marked interval and let $\Gamma \subseteq \Sigma$ be a graded full (resp. finitely-full) formal arc system. Define $\mathcal{F} = \mathcal{F}(\Gamma)$ as the k-linear graded category with $\mathrm{Ob}(\mathcal{F}) = \Gamma$ and morphism spaces

$$\mathrm{Hom}^\bullet_{\mathcal{F}}(\gamma_1, \gamma_2) := k\,\mathrm{Flow}(\gamma_1, \gamma_2),$$

for all γ_1, γ_2, where each $f \in \mathrm{Flow}(\gamma_1, \gamma_2)$ is regarded as a homogeneous element of degree $|f|$. The composition law for morphisms is the unique k-linear extension of the composition of flows so that $g \circ f = 0$ whenever the start point of the flow g does not agree with the end point of f. The **partially wrapped Fukaya category** of Σ is the category $\mathrm{Fuk}(\Sigma) := \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma))$.

Example 9.2.2. Let $n \in \mathbb{Z}$ and $A = k[t]$ with $|t| = n$. Then $A = \mathcal{F}(\{\gamma\})$, where $\gamma \subseteq \Sigma$ is any embedded arc on an annulus with one fully marked component and one marked interval which connects the fully marked component and the marked interval. The degree of $|t|$ is controlled by the line field.

Although a priori $\mathrm{Fuk}(\Sigma)$ depends on the choice of an arc system, it turns out that its Morita equivalence class is well-defined, cf. Proposition 9.2.5 below.

We note that $\mathcal{F}(\Gamma)$ is canonically augmented: for each object $\gamma \in \Gamma = \mathrm{Ob}(\mathcal{F}(\Gamma))$, the augmentation map is the projection $k\,\mathrm{Flow}(\gamma, \gamma) \to k$ onto the basis element of the constant flow. In particular, this induces a canonical augmentation over a product of fields on the category algebra of $\mathcal{F}(\Gamma)$, that is, the algebra

$$\bigoplus_{\gamma, \gamma' \in \Gamma} k\,\mathrm{Flow}(\gamma, \gamma'),$$

obtained by taking the (finite) direct sum of all morphism spaces in $\mathcal{F}(\Gamma)$.

The following proposition recalls a few useful properties of the previous constructions. It follows from [HKK17, Proposition 3.5] and Definition 9.2.1.

Proposition 9.2.3. *Let Γ be a graded full or finitely-full formal arc system on a graded marked surface Σ with at least one marked interval. Then the following hold:*

(1) *$\mathcal{F}(\Gamma)$ is proper if and only if Γ is finitely-full;*
(2) *$\mathcal{F}(\Gamma)$ is a smooth dg category if and only if Γ is full; in this setting,* $\mathrm{Fuk}(\Sigma)$ *is also smooth.*
(3) $\mathrm{Fuk}(\Sigma)$ *is smooth and proper if and only if Σ has no fully marked components. Equivalently, this is the case if and only if Σ admits a formal arc system which is simultaneously full* and *finitely-full.*

The category algebra of $\mathcal{F}(\Gamma)$ can be described by a quiver with quadratic relations and vertex set Γ whose set of arrows is given by the set of irreducible flows. The set of relations consists of all paths gf of length 2 such that $g \circ f = 0$ in $\mathcal{F}(\Gamma)$. Algebras of this kind are graded and possibly infinite dimensional analogoues of

gentle algebras which were first introduced in [AS87]. Every smooth graded gentle algebra A arises as the category algebra of $\mathcal{F}(\Gamma)$ for some full formal arc system Γ on a graded marked surface Σ_A. Likewise, every proper graded gentle algebra B is the category algebra of $\mathcal{F}(\Gamma)$ for some finitely-full formal arc system Γ on a surface Σ_B. For an explicit construction of the surfaces associated to a graded gentle algebra we refer the reader to [HKK17, OPS18, LP20]. The definition of $\mathcal{F}$ can be extended to non-formal arc systems which are full or finitely-full [HKK17] and hence $\mathrm{Fuk}(\Sigma)$ can be defined for all graded marked surfaces Σ. In the non-formal case, $\mathcal{F}(\Gamma)$ is a minimal A_∞-category whose underlying graded k-linear category is constructed in the same way as in the formal case. However, in addition to composition, higher A_∞-compositions are contributed by those disks in $\Sigma \setminus \Gamma$ which do *not* contain a boundary segment, or in other words, disks which are bounded by alternating sequences of marked intervals and arcs of Γ. For a formal arc system Γ the complement $\Sigma \setminus \Gamma$ does not contain such disks, and the A_∞-category $\mathcal{F}(\Gamma)$ is formal and reduces to the one of Definition 9.2.1. Next, we recall an important property of the construction $\Gamma \mapsto \mathcal{F}(\Gamma)$.

Proposition 9.2.4 ([HKK17, Lemma 3.2])**.** *Let $\Gamma \subseteq \Gamma' \subseteq \Sigma$ be full (resp. finitely-full) arc systems. Then the canonical inclusion $\mathcal{F}(\Gamma) \hookrightarrow \mathcal{F}(\Gamma')$ is a Morita equivalence, that is, the induced A_∞-functor $\mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma)) \to \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma'))$ is an equivalence of triangulated categories.*

We note that while the previous proposition appears in [HKK17] only for full arc systems, its proof extends immediately to the finitely-full case. It implies the following.

Proposition 9.2.5. *Let $\Gamma, \Gamma' \subseteq \Sigma$ be full (resp. finitely-full) arc systems. Then $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Gamma')$ are Morita equivalent.*

Proof. For full arc systems, this was proved in [HKK17]. It relies on the fact that the category of full arc systems, with morphisms given by inclusions, is contractible, which in turn is a consequence of the contractibility of the arc complex of Σ. Likewise, [Hat91, Corollary] shows that every two inclusion maximal arc systems on a graded marked surface ("triangulations") are connected by a sequence of arc inclusions and removals, or in other words, a zig-zag in the category of arc systems. Since every arc system sits inside an inclusion maximal one, this shows that any two full arc systems can be connected through a zig-zag of inclusions. In fact, the result in *op. cit.* does not require the set of markings $\mathcal{M}$ to intersect every boundary component, and when applied to the "marked" surface obtained by removing all fully marked components from $\mathcal{M}$, the previous arguments imply that any two finitely-full arc systems are connected by a zig-zag of inclusions passing through finitely-full arc systems. Now Proposition 9.2.4 implies that $\mathcal{F}(\Gamma)$ are $\mathcal{F}(\Gamma')$ are Morita equivalent. $\square$

9.3. **Koszul duality between smooth and proper gentle algebras.** We recall the precise relationship between proper and smooth graded gentle algebras. In what follows Σ denotes a graded marked surface.

Suppose that $\Gamma \subseteq \Sigma$ is a graded full formal arc system. We construct a dual finitely-full arc system $\Gamma^! \subseteq \Sigma$ from Γ as follows. For every arc $\gamma \in \Gamma$, denote by $\gamma^\perp$ any arc which intersects γ transversally in the interior but no other arc $\delta \in \Gamma \setminus \{\gamma\}$ and whose end points lie in the unique boundary segment of the two disks of $\Sigma \setminus \Gamma$

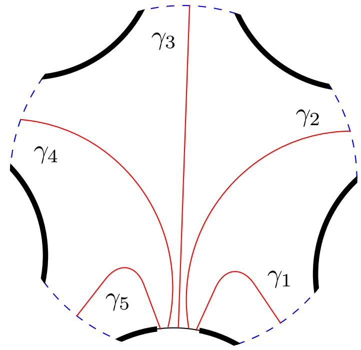

FIGURE 1. Marked intervals (black solid) and arcs $\gamma_1, \ldots, \gamma_5$ (blue dashed) and $\gamma_1^\perp, \ldots, \gamma_5^\perp$ (red solid). The duals $\gamma_i^!$ are obtained by sliding the end points of the arcs $\gamma_i^\perp$ on the boundary segment onto the neighbouring marked interval to their right while keeping the linear order among each other.

which border γ from either side, see Figure 1. Then let $\gamma^!$ denote any simple arc obtained from $\gamma^\perp$ by sliding both its end points along $\partial\Sigma$, following the induced orientation of $\partial\Sigma$, until the end points land in the marked interval next to the respective boundary segment. The collection $\Gamma^! \coloneqq \{\gamma^! \mid \gamma \in \Gamma\}$ is well defined up to isotopy. By deforming $\gamma^!$ if necessary, we can arrange that $\Gamma^!$ is an arc system (which is then finitely-full) and that for all $\delta \in \Gamma$, $\gamma^!$ and δ are in minimal position, that is, all their intersections are transversal and the number of their intersections is minimal within their respective homotopy classes.

Note that by construction, either $|\operatorname{Flow}(\gamma, \gamma^!)| = 1$ or γ and $\gamma^!$ intersect in a single (interior) point. On the other hand, for all other $\delta \in \Gamma \setminus \{\gamma\}$, δ and $\gamma^!$ are disjoint and $\operatorname{Flow}(\delta, \gamma^!) = \emptyset$. In fact these properties uniquely characterise the isotopy classes in the collection $\Gamma^!$. Because each step of the process of passing from γ to $\gamma^!$ is reversible, the assignment $\Gamma \mapsto \Gamma^!$ defines a bijection between the sets of full formal arc systems and finitely-full and formal arc systems on Σ up to isotopy.

Although it will not be very important in this paper, $\gamma^!$ inherits a canonical grading from γ by requiring that the unique flow $f \in \operatorname{Flow}(\gamma, \gamma^!)$ or the unique intersection of γ and $\gamma^!$ is of degree 0. Moreover, with the given grading, there is a bijection

$$\operatorname{Flow}(\gamma_1, \gamma_2) \xrightarrow{\sim} \operatorname{Flow}(\gamma_2^!, \gamma_1^!),$$

$$f \longmapsto f^!$$

such that $|f^!| = 1 - |f|$. This arc system duality is a topological incarnation of Koszul duality:

Proposition 9.3.1 ([OPS18, Appendix C])**.** *Let $\Gamma \subseteq \Sigma$ be a full formal arc system and let A denote the category algebra of $\mathcal{F}(\Gamma)$ endowed with its canonical augmentation. Then $A^!$ is quasi-isomorphic to the category algebra of $\mathcal{F}(\Gamma^!)$ with $\Gamma^!$ endowed with the induced grading.*

Example 9.3.2. Let A and Σ be as in Example 9.2.2. Then $A^!$ is quasi-isomorphic to $k[x]/(x^2)$ with $|x| = 1 - |t|$. The corresponding arc system of $A^!$ on the annulus is the embedded arc which starts and ends on the marked interval and which traverses the fully marked component once.

In particular, Koszul duality induces a bijection between the sets of isomorphism classes of smooth graded gentle algebras and the set of isomorphism classes of proper graded gentle algebras. To state the precise Koszul duality on the level of categories, we recall that every embedded, oriented closed curve γ on Σ inherits a **winding number** $\omega(\gamma) \in \mathbb{Z}$ thanks to the line field η, cf. [LP20]. The winding number of a boundary component $B \subseteq \partial\Sigma$ is by definition the winding number of the closed curve corresponding to the *orientation preserving* identification $S^1 \cong B \subset \partial\Sigma$. Here, B is endowed with the induced orientation which it inherits from the orientation of Σ in the usual way. If B is a fully marked component, $\omega(B)$ has a particularly simply algebraic interpretation: for any arc γ with an endpoint p on B, $\omega(B)$ coincides with the degree of the flow which starts and ends at p and traverses B exactly once. On the level of categories, Koszul duality gives us the following relationship.

Proposition 9.3.3 ([OPS18, Appendix C, Step 6])**.** *Let $\Gamma \subseteq \Sigma$ be a full formal graded arc system and $\Gamma^! \subseteq \Sigma$ its dual equipped with any grading. Let further $\mathcal{T} \subseteq \mathcal{D}(\mathcal{F}(\Gamma^!))$ denote the thick subcategory associated to the augmentation of $\mathcal{F}(\Gamma)^!$, that is, $\mathcal{T} \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma)^{!!})$. Then there exists an essentially surjective exact functor*

$$\mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma^!)) \longrightarrow \mathcal{T}.$$

which is an equivalence if and only if Σ contains no fully marked components with vanishing winding number. Moreover, this is the case if and only if $\mathcal{F}(\Gamma)$ is derived complete.

In fact, as shown in [OPS18], the functor in the previous proposition is equivalent to the derived completion functor $\mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma)) \to \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma)^{!!})$.

Corollary 9.3.4. *If Σ contains no fully marked components with vanishing winding number then*

$$\mathrm{Fuk}(\Sigma) \simeq \mathbf{thick}_A A/\operatorname{rad}(A)$$

for some proper graded gentle algebra A with graded marked surface $\Sigma_A \cong \Sigma$.

9.4. **Semiorthogonal decompositions, reflexivity and $\mathcal{D}_{\mathrm{fd}}$-generators.**

Proposition 9.4.1. *Let A be a proper graded gentle algebra and let $\{B_1, \dots, B_n\} \subset \partial\Sigma_A$ denote its set of fully marked components. Then there exists a semiorthogonal decomposition of the form*

$$\mathcal{D}^{\mathrm{perf}}(A) \simeq \langle \mathcal{D}^{\mathrm{perf}}(A^{(n)}), \mathcal{D}^{\mathrm{perf}}(k[x_1]/(x_1^2)), \dots, \mathcal{D}^{\mathrm{perf}}(k[x_n]/(x_n^2)) \rangle,$$

where $|x_i| = 1 - \omega(B_i)$ and $A^{(n)}$ is a smooth proper graded gentle algebra.

Proof. Set $\Sigma = \Sigma_A$. Because A is proper, there exists a marked interval $I \subseteq \partial\Sigma_A$. Now choose a simple arc $\gamma_n \colon [0, 1] \to \Sigma_A$ which starts in I and ends on B_n. Then by concatenating γ_n first with the closed embedded boundary path with endpoints $\gamma_n(1)$ which traverses B_n in counter-clockwise direction and then with the inverse of γ_n, we obtain a path δ_n which can be homotoped relative to $\partial\Sigma$ into a simple arc with endpoints in $x, y \in I$. The natural orientation of $I \subseteq \partial\Sigma$ then induces a total order on the set $\{x, y\}$, say $x < y$. The complement of δ_n in Σ has two connected components, one of which, which we shall call U, is a half-open cylinder

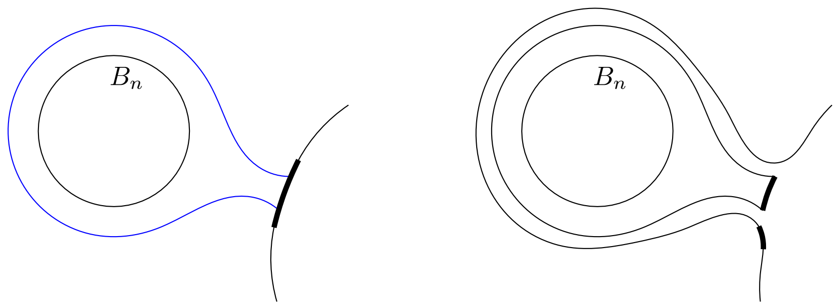

FIGURE 2. The arc δ_n on the left (blue) and the two connected components of $\Sigma^{(1)}$ on the right. The connected component on the right containing B_n gives rise to the algebra $k[x_n]/(x_n^2)$.

$C \cong [0,1) \times S^1$ whose boundary $\{0\} \times S^1$ coincides with B_n. After smoothing out the complement of δ_n, we obtain a graded surface $\Sigma^{(1)} := \Sigma \setminus \delta_n \subset \Sigma$ whose line field is obtained by restriction of η_A. We endow it with the set of markings obtained from the set of markings of Σ by removing any half-open interval $J \subsetneq I$ which contains y but not x and such that $I \setminus J$ is connected, cf. Figure 2. Choose now a finitely-full formal graded arc system $\mathcal{A}$ of $\Sigma^{(1)}$. Note that, there is a unique arc, say $\gamma_U \in \mathcal{A}$ which lies in in U. Equivalently, we may regard $\mathcal{A}$ as a finitely-full and formal arc system on the original surface Σ. Let $A^{(1)}$ (resp. A') denote the proper graded gentle algebra associated to the arc system $\mathcal{A}^{(1)} := \mathcal{A} \setminus \{\gamma_U\}$ (resp. $\mathcal{A}$). The important point is that, by construction, there are no flows from γ_U to any of the arcs in $\mathcal{A} \setminus \{\gamma_U\}$ and hence we obtain a semiorthogonal decomposition of the form

$$\mathcal{D}^{\mathrm{perf}}(A') = \langle \mathcal{D}^{\mathrm{perf}}(A^{(1)}), \mathcal{D}^{\mathrm{perf}}(k[x_n]/(x_n^2)) \rangle,$$

where $k[x_n]/(x_n^2)$ is the graded gentle algebra associated to γ_U. Furthermore, by Proposition 9.2.5, A and A' are Morita equivalent, so that $\mathcal{D}^{\mathrm{perf}}(A)$ has the same semiorthogonal decomposition. Now, we simply iterate the previous constructions, starting with the connected component of $\Sigma^{(1)}$ which contains the arc system $\mathcal{A}^{(1)}$ corresponding to $A^{(1)}$ and finding a semiorthogonal decomposition of $\mathcal{D}^{\mathrm{perf}}(A^{(1)})$ of the same shape. Induction over the number of fully marked components together with Proposition 9.2.3 then eventually yields a choice of a graded finitely-full formal arc system on Σ whose associated graded proper gentle algebra B is Morita equivalent to A and such that $\mathcal{D}^{\mathrm{perf}}(B)$ admits the desired semiorthogonal decomposition with $A^{(n)}$ and each algebra $k[x_i]/(x_i^2)$ corresponding to an idempotent subalgebra of B. □

Lemma 9.4.2. *Let A be the dg algebra $k[x]/(x^2)$ with $|x| = n \in \mathbb{Z}$. Then A is reflexive and the unique simple A-module is a $\mathcal{D}_{\mathrm{fd}}$-generator.*

Proof. It is easy to check that A is derived complete at k. Hence for $n < 0$, reflexivity follows from Theorem 3.3.1 and the generation statement from Proposition 3.1.5. For $n \geq 0$, reflexivity follows from Proposition 3.4.1 and the generation statement from Proposition 2.2.2. □

Theorem 9.4.3. *For every proper graded gentle algebra A, $\mathcal{D}^{\mathrm{perf}}(A)$ and $\mathcal{D}_{\mathrm{fd}}(A)$ are reflexive dg categories.*

Proof. This follows from the repeated application of Theorem 8.0.3 to Proposition 9.4.1, combined with the fact that smooth proper dg categories are reflexive (Example 2.1.3) and the dual numbers are reflexive (Lemma 9.4.2). □

The decomposition of Proposition 9.4.1 can also be used to show that the maximal semisimple quotient of any proper graded gentle algebra is a $\mathcal{D}_{\mathrm{fd}}$-generator:

Theorem 9.4.4. *Let A be a proper graded gentle algebra. Then $A/\operatorname{rad}(A)$ is a $\mathcal{D}_{\mathrm{fd}}$-generator for A. In other words, the thick subcategory of $\mathcal{D}(A)$ generated by the simple A-modules coincides with $\mathcal{D}_{\mathrm{fd}}(A)$.*

Proof. We know from Proposition 9.4.1 and its proof that A is Morita equivalent to a proper graded gentle algebra B which admits a semiorthogonal decomposition of the form

$$\mathcal{D}^{\mathrm{perf}}(B) \simeq \langle \mathcal{D}^{\mathrm{perf}}(B'), \mathcal{D}^{\mathrm{perf}}(k[x_1]/(x_1^2)), \dots, \mathcal{D}^{\mathrm{perf}}(k[x_n]/(x_n^2))\rangle,$$

so that B' and all graded dual numbers correspond to idempotent subalgebras of B. Moreover, B' is a smooth proper graded gentle algebra. We claim that this is sufficient to conclude that $\mathcal{D}_{\mathrm{fd}}(B)$ is generated by $B/\operatorname{rad}(B)$. Indeed, the above semiorthogonal decomposition induces a semiorthogonal decomposition of $\mathcal{D}_{\mathrm{fd}}(B)$ (with orders reversed). On the other hand, by Lemma 9.4.2, $\mathcal{D}_{\mathrm{fd}}(k[x_i]/(x_i^2))$ is generated by the simple $k[x_i]/(x_i^2)$-module and, as a thick subcategory of $\mathcal{D}_{\mathrm{fd}}(B)$ is generated by the simple A-module corresponding to the idempotent of A associated with the subalgebra $k[x_i]/(x_i^2)$. Because B' is smooth and proper (and hence reflexive), $\mathcal{D}_{\mathrm{fd}}(B') \simeq \mathcal{D}^{\mathrm{perf}}(B')$ is generated by the simple B'-modules. As before, these can equally be considered as simple A-modules which generate the thick subcategory of $\mathcal{D}_{\mathrm{fd}}(B)$ corresponding to $\mathcal{D}_{\mathrm{fd}}(B')$. Altogether, this shows that $B/\operatorname{rad}(B)$ generates $\mathcal{D}_{\mathrm{fd}}(B)$.

Since B is reflexive by Theorem 9.4.3 and derived complete (it is proper), is thus follows from Lemma 2.3.12 that $B^!$ is reflexive and $\mathbf{thick}_{B^!}(B/\operatorname{rad}(B)) = \mathcal{D}_{\mathrm{fd}}(B^!)$. Hence, in order to show that $A/\operatorname{rad}(A)$ is a generator of $\mathcal{D}_{\mathrm{fd}}(A)$, it suffices by Lemma 2.3.12 to show that there is an equivalence $\mathcal{D}^{\mathrm{perf}}(A^!) \simeq \mathcal{D}^{\mathrm{perf}}(B^!)$ which identifies $\mathbf{thick}_{A^!}A/\operatorname{rad}(A)$ with $\mathbf{thick}_{B^!}B/\operatorname{rad}(B)$. We recall from Proposition 9.3.1, that, up to passing to a category algebra, A and B are the Koszul duals of $\mathcal{F}_A = \mathcal{F}(\Gamma_A)$ and $\mathcal{F}_B = \mathcal{F}(\Gamma_B)$ for full formal arc systems $\Gamma_A, \Gamma_B \subseteq \Sigma$ on a graded marked surface Σ. In other words, $A^!$ and $B^!$ are the derived completions of $\mathcal{F}_A$ and $\mathcal{F}_B$ at the thick subcategories $\mathcal{B}_A$ and $\mathcal{B}_B$ generated by the respective canonical augmentations. But now we recall from [OPS18, Appendix C, Step 6 (3)] that under the equivalence $\mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma_A)) \simeq \mathcal{D}^{\mathrm{perf}}(\mathcal{F}(\Gamma_B))$ from Proposition 9.2.5, $\mathcal{B}_A$ and $\mathcal{B}_B$ are mapped to each other. Our assertion therefore follows from the Morita invariance of derived completion, cf. Remark 2.3.4(2). □

By Corollary 9.3.4, we also obtain the following:

Corollary 9.4.5. *If Σ is a graded marked surface with at least one marked interval and without fully marked components of vanishing winding number, then* $\operatorname{Fuk}(\Sigma)$ *is reflexive.*

Example 9.4.6. One cannot drop the assumption on the winding numbers. Indeed, the polynomial ring $k[x]$ with x in degree zero is not reflexive by Theorem 4.0.4.

The associated surface is an annulus with one marked interval and one fully marked component whose winding numbers vanish.

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